The Remark on the Fractal Dimensions
- 후랙탈 차원에 관하여 -

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요 지

Julia set, Fatou set와 Mandelbrot set 가 컴퓨터에 의하여 도형화된후 부터 혼돈 역학체계(chaotic dynamical system)에 대한 연구가 모든 학계에 비상한 관심을 모으고 있으며 특히 수학자들에 의하여 많은 연구가 이루어지고 있다. 또한 혼돈 역학체계를 기초로 하여 컴퓨터 그래픽스를 이용한 후랙탈(fractal)들의 매혹적인 시각적 표현으로 인하여 최근들어 과학자들 뿐 아니라 일반대중의 후랙탈에 대한 관심이 매우 높아지고 있다. 후랙탈이란 말은 라틴어 fractus(부서진 상태를 뜻함)에서 유래되었으며 1975년 Mandelbrot가 수학및 자연계의 비 정규적 패턴들에 대한 체계적 고찰을 담은 자신의 에세이의 표제를 주기 위해서 만들었다([6]). 후랙탈을 기술하는데 있어서 가장 중요한 양은 차원(dimension)으로, 예컨데 Cantor 1/3 집합은 길이 1인 선분으로 부터 시작하야 매단계마다 모든 선분들의 가운데 1/3을 잘라내는 것을 무한히 반복함으로써 얻어지는데 이 집합의 Lebesgue measure는 0이지만 후랙탈 차원은 log2/log3 로 정수차원이 아닌 실수차원을 갖으며 또한 Cantor 1/3집합은 연속이 아니면서 점도 선도 아닌 집합인 것이다.
이 논문에서는 Box counting dimension 과 Hausdorff dimension에 대한 몇가지 정의를 하고 정리 2.6, 정리2.7 및 정리 3.3을 증명함으로써 어떤 성질을 갖는 후랙탈의 가장 중요한 양인 후랙탈 차원에 대하여 논의 하고자 한다.

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1.Introduction

In the last few years fractals have become enormously popular as an art form, with the advent of computer graphics, and as a model of a wide variety of physical phenomena. Whilst it is possible in some ways to appreciate fractals with little of no knowledge of their mathematics, an understanding of the mathematics that can be applied to such a diversity of objects certainly enhances one's appreciation. The word 'fractal' was coined by Mandelbrot in his fundamental essay from the Latin fractus, meaning broken, to describe objects that were too irregular to fit into a traditional geometrical setting. The main tool of fractal is dimension in its many forms ([6]). Here, the fractal dimension of set is a number which tells how densely the set occupies the metric space in which it lies. The purpose of this paper is to study some properties of fractal and it's dimension. In particular, fractal dimension for non-smooth and irregular sets might be a real number that is not an integer number. In this paper, we will define the Box counting dimension and Hausdorff dimension etc, and will prove theorem 2.6, theorem2.7 and theorem3.3.

2.Hausdorff and Box counting dimensions

Definition 2.1. Let U be any non-empty subset of n-dimensional Euclidean space $\mathbb{R}^n$. A subset U of $\mathbb{R}^n$ is *connected* if it consists of just one 'piece'. The set U is *totally disconnected* if the connected component of each point consists of just that point ([1],[6],[7]).

Definition 2.2. Let (X,d) denote a complete metric space, $H(X)$ be the Hausdorff measure of X and A be non-empty compact subset of X. For any $\varepsilon > 0$, let $B_\varepsilon(x)$ be the closed ball of radius ε and center at a point $x \in X$. Let $N(A,\varepsilon)$ be the least number of closed balls of radius ε to cover the set A. That is, $N(A,\varepsilon)$ is the smallest positive integer M such that $A \subset \bigcup_{n=1}^{M} B_\varepsilon(x_n)$, $x_1, x_2, \cdots, x_n \in X$. We define the *Box counting dimension* or *Box dimension* of A as follows ; (1) $\dim_B A = \lim_{\delta \to 0} \frac{\log(N(A,\delta))}{-\log \delta}$.

Roughly speaking, (1) says that $N(A,\delta) \simeq \delta^{-s}$, for small δ, where $s = \dim_B A$ ([1],[6]).

Definition 2.3. Let U be any non-empty subset of n-dimensional Euclidean space $\mathbb{R}^n$ and the *diameter* of U defined as $|U| = \sup\{|x-y| ; x,y \in U\}$. If $\{U_i\}$ is a countable (of finite) collection of sets of diameter at most δ that cover of F, then we say that $\{U_i\}$ is a *δ-cover* of F. Let F be a subset of $\mathbb{R}^n$ and let S be a non-negative number. For any $\delta > 0$, we define $H^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s ; \{U_i\}$ is a δ-cover of F$\}$ and

$H^s(F) = \lim_{\delta \to 0} H^s_\delta(F)$.

We call $H^s(F)$ the *s-dimensional Hausdorff measure* of F. If t>s and $\{U_i\}$ is a δ-cover of F, we have $\sum_i |U_i|^s \le \delta^{t-s} \sum_i |U_i|^s$. Letting $\delta \to 0$, we see that if $H^s(F) < \infty$ then $H^t(F) = 0$ for t>s. Here we known that there is a critical value of s at which $H^s(F)$ 'jumps' from ∞ to 0 (See Figure 1). This critical value is called *the Hausdorff dimension of* F and written $\dim_H F$.

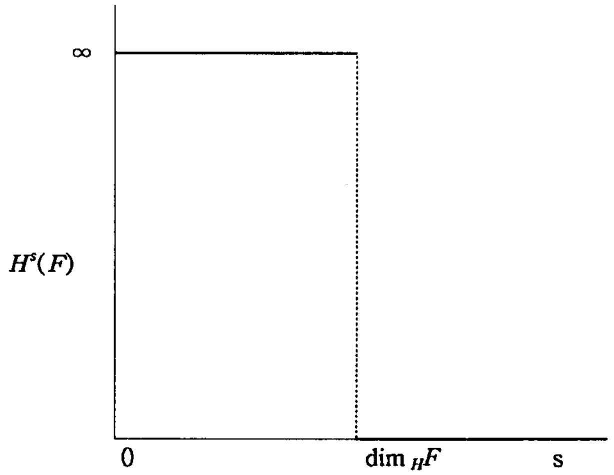

Figure 1. Graph of $H^s(F)$ against s for a set F. The Hausdorff dimension is the value of s at which the 'jump' from ∞ to 0 occure.

Accordingly, $\dim_H F = \inf\{s; H^s(F) = 0\} = \sup\{s; H^s(F) = \infty\}$, so that

$$H^s(F) = \begin{cases} \infty & \text{if} \quad s < \dim_H F \\ 0 & \text{if} \quad s > \dim_H F \end{cases} \qquad ([1],[6]).$$

Definition 2.4. Let U be a closed subset of $\mathbb{R}^n$. A mapping $S : U \to U$ is called a *contraction* on U if there is a number c with 0<c<1 such that $|S(x) - S(y)| \le c|x-y|$, for all $x, y \in U$ ([6]).

Proposition 2.5. *A* set $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.

Proof. Let x and y be distinct points of F. Define a mapping $f : \mathbb{R}^n \to [0, \infty)$ by $f(z) = |z - x|$. Since $|f(z) - f(w)| \le |z - w|$, we have $\dim_H f(F) \le \dim_H F < 1$. Thus $f(F)$ is a subset of $\mathbb{R}$ of H^1-measure or length zero. Choosing r with $r \notin f(F)$ and $0 < r < f(y)$.

It follows that $F=\{ z \in F ; |z-x| < r \} \cup \{ z \in F ; |z-x| > r \}$. Thus F is contained in two disjoint open sets with x in one set and y in the other. So that x and y lie in different connected components of F. ///

Theorem 2.6. Let F be the middle third Cantor set. (See Figure 2). Let s=log2/log3= 0.6309⋯ . Then $\dim_H F = s$ and $\frac{1}{2} \le H^s(F) \le 1$.

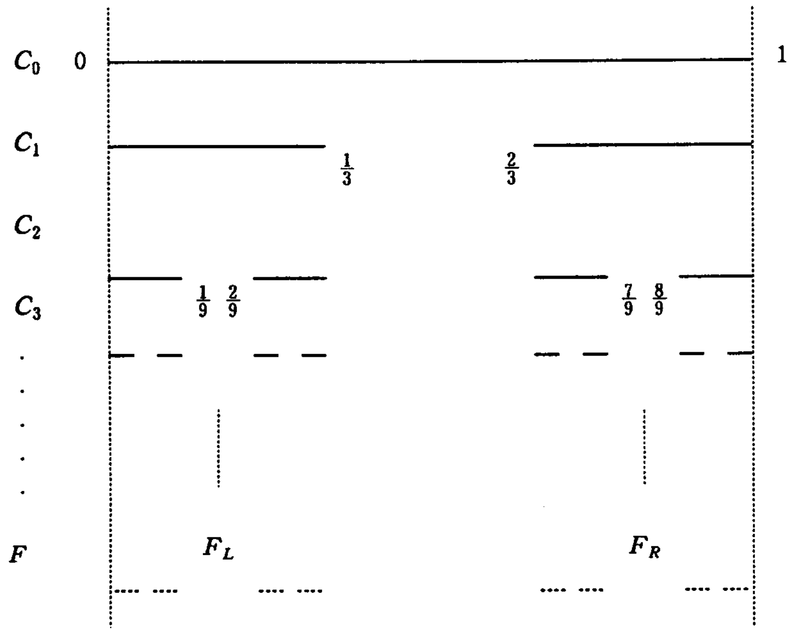

Figure 2. Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note F_L and F_R the left and right parts of F are copies of F scaled by a factor $\frac{1}{3}$.

Heuristic calculation. The middle third Cantor set is divided into two parts; one is the left part $F_L = F \cap [0, \frac{1}{3}]$ and the other right part is $F_R = F \cap [\frac{2}{3}, 1]$. These two parts are similar to F with contractive factor $\frac{1}{3}$ and $F = F_L \cup F_R$. Appling to the Hausdorff measure, $H^s(F) = H^s(F_L) + H^s(F_R) = (\frac{1}{3})^s H^s(F) + (\frac{1}{3})^s H^s(F)$. We get $1 = 2(\frac{1}{3})^s$ or s=log2/log3.

Generous calculation. The covering $\{ U_i \}$ of F consisting of the 2^k interval of C_k of the length 3^{-k} gives that $H^s_{3^{-k}}(F) \le \sum_i |U_i|^s = 2^k 3^{-ks} = 1$ if s=log2/log3. Letting k→∞ gives

$H^s(F) \le 1$. If $|U_i|^s = \frac{1}{2} = 3^{-s}$, then we can prove $H^s(F) \ge \frac{1}{2}$. For each U_i, let k be the integer such that $3^{-(k+1)} \le |U_i| < 3^{-k}$. If $j \ge k$ then U_i intersects at most $2^{j-k} = 2^j \cdot 2^{-k} = 2^j \cdot 3^{-sk} \le 2^j 3^s |U_i|^s$ with basic intervals of C_j. Choose j large enough so that $3^{-(j+1)} \le |U_i|$, for all U_i, then counting intervals gives $2^j \le \sum_i 2^j 3^s |U_i|^s$. Therefore $H^s(F) \ge \frac{1}{2}$. ///

Theorem 2.7. Let $A \subset \mathbb{R}^n$ be a compact subset and let $0<r<1$. Then $\dim_B A = \liminf_{j\to\infty} \frac{\log(N(A, r^j))}{j\log(\frac{1}{r})}$.

Proof. For any $\varepsilon > 0$, there is a j such that $r^{j+1} < \varepsilon \le r^j$. Then $r \cdot r^{-j-1} \le \varepsilon^{-1} < r^{-1} r^{-j}$, so that $\log r^{-1} + j\log r^{-1} > \log \varepsilon^{-1} \ge \log r + (j+1)\log r^{-1}$ and

$$(\log r^{-1} + j\log r^{-1})^{-1} < (\log \varepsilon^{-1})^{-1} \le [\log r + (j+1)\log r^{-1}]^{-1} .$$

Since $N(A, \varepsilon) \ge N(A, \delta)$ whenever $\varepsilon < \delta$, $N(A, r^j) \le N(A, \varepsilon) \le N(A, r^{j+1})$.

Therefore $\liminf_{j\to\infty} \frac{\log(N(A, r^j))}{j\log(r^{-1})}$

$$= \liminf_{j\to\infty} \frac{\log(N(A, r^j))}{\log(r^{-1}) + j\log(r^{-1})}$$

$$\le \liminf_{j\to\infty} \frac{\log(N(A, \varepsilon))}{\log(\varepsilon^{-1})}$$

$$= \dim_B A$$

$$\le \liminf_{j\to\infty} \frac{\log(N(A, r^{j+1}))}{\log r + (j+1)\log(r^{-1})}$$

$$= \liminf_{j\to\infty} \frac{\log(N(A, r^{j+1}))}{(j+1)\log(r^{-1})}$$

$$= \liminf_{j\to\infty} \frac{\log(N(A, r^j))}{j\log(r^{-1})} . \ ///$$

Example 2.8. Let C be the middle-α Cantor set in the line. Let β be chosen so that $2\beta + \alpha = 1$, so that $0 < \beta < \frac{1}{2}$. Then in the construction of the Cantor set, there are 2^j intervals of length β^j which cover C, so $N(C, \beta^j) = 2^j$.

Therefore $\dim_B C = \liminf_{j\to\infty} \frac{\log(N(C, \beta^j))}{j\log(\beta^{-1})}$

$$= \liminf_{j\to\infty} \frac{\log(2^j)}{j\log(\beta^{-1})}$$

$$= \frac{\log 2}{\log(\beta^{-1})} .$$

Here we note that $0 < \dim_B C < 1$.

3.Potential dimensions.

The ideas of potential and energy will be familiar to us with a knowledge of gravitation or electrostatics. Let μ be a mass distribution on F with $I_s(\mu) < \infty$ where $I_s(\mu)$ is the s-energy of μ.

Definition 3.1. The *s-potential* at a point x of $\mathbb{R}^n$ is defined as follows ; $\varphi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}$ for $S \geq 0$. the s-energy of μ is defined ; $I_s(\mu) = \int \varphi_s(x) du(x) = \int\int \frac{du(x)du(y)}{|x-y|^s}$ [6].

The following is well-known ([2],[6],[7]) :

1° . Let $F \subset \mathbb{R}^n$ Borel set and $0<c<\infty$ be a constant. If $\lim_{r\to 0} \frac{\mu(B_r(x))}{r^s} < c$ for all $x \in F$, then $H^s(F) \geq \frac{\mu(F)}{c}$.

2° . Let $F \subset \mathbb{R}^n$ Borel set and $0<c<\infty$ be a constant. If $\lim_{r\to 0} \frac{\mu(B_r(x))}{r^s} > c$ for all $x \in F$, then $H^s(F) \leq 2^s \mu(\mathbb{R}^n)/c$.

3° . Let F be a Borel set with $0 < H^s(F) < \infty$, then there exists a constant b and a compact set $E \subset F$ with the $H^s(E) > 0$ such that $H^s(E \cap B_r(x)) \leq br^s$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Lemma 3.2. Let F be a Borel subset with $H^s(F) = \infty$, then there exists a compact set $E \subset F$ with $0 < H^s(F) < \infty$ and such that for some constant b. $H^s(E \cap B_r(x)) \leq br^s$ for all $x \in \mathbb{R}^n$ and $r \geq 0$.

Proof. Let μ satisfying $\mu(A)=H^s(F\cap A)$. Then, by 2^0, if $F_1=\{x\in \mathbb{R}^n;\ \lim_{r\to 0}\frac{H^s(F\cap B_r(x))}{r^s}>2^{1+s}\}$, then

$$H^s(F_1)\leq 2^s 2^{-1-s}\mu(\mathbb{R}^n)=\frac{1}{2}\mu(\mathbb{R}^n)=\frac{1}{2}H^s(\mathbb{R}^n\cap F)=\frac{1}{2}H^s(F).$$

Thus $H^s(F_1)\leq\frac{1}{2}H^s(F)$. 1Now, $H^s(F\backslash F_1)=H^s(F)\backslash H^s(F_1)\geq H^s(F)-\frac{1}{2}H^s(F)=\frac{1}{2}H^s(F)$.

Let $E_1=F-F_1$ and $H^s(E_1)>0$, then $\overline{\lim_{r\to 0}}\frac{H^s(F\cap B_r(x))}{r^s}\leq 2^{1+s}$ for some $x\in F_1$. Therefore by 3^0, there exists $E\subset E_1$ with $H^s(E)>0$ and a number r_0 such that $\frac{H^s(F\cap B_r(x))}{r^s}\leq 2^{s+2}$ for all $x\in E$ and $0\leq r\leq r_0$. But $\frac{H^s(F\cap B_r(s))}{r^s}\leq\frac{H^s(F)}{r_0^s}$, where $\frac{H^s(F)}{r_0^s}=b$. Thus $H^s(E\cap B_r(x))\leq br^s$. ///

Theorem 3.3. Let $F\subset\mathbb{R}^n$. If there is a mass distribution μ on F with $I_s(\mu)<\infty$, then $H^s(F)=\infty$ and $\dim_H F\geq S$.

Proof. Suppose that $I_s(\mu)<\infty$ for some mass distribution μ with support contained in F. Define $F_1=\{x\in F:\ \lim_{r\to 0}\frac{\mu(B_r(x))}{r^s}>0\}$. Then $\frac{\mu(B_r(x))}{r^2}=\rho\pi$, where ρ is density. If $x\in F_1$, for each $\varepsilon>0$, there exists $\{r_i\}$ decreasing to 0 such that $\mu(B_r(x))\geq\varepsilon r_i^s$. We claim $\mu(F_1)=0$.

First, if $\mu(\{x\})>0$, $I_s(\mu)=\int\varphi_s(x)du(x)=\int\int\frac{du(x)du(y)}{|x-y|^s}$

$$\geq c\int\mu(x)du\geq c\mu(x)\int du,\quad \int 1du=\infty,$$

therefore $I_s(\mu)=\infty$. Now, if $\mu(\{x\})\leq 0$, i.e. $\mu(\{x\})=0$, from continuity of μ be taking $q_i\ (0<q_i<r_i)$ small enough, $\mu(A_i)\geq\frac{1}{4}\varepsilon r_i^s$ (i=1,2,$\cdots$), where $A_i=B_{r_i}(x)-B_{q_i}(x)$. Then $q_i<r_i$ such that $\mu(B_{q_i}(x))<\delta=\frac{3}{4}\varepsilon r_i^s$, $\mu(A_i)=\mu(B_{r_i}(x))-\mu(B_{q_i}(x))\geq\frac{1}{4}\varepsilon r_i^s$. Taking subsequence if necessary, suppose that $r_{i+1}<q_i$ for all i, where A_i are disjoint annulus centered at x.

For $x\in F_1$, $\varphi_s(x)=\int\frac{du(y)}{|x-y|^s}\geq\sum_{i=0}^{\infty}\frac{1}{4}\varepsilon r_i^s r_i^{-s}=\infty$. For $q_i\leq|x-y|\leq r_i$,

$\frac{1}{q_i} \geq \frac{1}{|x-y|} \geq \frac{1}{r_i} \cdot \frac{1}{|x-y|^s} \geq \frac{1}{(r_i)^s}$, and then A_i are disjoint annulus centered at x.

$$\int_{A_i} \frac{du(y)}{|x-y|^s} \leq \int_{A_i} \frac{du(y)}{r_i^s} = \frac{1}{r_i^s} \cdot \frac{1}{4} \varepsilon r_i^s = \frac{1}{4} \varepsilon .$$

But $I_s(\mu) = \int \varphi_s(x) du(x) < \infty$, so $\varphi_s(x) < \infty$,

$$\varphi_s(x) = \int \frac{du(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{du(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon .$$

Therefore $\varphi_s(x) = \infty$ for μ-almost all x. Therefore $\mu(F_1) = 0$, Since $\lim_{r \to 0} \frac{\mu(B_r(x))}{r^s} = 0$,

for $x \in F - F_1$. By 1° and since $\frac{\mu(F_1)}{c} = 0$,

$H^s(F) \geq \frac{H^s(F-F_1)}{c} \geq \frac{\mu(F)}{c} - \frac{\mu(F_1)}{c} = \frac{\mu(F)}{c}$. But $\lim_{c \to 0} \frac{u(F)}{c} = \infty$. Therefore

$H^s(F) \geq \frac{\mu(F)}{c} = \infty$. Hence $H^s(F) = \infty$. ///

Important application of this note will be given in the proof of the projection theorems and in the determination of the dimension of Brownian paths. The theory of the fractal and chaotic dynamical system tries to explain all problems of natural phenomena as mathmatical model and have a characterstic of study of various field . The fractal have given many influences to scholars that have recognition of irregular pattern and provided new aspects to understand natural and been required with study beyond major.

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