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# Estimation of Manoeuvring Coefficients of a Submerged Body using Parameter Identification Techniques

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**Abstract**

This paper describes parameter identification techniques formulated for the estimation of maneuvering coefficients of a submerged body. The first part of this paper is concerned with the identifiability of the system parameters. The relationship between a stochastic linear time-invariant system and the equivalent dynamic system is investigated. The second is concerned with the development of the numerically stable identification technique. Two identification techniques are tested; one is the maximum likelihood (ML) methods using the Nelder & Mead simplex search method and using the modified Newton-Raphson method, and the other is the modified extended Kalman filter (MEKF) method with a square-root algorithm, which can improve the numerical accuracy of the extended Kalman filter.

As a results, it is said that the equations of motion for a submerged body have higher probability to generate simultaneous drift phenomenon compared to general state equations and only the ML method using the Nelder & Mead simplex search method and the MEKF method with a square-root algorithm gives acceptable results.

## 1 Introduction

The manoeuvring coefficients in the equations of motion for a submerged body are used as basic input data for motion simulation, hull form design and controller design and they are normally obtained by model test, theoretical calculation or empirical formulas. However, it is expected that the simulation results based on such manoeuvring coefficients are different from sea trial results. Thus, the parameter identification technique based on real observations should be applied to the reestimation of the manoeuvring coefficients. As a result, parameters of a mathematical model, which are obtained from the parameter identification technique combined with the measured data, permit reasonably predicted results for real application. There are three major steps involved in solving the system identification problem; The first step is to build a mathematical model which can clearly express physical phenomena[1]. The second is to check

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the identifiability of system parameters[2]. And the last step is to select the proper identification technique for the identification of the parameters[3].

The identifiability implies whether parameters in the candidate mathematical model can be uniquely determined from the observed data. The simultaneous drift phenomenon, one of the serious defects of current identification technique, results from the lack of identifiability of the system parameters. And the sensitivity of the estimated parameters subject to the change of the initial guess also depends on the local identifiability. Even though the identifiability of the system could be guaranteed, the success of the parameter estimation still depends on the convergence of the identification technique since the numerical instability may leads to divergence from real solution. Hence it is essential to check the identifiability of the system parameters at first and then to seek the numerically stable technique. The first part of this paper is concerned with the identifiability of the system parameters. To this end, the relationship between a stochastic linear time invariant system and the equivalent dynamic system is investigated using the identifiability concept introduced by Tse and Anton[4]. Also, the relationship between maneuvering coefficients of dynamic system and those of the equivalent system are investigated. As a result, it is found that linear equations of motion of a submerged body cannot have global identifiability and have higher probability to generate simultaneous drift phenomenon due to the existence of the inertia matrix compared with general state equations. The second is concerned with the development of the numerically stable identification technique. Two identification methods are tested; they are the maximum likelihood method which is one of the typical off-line techniques and the extended Kalman filter method which is one of the typical on-line techniques. In the case of the maximum likelihood method, the maneuvering coefficients are identified by two methods : one is the combination of the Newton-Raphson method and the steepest descent method as a gradient method, and the other is the Nelder & Mead simplex search method as a direct search method. To improve the numerical accuracy of the extended Kalman filter method, the modified extended Kalman filter method with square-root algorithm is used for the identification of the maneuvering coefficients. It is found from the identified results that the maximum likelihood technique in combination with the steepest descent method and the Newton-Raphson method gives the unsatisfactory results, but the maximum likelihood technique of the Nelder & Mead simplex method and the modified extended Kalman filter method with a square-root algorithm are less.

# 2 6-DOF Equations of Motion

Although trajectories of the body can be described by referring to the coordinate system fixed to the ground, in order to make the calculation of hydrodynamic loading easier, we take the coordinate system fixed in the body shown in Fig.1.

The origin of the right-handed coordinate system which the equations of motion are referred, coincides with the center of gravity. And the equations of motion and the

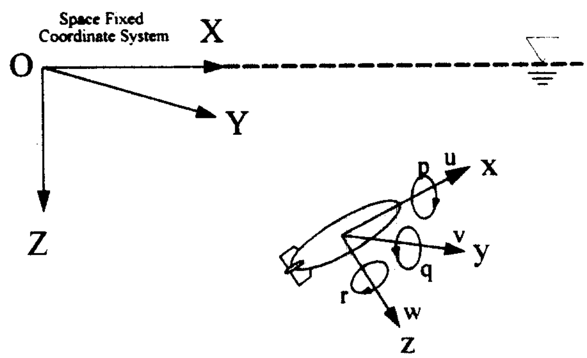



Figure 1: Coordinate System

measurement equations are as follows:

$$\begin{aligned}\tilde{M}\dot{\bar{x}} &= \tilde{A}\bar{x} + \tilde{B}\bar{u} + \tilde{S} + \bar{w}_1,\\ \Rightarrow \dot{\bar{x}} &= \tilde{M}^{-1}(\tilde{A}\bar{x} + \tilde{B}\bar{u} + \tilde{S}) + \bar{w}\\ &\equiv A\bar{x} + B\bar{u} + S + \bar{w},\\ \bar{z}_k &= C\bar{x}_k + \bar{v}, \end{aligned} \tag{1}$$

where $\tilde{M}$ is the inertia matrix, $\tilde{A}$ is the damping matrix of system, $\tilde{B}$ is the input matrix, $\tilde{S}$ is the restoring matrix, and $C$ is the measurement matrix. Detailed expressions of Eqn.(1) are shown in the appendix.

The equations of motion in Eqn.(1), are continuous-time system physically. But, the discrete state equations would make it convenient to use digital computers. Parameter identification techniques using Kalman filter or using gradient optimization method contains the derivatives of state variables with respect to system parameters. It might make serious error to calculate the first or the second derivatives numerically. In this paper, the linear equations of motion were discretized using Cayley-Hamilton principle and the new state equation, in which differentiated state variables are regarded as state variables, was built up as follows:

$$\bar{x}_k = e^{A\Delta k}\bar{x}_{k-1} + A^{-1}(e^{A\Delta k} - I)\cdot(B\bar{u}_{k-1} + S) + A^{-1}(e^{A\Delta k} - I)\bar{w}_{k-1} \tag{2}$$

$$\begin{aligned}\frac{\partial\dot{\bar{x}}(t)}{\partial\theta} &= A\frac{\partial\bar{x}(t)}{\partial\theta} + \frac{\partial A}{\partial\theta}\bar{x}(t) + \frac{\partial B}{\partial\theta}\bar{u}(t) + \frac{\partial S}{\partial\theta},\\ \frac{\partial^2\dot{\bar{x}}(t)}{\partial\theta_1\theta_2} &= A\frac{\partial^2\bar{x}(t)}{\partial\theta_1\theta_2} + \frac{\partial A}{\partial\theta_1}\frac{\partial\bar{x}(t)}{\partial\theta_2} + \frac{\partial A}{\partial\theta_2}\frac{\partial\bar{x}(t)}{\partial\theta_1}\\ &\quad + \frac{\partial^2 A}{\partial\theta_1\partial\theta_2}\bar{x}(t) + \frac{\partial^2 B}{\partial\theta_1\partial\theta_2}\bar{u}(t) + \frac{\partial S}{\partial\theta_1\partial\theta_2},\end{aligned} \tag{3}$$

where the initial conditions are such that

$$\frac{\partial\bar{x}(0)}{\partial\theta} = \frac{\partial^2\bar{x}(0)}{\partial\theta_1\theta_2} = 0.$$

# 3 Identifiability of Equivalent Dynamic System

Let $\bar{x}_1 = P^{-1}\bar{x}$ and $P$ be a nonsingular matrix. Then equations of motion for a submerged body can be written such that

$$\begin{aligned} \tilde{M}_1\dot{\bar{x}} &= \tilde{A}_1\bar{x} + \tilde{B}\bar{u} + \tilde{S} + \bar{w}_1, \\ \bar{z}_k &= C_1\bar{x}_k + \bar{v}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \tilde{M}_1 &= \tilde{M}P, \\ \tilde{A}_1 &= \tilde{A}P, \\ C_1 &= CP. \end{aligned}$$

Then

$$C\bar{x} = C_1P^{-1}P\bar{x}_1 = C_1\bar{x}_1,$$

for any input and two dynamic systems parametrized by $\theta$ and $\theta_1$ which are respectively equivalent. Therefore, introducing the definition of the identifiability by the probability density function concept of Tse and Anton[4]

$$\begin{aligned} p(Z_M;\bar{\theta}_0) &= const \cdot \exp\left[-\frac{1}{2}\sum_{k=0}^{M}(\hat{\bar{z}}_k - C\bar{x}_k;\bar{\theta}_0)^T R^{-1}(\hat{\bar{z}}_k - C\bar{x}_k;\bar{\theta}_0)\right] \\ &= const \cdot \exp\left[-\frac{1}{2}\sum_{k=0}^{M}(\hat{\bar{z}}_k - C_1\bar{x}_{1k};\bar{\theta}_1)^T R^{-1}(\hat{\bar{z}}_k - C_1\bar{x}_{1k};\bar{\theta}_1)\right] \\ &= p(Z_M;\bar{\theta}_1). \end{aligned} \tag{5}$$

This relation means that the conditional probability density functions of two systems are identical and $\bar{x}$ and $\bar{x}_1$ are both unresolvable. Consequently, it can be said that the dynamic system which could be equivalently transformed has not the global identifiability and at most can be locally identifiable. Applying the above to Eqn.(1) for a submerged body system, the equivalent transformation matrices which are non-identical and nonsingular, can be found. From the relations of two dynamic systems, it can be said that the parameters with respect to the inertia terms and the damping terms would be drift simultaneously. We can say that this makes parameter identification for the equations of motion more difficult than the general state equations. The relation of Eqn.(6) is one of an example of these phenomena.

$$\begin{aligned} Y'_{\dot{p}} &= (M - Y_{\dot{v}})P_{24} - Y_{\dot{p}}, \\ Y'_p &= Y_v P_{24} + Y_p, \end{aligned} \tag{6}$$

where $P_{24}$ means the (2,4) element of $P$ matrix. $Y'_{\dot{p}}$ and $Y'_p$ are corresponding to sway added mass $Y_{\dot{p}}$ and sway damping $Y_p$ due to roll in the equivalent system, respectively.

# 4 The Maximum Likelihood Method

The maximum likelihood method is to take $\bar{x}_k$ estimate which maximizes the probability of the measurement $\bar{z}_k$ that actually occurred, taking into account of known statistical properties of $\bar{v}$ taken as a zero mean, gaussian distributed observation with covariance matrix $R$. Therefore, the ML estimation problem can be formulated in a probabilistic manner by defining the likelihood function as the probability density functions of the measurement $\bar{z}_k$ given $\bar{\theta}$ and $R$.

$$\begin{aligned} p(Z_M|\bar{\theta}) &= p(\bar{z}_k|Z_{M-1},\bar{\theta})\cdot p(Z_{M-1}|\bar{\theta}) \\ &= \prod_{k=0}^{M} p(\bar{z}_k|Z_{k-1},\bar{\theta}), \end{aligned} \tag{7}$$

where the conditional probability density function can be given such that

$$p(\bar{z}_k|Z_{k-1},\bar{\theta}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|R|}} \exp\left[-\frac{1}{2}(\bar{z}_k - C\bar{x}_k)^T R^{-1}(\bar{z}_k - C\bar{x}_k)\right].$$

Maximization of the likelihood function w.r.t. $\bar{\theta}$ and $R$ can also be achieved by minimizing the negative log-likelihood function.

$$-\ln p(Z_M|\bar{\theta}) = const + \frac{1}{2}\sum_{k=0}^{M}(\bar{z}_k - C\bar{x}_k)^T R^{-1}(\bar{z}_k - C\bar{x}_k). \tag{8}$$

Minimization of $-\ln p(Z_M|\bar{\theta})$ w.r.t. $R$ results in

$$J_M(\bar{\theta}) = \frac{1}{M+1}\sum_{k=0}^{M}(\bar{z}_k - C\bar{x}_k)^T R^{-1}(\bar{z}_k - C\bar{x}_k).$$

This leads to a system of nonlinear equations which can be solved by some numerical methods. In this paper, two methods are applied; one is gradient method known as the modified Newton-Raphson method and the other is direct method known as the Nelder and Mead simplex search method.

## 4.1 The modified Newton Raphson method

Typical technique to find the maximum likelihood estimate is to use root-finding routines known as Newton-Raphson method. The correction of this technique is given by

$$\Delta\bar{\theta} \approx -\left[\frac{\partial^2 J_M(\bar{\theta})}{\partial\bar{\theta}^2}\right]^{-1}\frac{\partial J_M(\bar{\theta})}{\partial\bar{\theta}}, \text{ at } \bar{\theta} = \bar{\theta}_k. \tag{9}$$

The Newton-Raphson method may converge very fast but may be sensitive to measurement noise and, in extreme cases, inaccuracy in the mathematical model may lead to the divergence. To ameliorate the convergence of this estimator the steepest descent method is combined with the Newton-Raphson method such that

$$\mu\frac{\partial J_M(\bar{\theta})}{\partial\bar{\theta}} + \left[\frac{\partial^2 J_M(\bar{\theta})}{\partial\bar{\theta}^2} + \lambda I_{m\times m}\right]\Delta\bar{\theta} = 0 \text{ at } \bar{\theta} = \bar{\theta}_k, \tag{10}$$

where

$$\begin{aligned}\lambda &= \text{Max}\left|\frac{\partial J}{\partial\bar{\theta}}\right|, \\ \mu &= 0.5 \text{ if } \lambda > 1, \\ \mu &= \frac{1}{\lambda+1} \text{ if } \lambda \leq 1.\end{aligned}$$

## 4.2 Nelder & Mead simplex search method

The maximum likelihood estimate using Nelder & mead simplex search method is a direct search method. This method involves constructing an $m+1$ cornered shape (simplex) and then allowing this shape to move toward the minimum point by sequent, replacing the corner of the current simplex having the highest value of $J_M(\bar{\theta})$ with a new point having a lower value of $J_M(\bar{\theta})$. The algorithm can be described as follows.

1. Sample $J_M$ at the $m+1$ corners of the simplex and establish the corners which yield the highest, $\bar{\theta}_h$, and lowest $\bar{\theta}_l$, $J_M(\bar{\theta})$ in the current simplex.

2. Sample $J_M$ at the centroid $\bar{\theta}_b$ of all $\bar{\theta}_i$ but $\bar{\theta}_h$.

3. Test the stopping condition; If $\frac{1}{m+1}\sum_{i=1}^{m+1}\{J_M(\bar{\theta}_i) - J_M(\bar{\theta}_b)\}^2\}^{1/2} \leq \epsilon$, stop and return $J_M(\bar{\theta}_l)$ as minimum; If not, continue.

4. Reflect $\bar{\theta}_h$ through $\bar{\theta}_b$ to give $\bar{\theta}_r = 2\bar{\theta}_b - \bar{\theta}_h$; Sample $J_M(\bar{\theta}_r)$.

5. If $J_M(\bar{\theta}_r) < J_M(\bar{\theta}_l)$, reflect $\bar{\theta}_b$ through $\bar{\theta}_r$ in order to give $\bar{\theta}_e = 2\bar{\theta}_r - \bar{\theta}_b$, and then sample $J_M(\bar{\theta}_e)$; If $J_M(\bar{\theta}_e) < J_M(\bar{\theta}_l)$, replace $\bar{\theta}_h$ by $\bar{\theta}_e$, and return to 2.; If not, replace $\bar{\theta}_h$ by $\bar{\theta}_r$, and return to 2.

6. If $J_M(\bar{\theta}_r) \geq J_M(\bar{\theta}_l)$, check $J_M(\bar{\theta}_r) \geq J_M(\bar{\theta}_i)$ for all $\bar{\theta}_i$ except $\bar{\theta}_h$; If false, replace $\bar{\theta}_h$ by $\bar{\theta}_r$, and return to 2; If true, continue.

7. Sample $J_M$ at $\bar{\theta}_c = (\bar{\theta}_r + \bar{\theta}_b)/2$, if $J_M(\bar{\theta}_r) < J_M(\bar{\theta}_h)$; If not, sample at $\bar{\theta}_c = (\bar{\theta}_h + \bar{x}_b)/2$.

8. If $J_M(\bar{\theta}_c) < J_M(\bar{\theta}_h)$, replace $\bar{\theta}_h$ by $\bar{\theta}_c$, and return to 2.; If not, reduce simplex toward $\bar{\theta}_l$ using $\bar{\theta}_i = (\bar{\theta}_i + \bar{\theta}_l)/2$, and return to 2.

# 5 The Modified EKF with a Square-Root Algorithm

The parameter identification technique using the EKF is generally used. In the EKF, the usual linear Taylor approximation of the nonlinear system is used. It is clear that the EKF is a real-time algorithm and efficient, but it does not always produce desirable result. Hence, a modified version of the EKF (MEKF) is introduced. In the MEKF, the center for each updated linear Taylor approximation is derived from an optimal Kalman filtering algorithm. That is, the optimal Kalman filter to update the state variables at the previous state estimate and the EKF to obtain parameters of the model at the corresponding predicted position should be solved in parallel, starting with the same initial estimate as shown in Fig.2.

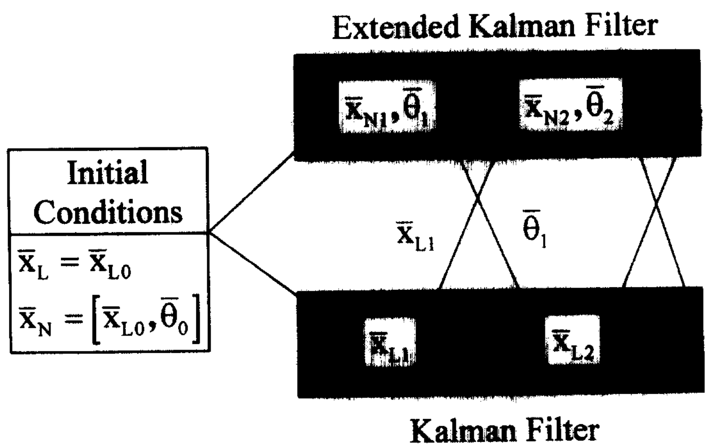



Figure 2: Parallel Algorithm for Modified Extended Kalman Filter

And it has been observed that the standard Kalman filter is in many cases numerically unstable. For this reason, the square-root filter is proposed. The square-root filter requires inversion of triangular matrices, and improves the computational accuracy by working with the square-root of possible very large or very small numbers. The square-root Kalman filtering algorithm can be stated as follows:

- The initial conditions

$$\begin{aligned} \hat{x}(t_0) &= \hat{x}_0, \\ J_{0,0} &= [\mathrm{Var}(\hat{x}_0)]^{1/2}. \end{aligned}$$

- Time-propagation

1. compute $J_{k,k-1}$, the square-root of matrix.
2. $$\begin{aligned} L_k &\equiv (C_k J_{k,k-1} J_{k,k-1}{}^T C_k{}^T + R_k)^c, \\ K_k &= J_{k,k-1} J_{k,k-1}{}^T C_k{}^T (L_k{}^T)^{-1} L_k{}^{-1}. \end{aligned}$$
3. $\hat{x}(t_i^-) = \Phi(t_i, t_{i-1}) \hat{x}(t_{i-1}^+) + B_d(t_{i-1})$.

- Measurement updating

$$\begin{aligned} J_{k,k} &= J_{k,k-1}\left[I - J_{k,k-1}{}^{T}C_k{}^{T}(L_k{}^{T})^{-1}\cdot(L_k + R_k{}^{c})^{-1}C_kJ_{k,k-1}\right], \\ \hat{x}(t_i^+) &= \hat{x}(t_i^-) + K(t_i)\left[\bar{z}_i - C(t_i)\hat{x}(t_i^-)\right]. \end{aligned}$$

In the above, the superscript c indicates the Cholesky decomposition of a positive-definite matrix, which is lower triangular and taking into account of the numerical stability, the singular value decomposition can be used to calculate $J_{k,k-1}$.

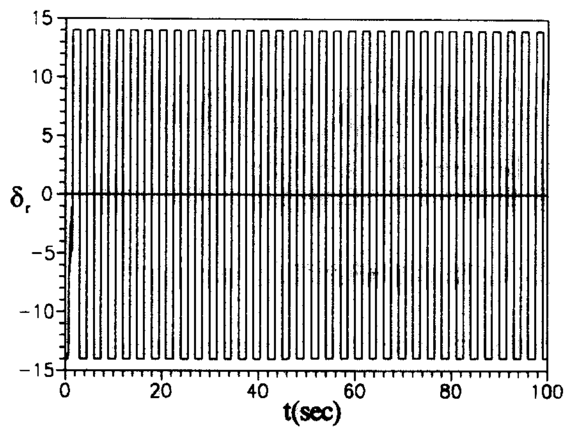
# 6 Calculation Results and Discussions

To identify maneuvering coefficients of a submerged body, two measurement data are produced from two scenarios. The first scenario is the one which is generally used in sea trial tests, and is composed of increasing or decreasing depth control command by 20m at an interval of 5sec, changing course rate command into $\pm 20^o$/sec at an interval of 1.5sec and zero roll command as shown in Fig.3. For the convenience, let's express the scenario as scenario I. The second scenario, scenario II, is composed of horizontal mode data and vertical mode data. To obtain the horizontal mode data, the operating depth is kept constant by using an automatic control of right and left elevator angles, and the rudder angle is commanded to follow the scenario built by pseudo random binary sequence. On the other hand, vertical mode data are obtained when the left and right elevator angles are commanded to follow the same scenario as above while rudder angle is fixed.

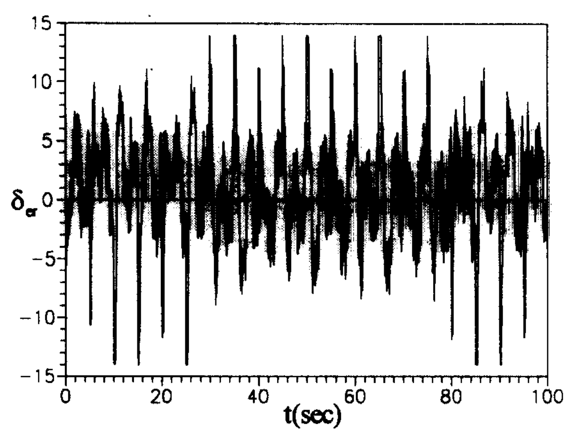
For the first measurements, maneuvering coefficients of a submerged body are identified by using the maximum likelihood technique with the Newton-Raphson & steepest descent method, the maximum likelihood technique with the Nelder & Mead simplex search method and the modified extended Kalman filter with a square-root algorithm. These results are compared in Fig.4, where the x-axis represents values of parameters.

As shown in Fig.4, the ML with Nelder & Mead simplex search method and the MEKF with a square-root algorithm give acceptable results but in the ML in combination with the Newton-Raphson method & the steepest descent method, some identified coefficients drift and in the worst case, seem to be losing convergency. This phenomenon is due to using the inverse of Hessian matrix and the sensitivity of accuracy of the mathematical model. Because absolute values are plotted in the figures, for some parameters whose magnitude are large, the estimated values show larger differences from the true value, even though their estimation errors are within the acceptable ranges compared to those of other parameters.

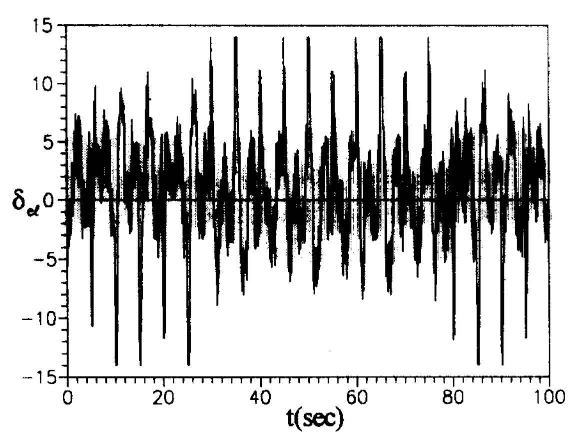
For the second measurements, maneuvering coefficients are identified by using the ML with the Nelder & Mead simplex search method and the MEKF with a square-root algorithm, and the identified results are compared in Fig.6. It is obvious that the measurement data from from scenario II are better than those from scenario I in order to identify maneuvering coefficients using the ML with the Nelder & Mead simplex search method, but for the MEKF with a square-root algorithm, two data give similar




(a) Rudder Angle

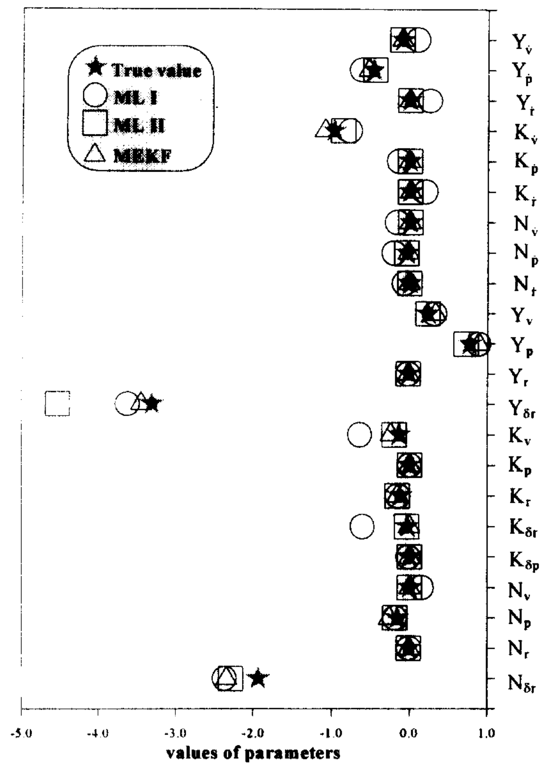



(b) Right Elevator Angle




(c) Left Elevator Angle

Figure 3: Time History of Input Signals for the Scenario I



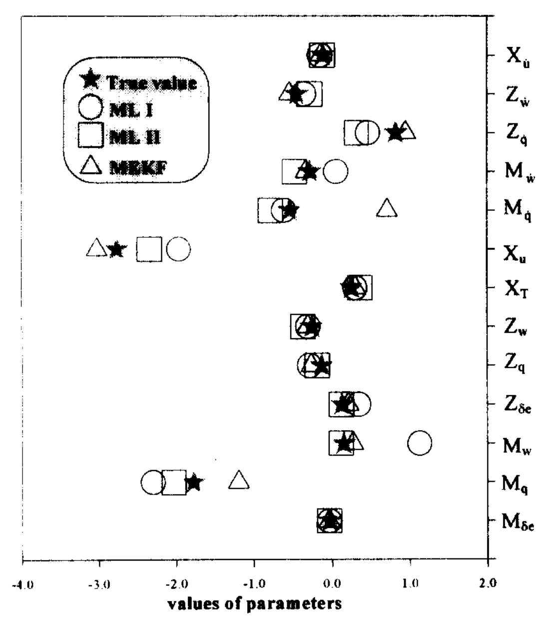
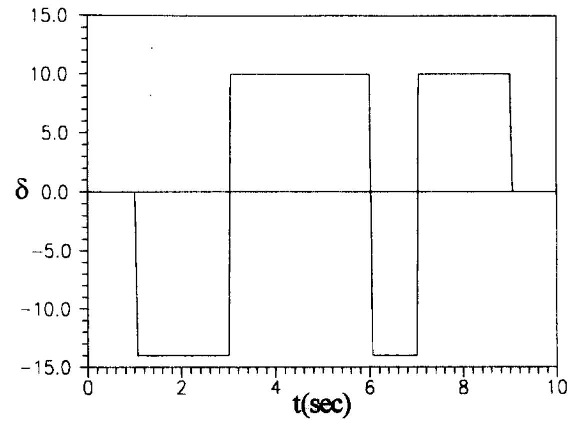
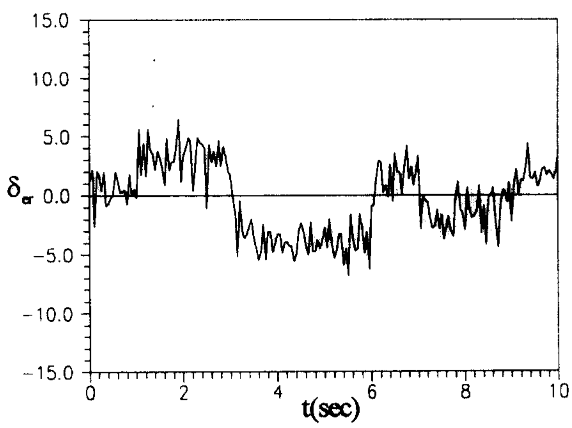



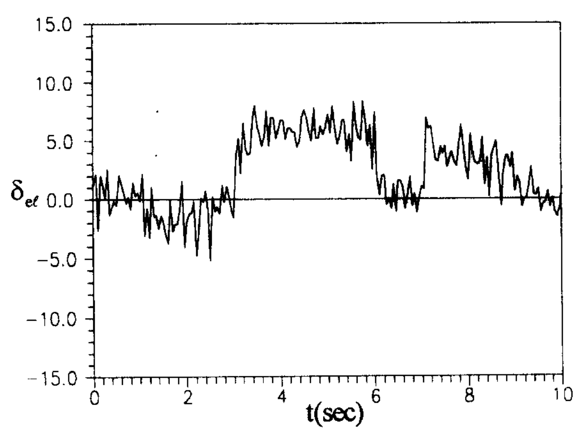

Figure 4: Comparison between Identified Results (Scenario I)




(a) Rudder Angles for the Horizontal Mode or Elevator Angles for the Vertical Mode




(b) Right Elevator Angles for the Horizontal Mode




(c) Left Elevator Angles for the Horizontal Mode

Figure 5: Time History of Input Signals for the Scenario II

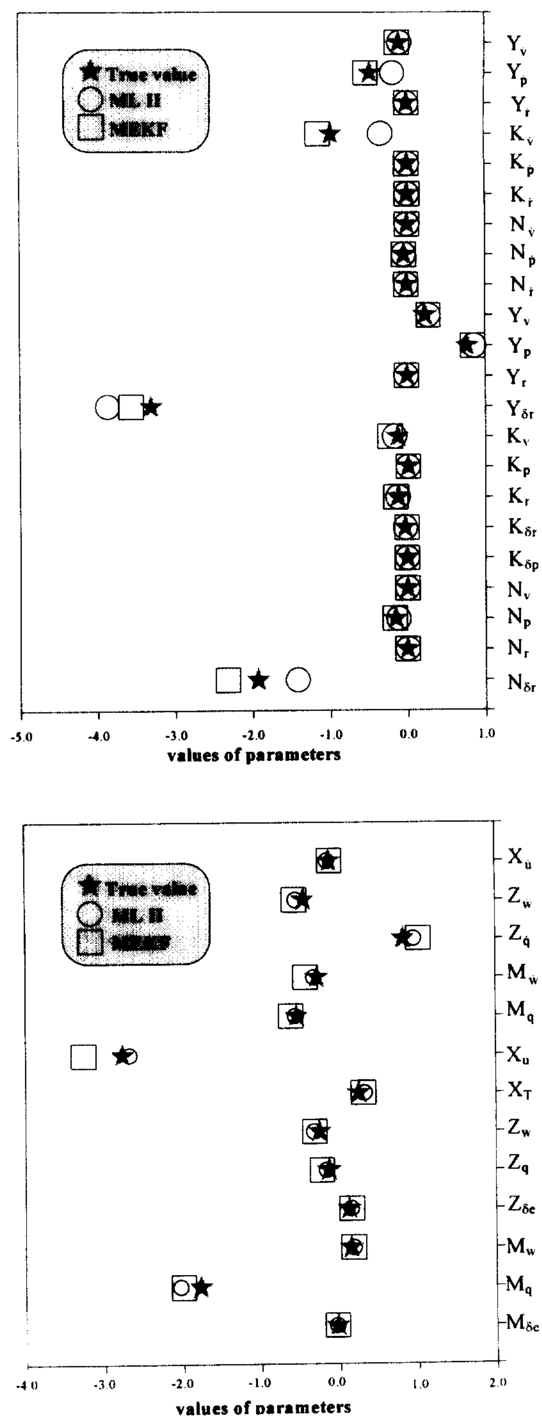



Figure 6: Comparison between Identified Results (Scenario II)

results. Therefore, it is said that the ML with the Nelder & Mead simplex search method could identify parameters satisfactorily if the input signals were designed using the concept of pseudo random binary sequence.

# 7 Conclusions

In this paper, the relationships between maneuvering coefficients of dynamic system and those of the equivalent system are investigated, and two identification techniques are tested. One of the identification techniques is off-line method such that the ML methods using the Nelder & Mead simplex search method and using the modified Newton-Paphson methods, and the other is the MEKF with a square-root algorithm as on-line method. As results, the following conclusions are drawn.

1. Linear equations of motion of a submerged body cannot have global identifiability and have higher probability to generate simultaneous drift phenomenon due to the existence of the inertia matrix compared to general state equations.

2. Identified results show that the ML with the Nelder & Mead simplex search method gives acceptable results but results of the ML in combination with the Newton-Raphson method and the steepest descent method are unsatisfactory.

3. The ML with the Nelder & Mead simplex search method gives satisfactory results if the input signals were designed using the concept of pseudo random binary sequence.

Among the areas for further investigation, it might be mentioned as follows:

1. the combination of on-line and off-line method

2. identification technique to guarantee the global identifiability such a genetic algorithm.

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# Appendix Equations of Motion

Equations of motion of a submerged body are written as follows:

- surge

$$(m - X_{\dot{u}})\dot{u} = X_u u + X_T - (W - B)\sin\theta.$$

- sway

$$\begin{aligned}(m - Y_{\dot{v}})\dot{v} - Y_{\dot{p}}\dot{p} - Y_{\dot{r}}\dot{r} &= Y_v v + Y_p p + (Y_r - mU_c)r + (W - B)\sin\phi\cos\theta \\ &\quad + Y_{\delta_r}\delta_r.\end{aligned}$$

- heave

$$\begin{aligned}(m - Z_{\dot{w}})\dot{w} - Z_{\dot{q}}\dot{q} &= Z_w w + (Z_q + mU_c)q + (W - B)\cos\phi\cos\theta \\ &\quad + Z_{\delta_e}\frac{\delta_{er} + \delta_{el}}{2}.\end{aligned}$$

- roll

$$\begin{aligned}-K_{\dot{v}}\dot{v} + (I_x - K_{\dot{p}})\dot{p} - K_{\dot{r}}\dot{r} &= K_v v + K_p p + K_r r - y_B B\cos\phi\cos\theta \\ &\quad + z_B B\sin\phi\cos\theta + K_{\delta_r}\delta_r + K_{\delta_{er}}\delta_{er} - K_{\delta_{el}}\delta_{el}.\end{aligned}$$

- pitch

$$\begin{aligned}-M_{\dot{w}}\dot{w} + (I_y - M_{\dot{q}})\dot{q} &= M_w w + M_q q + x_B B\cos\phi\cos\theta + z_B B\sin\theta \\ &\quad + M_{\delta_e}\frac{\delta_{er} + \delta_{el}}{2}.\end{aligned}$$

- yaw

$$\begin{aligned}-N_{\dot{v}}\dot{v} - N_{\dot{p}}\dot{v} + (I_z - N_{\dot{r}})\dot{r} &= N_v v + N_p p - x_B B\sin\phi\cos\theta \\ &\quad - y_B B\sin\theta + N_{\delta_r}\delta_r.\end{aligned}$$

- Euler angles

$$\begin{aligned}\dot{\psi} &= \frac{r + q\tan\phi}{\cos\theta\cos\phi(1 + \tan^2\phi)}, \\ \dot{\theta} &= q\sec\phi - \dot{\psi}\cos\theta\tan\phi, \\ \dot{\phi} &= p + \dot{\psi}\sin\theta.\end{aligned}$$