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A Note on the Hausdorff Dismensions
- Hausdorff 차원에 관하여 -

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요 지

프랙탈이란 말은 라틴어 Fractus(부서진 상태를 뜻함)에서 유래되었으며, 1975년 Mandelbrot가 수학및 자연계의 비 정규적 패턴들에 대한 체계적 고찰을 담은 자신의 에세이의 표제를 주기 위해서 만들었다 ([6]). 프랙탈을 기술하는데 있어서 가장 중요한 양은 차원(dimension)으로 프랙탈 차원은 정수차원이 아닌 실수 차원을 갖는다. 이 논문에서는 box counting 차원, Hausdorff 차원, s-potential및 s-energy등에 대한 정의를 하고 정리 2.8, 정리 3.1및 정리 3.2을 증명하고자 한다.

1.Introduction

The word 'fractal' was coined by Mandelbrot in his fundamental essay from the Latin fractus, meaning broken, to describe objects that were too irregular to fit into a traditional geometrical setting([6]). The main tool of fractal is dimension in its many forms. Here, the fractal dimension of set is a number which tells how densely the set occupies the metric space in which it lies. In particular, fractal dimension for non-smooth and irregular sets might be a real number that is not an integer number. In this paper, we will define box counting dimension, Hausdorff dimension, s-potential and s-energy of μ etc, and will prove theorem 2.8, theorem 3.1 and theorem 3.2.

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2.Preliminaries

Let (X,d) denote a complete metric space, and let $H(X)$ be the Hausdorff measure of X and A be non-empty compact subset of X. For any $\varepsilon>0$, let $B_\varepsilon(x)$ be the closed ball of radius ε and center at a point $x\in X$, and let $N(A,\varepsilon)$ be the smallest positive integer M such that $A\subset \bigcup_{n=1}^{M} B_\varepsilon(x_n)$, $x_1,x_2,\cdots,x_n\in X$.

Definition 2.1. Define the *box-counting dimension* or *box dimension* of A as follows ;

(1) $\dim_B A=\lim_{\delta\to 0}\frac{\log(N(A,\delta))}{-\log\delta}$.

Roughly speaking, (1) says that $N(A,\delta)\simeq\delta^{-s}$ for small δ, where $s=\dim_B A$.

Define *the lower box counting dimension* of A denoted by $\underline{\dim}_B A$ as follows;

$\underline{\dim}_B A=\lim_{\delta\to 0}\inf\frac{\log(N(A,\delta))}{-\log\delta}$.

Define *the upper box counting dimension* of A denoted by $\overline{\dim}_B A$ as follows;

$\overline{\dim}_B A=\lim_{\delta\to 0}\sup\frac{\log(N(A,\delta))}{-\log\delta}$ ([1],[5],[6]).

Definition 2.2. Let U be any non-empty subset of n-dimensional Euclidean space $\mathbb{R}^n$ and *the diameter of* U defined as $|U| = \sup\{\,|x-y|\,;x,y\in U\}$.

Let F be a subset of $\mathbb{R}^n$ and let s be a non-negative number. For any $\delta>0$, we define $H^s_\delta(F)=\inf\{\sum_{i=1}^{\infty}|U_i|^s;\{U_i\}$ is a δ-cover of F $\}$ and $H^s(F)=\lim_{\delta\to 0}H^s_\delta(F)$. We call $H^s(F)$ the *s-dimensional Hausdorff measure* of F. If t>s and $\{U_i\}$ is a δ-cover of F, we have $\sum_i|U_i|^s\le\delta^{t-s}\sum_i|U_i|^s$.

Here we known that there is a critical value of s at which $H^s(F)$ 'jumps' from ∞ to 0. This critical value is called *the Hausdorff dimension* of F and written $\dim_H F$. Accordingly,

$\dim_H F=\inf\{s\,;H^s(F)=0\}=\sup\{s\,;H^s(F)=\infty\}$

so that $H^s(F)=\begin{cases}\infty & \text{if } s<\dim_H F\\ 0 & \text{if } s>\dim_H F\end{cases}$ ([1],[2],[5],[6],[7]).

Definition 2.3. Let μ be a mass distribution on F with $I_s(\mu)<\infty$, where $I_s(\mu)$ is the s-energy of μ.

(1) The *s-potential* at a point x of $\mathbb{R}^n$ is defined as follows;

$\varphi_s(x)=\int\frac{du(y)}{|x-y|^s}$, for $s\ge 0$.

(2) The *s-energy of* μ is defined as follows ;

$$I_s(\mu)=\int \varphi_s(x)du(x)=\int\int \frac{du(x)du(y)}{|x-y|^s} \quad ([6]).$$

Definition 2.4. Let (X,ρ) be a metric space and B the set of all open balls of X. A countable family of bounded subsets of X will be denoted by $\mathfrak{R}$ and $D(\mathfrak{R})=\sup_{E\in\mathfrak{R}}|E|$.

If $E\subset X$, $r>0$, $A(E,r)=\{\mathfrak{R}/D(\mathfrak{R})\le r,\ E\subset\cup\mathfrak{R}\}$ and $B(E,r)=\{\mathfrak{R}\subset B/D(\mathfrak{R})\le r,$ the elements of $\mathfrak{R}$ do not overlap and $^\forall B\in\mathfrak{R};\ \rho\ (B,E)=0\}$. Let $\Lambda=id_{\mathrm{IR}^+}$ $(\Lambda(t)=t,$ $\Lambda^a(t)=t^a,\ a\in\mathrm{IR}^+)$.

$$\begin{aligned}(\Lambda^a-M)(E)&=\sup\{\Lambda^a(\mathfrak{R})\mid \mathfrak{R}\in B(E,r)\}\\ &=\sup\{\textstyle\sum\Lambda^a(t)\mid t\in\mathrm{IR}^+,\mathfrak{R}\in B(E,r)\}\\ &=\sup\{\textstyle\sum t^a\mid t\in\mathrm{IR}^+,\mathfrak{R}\in B(E,r)\}\quad([2]).\end{aligned}$$

Definition 2.5.

$$\begin{aligned}\Delta(\mathrm{E})&=\inf\{a\in\mathrm{IR}^+/\Lambda^a\text{-}M(E)=0\}\\ &=\sup\{a\in\mathrm{IR}^+/\Lambda^a\text{-}M(E)=\infty\}\end{aligned}$$

with the following properties;

(A_1) Δ is monotone ; $E_1\subset E_2\Rightarrow\Delta(E_1)\le\Delta(E_2)$.

(A_2) Δ is stable ; $\Delta(E_1\cup E_2)=\max(\Delta(E_1),\Delta(E_2))$; this follows from

$$(H-M)(E_1\cup E_2)\le(H-M)(E_1)+(H-M)(E_2).$$

(A_3) $\Delta(E)=\Delta(\overline{E})$ ([2]).

Lemma 2.6. (Mass distribution principle)
Let μ be a mass distribution on F and suppose that for some s there are numbers c>0 and $\delta>0$ such that $\mu(U)\le c|U|^s$ for all sets U with $|U|\le\delta$. Then $H^s(F)\ge\mu(F)/c$ and $s\le\dim_H F\le\underline{\dim}_B F\le\overline{\dim}_B F$.

Proof. The theoretic proof is easy. This follows the proof of Falconer's Mass distribution principle 4.2. ([6]). ///

Lemma 2.7. Let $\mathfrak{E}$ be a family of balls contained in some region of IR^n. Then there is a (finite or countable) disjoint subcollection $\{B_i\}$ such that $\bigcup_{B\in\mathfrak{E}}B\subset\bigcup_i\widehat{B_i}$, where $\widehat{B_i}$ is the closed ball concentric with B_i and of four times the radius.

Proof. The theoretic proof is easy. This follows the proof of Falconer's Covering lemma 4.8. ([6]). ///

Theorem 2.8. Let μ be a mass distribution on $\mathbb{R}^n$. Let $F\subset\mathbb{R}^n$ be a Borel set and $0<c<\infty$ be a constant. If $\overline{\lim_{r\to 0}}\frac{\mu(B_r(x))}{r^s}>c$, for all $x\in F$, then $H^s(F)\le 2^s\mu(\mathbb{R}^n)/c$.

Proof. Let $F\subset\bigcup_{B\in\mathcal{B}} B$. By Lemma 2.7., there exists $B_i\in\mathcal{B}$ such that $\bigcup_{B\in\mathcal{B}} B\subset\bigcup_i \widetilde{B_i}$, where $\widetilde{B_i}$ is the closed ball concentric with B_i and of four times the radius.

So $\{\widetilde{B_i}\}$ is an 8δ-cover of F,

$$H^s_{8\delta}(F)\le\sum_i|\widetilde{B_i}|^s\le 4^s\sum_i|B_i|^s\le 4^s\sum_i c^{-1}2^s\mu(B_i(x))$$
$$=4^s2^sc^{-1}\sum_i\mu(B_i(x))=8^sc^{-1}\sum_i\mu(B_i(x))$$
$$=8^sc^{-1}\mu(\bigcup_i B_i)\le 8^sc^{-1}\mu(\mathbb{R}^n)<\infty .$$

Therefore $H^s(F)\le 8^sc^{-1}\mu(\mathbb{R}^n)<\infty$. Finally, if F is unbounded and $H^s(F)>8^sc^{-1}\mu(\mathbb{R}^n)$, the H^s-measure of some bounded subset of F will also exceed this value, contrary to the above. Therefore $H^s(F)\le 2^s\mu(\mathbb{R}^n)/c$. ///

Lemma 2.9. Let F be a Borel set with $0<H^s(F)<\infty$, then there exists a constant b and a compact set $E\subset F$ with $H^s(E)>0$ such that $H^s(E\cap B_r(x))\le br^s$, for all $x\in\mathbb{R}^n$and $r>0$.

Proof. Let μ be a mass distribution on $\mathbb{R}^n$, let $F\subset\mathbb{R}^n$ be a Borel set and let $0<c<\infty$ be a constant.

If $\overline{\lim_{r\to 0}}\frac{\mu(B_r(x))}{r^s}>c$ for all $x\in F$, then $H^s(F)\le 2^s\mu(\mathbb{R}^n/c)$ ·········(A)

Let $F_1=\{x\in\mathbb{R}^n;\overline{\lim_{r\to 0}}\frac{H^s(F\cap B_r(x))}{r^s}>2^{1+s}\}$ ·······························(B)

Applying to (A),(B), the Lemma will be proved easily. ///

Lemma 2.10. Let F be a Borel subset of $\mathbb{R}^n$ with $H^s(F)=\infty$, then there exists a compact set $E\subset F$ with $0<H^s(F)<\infty$ and such that for some constant b, $H^s(E\cap B_r(x))\le br^s$, for all $x\in\mathbb{R}^n$ and $r\ge 0$.

Proof. Take μ satisfying $\mu(A)=H^s(F\cap A)$. Then by Lemma 2.9, if $F_1=\{x\in\mathbb{R}^n;$

$\overline{\lim_{r\to 0}} \frac{H^s(F\cap B_r(x))}{r^s} > 2^{1+s}\}$, $H^s(F_1) < 2^s 2^{-1-s}\mu(\mathbb{R}^n) = \frac{1}{2}H^s(\mathbb{R}^n \cap F) = \frac{1}{2}H^s(F)$.

Thus $H^s(F_1) \le \frac{1}{2}H^s(F)$. Now, $H^s(F-F_1) = H^s(F) - H^s(F_1) \ge H^s(F) - \frac{1}{2}H^s(F) = \frac{1}{2}H^s(F)$.

Let $E = F - F_1$ and $H^s(E_1) > 0$, then $\overline{\lim_{r\to 0}} \frac{H^s(F\cap B_r(x))}{r^s} \le 2^{1+s}$, for some $x \in E_1$.

Therefore, by Lemma 2.9, there exists $E \subset E_1$, with $H^s(E) > 0$ and a number r_0 such that $\frac{H^s(F\cap B_r(x))}{r^s} \le 2^{s+2}$ for all $x \in E$ and $0 < r \le r_0$. But $\frac{H^s(F\cap B_r(x))}{r^s} \le \frac{H^s(F)}{r_0^s}$, where

$\frac{H^s(F)}{r_0^s} = b$. Thus $H^s(E\cap B_r(x)) \le br^s$. ///

3.Main theorem

Theorem 3.1. Let $F \subset \mathbb{R}^n$. If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $H^s(F) = \infty$ and $\dim_H F \ge s$.

Proof. Suppose that $I_s(\mu) < \infty$ for some mass distribution μ with support contained in F. Define $F_1 = \{x \in F ; \overline{\lim_{r\to 0}} \frac{\mu(B_r(x))}{r^s} > 0\}$. Then $\frac{\mu(B_r(x))}{r^2} = \rho\pi$, where ρ is density. If $x \in F_1$ for each $\varepsilon > 0$, there exists a sequence $\{r_i\}$ of numbers decreasing to 0 such that $\mu(B_r(x)) \ge \varepsilon r_i^s$, $\forall_i$. We claim $\mu(F_1) = 0$. First, if $\mu(\{x\}) > 0$,

$$I_s(\mu) = \int \varphi_s(x)du(x) \quad = \int\int \frac{du(x)du(y)}{|x-y|^s}$$

$$\ge c\int \mu(x)du \ge c\mu(x)\int du, \quad \int du = \infty, \text{ therefore } I_s(\mu) = \infty.$$

Now if $\mu(\{x\}) \le 0$, i.e. $\mu(\{x\}) = 0$, from continuity of μ by taking q_i $(0 < q_i < r_i)$ small enough, $\mu(A_i) \ge \frac{1}{4}\varepsilon r_i^s$ $(i=1,2,\cdots)$, where $A_i = B_{r_i}(x) - B_{q_i}(x)$. $q_i < r_i$ such that $\mu(B_{q_i}(x)) < \delta = \frac{3}{4}\varepsilon r_i^s$, $\mu(A_i) = \mu(B_{r_i}(x)) - \mu(B_{q_i}(x)) \ge \frac{1}{4}\varepsilon r_i^s$. Taking subsequence if necessary, suppose that $r_{i+1} < q_i$, for all i, where A_i are disjoint annulus centered at x. For $x \in F_1$, $\varphi_s(x) = \int \frac{du(y)}{|x-y|^s} \ge \sum_{i=0}^{\infty} \frac{1}{4}\varepsilon r_i^s r_i^{-s} = \infty$.

For $q_i \le |x-y| \le r_i$, $\frac{1}{q_i} \ge \frac{1}{|x-y|} \ge \frac{1}{r_i} \cdot \frac{1}{|x-y|^s} \ge \frac{1}{(r_i)^s}$ and then A_i are disjoint annulus centered at x. $\int_{A_i} \frac{du(y)}{|x-y|^s} \leq \int_{A_i} \frac{du(y)}{r_i^s} = \frac{1}{r_i^s} \cdot \frac{1}{4} \varepsilon r_i^s = \frac{1}{4} \varepsilon$.

But $I_s(\mu) = \int \varphi_s(x) du(x) < \infty$, so $\varphi_s(x) < \infty$,

$\varphi_s(x) = \int \frac{du(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{du(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon$.

Therefore $\varphi_s(x) = \infty$ for μ-almost all x. Therefore $\mu(F_1) = 0$, since $\lim_{r\to 0} \frac{\mu(B_r(x))}{r^s} = 0$,

for $x \in F - F_1$. By Lemma 2.8 and since $\frac{\mu(F)}{c} = 0$, $H^s(F) \geq \frac{H^s(F-F_1)}{c}$

$\geq \frac{\mu(F)}{c} - \frac{\mu(F_1)}{c} = \frac{\mu(F)}{c}$. But $\lim_{c\to 0} \frac{\mu(F)}{c} = \infty$. Therefore $H^s(f) \geq \frac{\mu(F)}{c} = \infty$.

Hence $H^s(f) = \infty$. ///

Theorem 3.2. Let E be a bounded subset of X and for each r>0, $M_r^*(E)$ be the greatest number of non-overlapping open balls of diameter $\in (\frac{r}{2}, r]$ such that ρ(B,E)=0. Then

$$\Delta(E) = \lim_{r\to 0} \sup \frac{\log M_r^*(E)}{-\log r}.$$

Proof. $(\Lambda^a - M)(E, r) = \sup \{ \Lambda^a(\mathfrak{R}) ; \mathfrak{R} \in B(E, r) \} = \{ \sum_{x\in\mathfrak{R}} |x|^a ; \mathfrak{R}$ is cover of E }.

Let $\mathfrak{R}_0 = \{ X ; X$ is non-overlapping open balls of diameter $[\frac{r}{2}, r]$, $n(x) = M_r^*(E)\}$. Then

$(\Lambda^a - M)(E, r) = \sup \{ \Lambda^a(\mathfrak{R}) ; \mathfrak{R} \in B(E, r) \} \geq \Lambda^a(\mathfrak{R}_0) = \sum_{x\in\mathfrak{R}_0} \Lambda^a(x)$.

Since $\Lambda^a(x) = t^a$, wher t is the diameter of X $(\frac{r}{2} < t \leq r)$, then $t \geq \frac{r}{2}$, i.e. $t^a \geq (\frac{r}{2})^a$.

$(\Lambda^a - M)(E, r) \geq \sum_{x\in\mathfrak{R}_0} (\frac{r}{2})^a = M_r^*(E)(\frac{r}{2})^a$, for each $a \in \mathbb{R}^+$, where $M_r^*(E)$ is the number of $\mathfrak{R}_0$. In the other direction, assume that $\Delta(E) \neq 0$, $0 < \beta < r < \Delta(E)$. By definition 2.5, $\Delta(E) = \sup \{ a \in \mathbb{R}^+ ; (\Lambda^a - M)(E) = \infty \}$. Hence $(\Lambda^r - M)(E) = \infty$ and for each r $\in$ (0,1), there exists $\mathfrak{R} \in B(E, r)$ such that $\Lambda^r(\mathfrak{R}) \geq 1$. Let $n \geq 0$ and k_n be the number of $B \in \mathfrak{R}$ such that $2^{-n-1} < |B| \leq 2^{-n}$, $\sum_{n=1}^{\infty} k_n 2^{-nr} \geq 1$. There exists $N \in Z$ such that

$$M_{2^{-N}}^*(E) \geq k_n \geq 2^{N\beta}(1 - 2^{\beta - r}).$$

Then $\log M_{2^{-N}}^*(E) \geq \beta \log 2^N + \log(1 - 2^{\beta - r})$.

So $\frac{\log M^{*}_{2^{-N}}(E)}{\log 2^{N}} \geq \beta + \frac{\log(1-2^{\beta-\gamma})}{\log 2^{N}} = \beta + \frac{1}{N}\frac{\log(1-2^{\beta-\gamma})}{\log 2} = \beta + Q(\frac{1}{N})$ [by $f(N) = \theta g(N)$ if and only if there exist a,b such that $a \leq |\frac{f(N)}{g(N)}| \leq b$ and $cg(N) = \theta g(N)$, where c is non-negative constant] , which concludes the proof. ///

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