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Wave Patterns Due to a Point Impulse Travelling over Free Surface of Water of Finite Depth

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Abstract

If a point impulse travels over free surface of water of finite depth, surface waves consist of divergent waves. The crestlines of those divergent waves are short and end on the cusp line if the impulse travels at a subcritical speed. But the crestlines become infinitely long and there are no cusps if the impulse travels at a supercritical speed.

1 Introduction

Havelock initiated study for the present problem. He ingeniously expressed the wave induced by a point impulse travelling over the free surface of water of finite depth in terms of Fourier integrals. Then he computed and gave two wave patterns for two subcritical speeds and one wave pattern for a supercritical speed[1]. His wave patterns for the subcritical speeds consist of transverse and divergent waves which are very similar to the well known Kelvin wave pattern except for wide cusp angles. Furthermore, his wave pattern for the supercritical speed consists of the crestlines of divergent waves which are concave to the x-axis. According to the note by editor[1, p.430], Inui recalculated the wave pattern for the supercritical speed and found that the creslines are not concave to the x-axis.

Recently the Kelvin wave pattern has turned out that its wave pattern consists of the creslines of divergent waves[2]. Based on the method of Chung and Lim which derived the new Kelvin wave pattern, the present problem is revisited. In the present study, the authors attempt to investigate wave patterns by applying the method of Green functions. The point impulse is represented by the point pressure in terms of Dirac delta function. Waves downstream from the point impulse are formulated by a Green function and approximated asymptotically by the principle of stationary phase.

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2 Formulation

We shall consider surface waves induced by a point pressure moving on the free surface of water of finite depth H at a constant speed U. We introduce the rectangular coordinate system (x, y, z) moving at the constant speed U. The origin of the coordinate system is fixed at the point pressure such that the x,y-plane coincides with the calm free surface and the z-axis is directed upwards. Let us introduce the following Green function:

$$\begin{aligned} G(x,y,z;\xi,\eta,\zeta) &= \frac{1}{R}+\frac{1}{R_1}-\frac{4}{\pi}\int_0^{\pi/2}\sec^2\sigma d\sigma\int_0^{\infty}\frac{\{e^{-kH}\mathrm{sech}kH\cosh k(z+H)\cosh k(\zeta+H)}{k-K\sec^2\sigma\tanh kH} \\ &\quad\times\cos[k(x-\xi)\cos\sigma]\cos[k(y-\eta)\sin\sigma](k\cos^2\sigma+K)-K\}dk \qquad (1) \\ &\quad+4\int_{\sigma_0}^{\pi/2}\frac{\{e^{-k_0H}\sec^2\sigma\,\mathrm{sech}k_0H\cosh k_0(z+H)\cosh k_0(\zeta+H)}{1-KH\sec^2\sigma\,\mathrm{sech}^2k_0H} \\ &\quad\times\sin[k_0(x-\xi)\cos\sigma]\cos[k_0(y-\eta)\sin\sigma](k_0\cos^2\sigma+K)-K\}d\sigma \end{aligned}$$

for (x, y) in downstream

where

$$\begin{aligned} K &= g/U^2, \\ R &= \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}, \\ R_1 &= \sqrt{(x-\xi)^2+(y-\eta)^2+(z+2H+\zeta)^2}, \\ \sigma_0 &= \begin{cases}\cos^{-1}\sqrt{KH} & \text{if } KH<1, \\ 0 & \text{if } KH>1\end{cases} \end{aligned}$$

and $k_0 = k_0(\sigma)$ is the positive root of

$$k_0 - K\sec^2\sigma\tanh k_0H = 0 \qquad \text{for } \sigma_0 < \sigma < \pi/2 \qquad (2)$$

The Green function in (1) is obtained by replacing $x-\xi$ in the Green function[3, p.150, Eq.(3.51b)] by $-(x-\xi)$ because the point source in the present case is moving in the negative direction. The surface displacement is given by

$$h(x,y) = \frac{U}{g}\phi_x(x,y,0) \qquad (3)$$

The point impulse is expressed as the point pressure given by $p(x,y) = p_0\delta(x)\delta(y)$ in terms of the Dirac delta function where $p_0 < 0$. Applying Green's theorem, we get the surface potential for the flow due to the point impulse as

$$\phi(x,y,0) = -\frac{p_0U}{2\pi\rho g}G_\xi(x,y,0;0,0,0) \qquad (4)$$

Let us introduce the polar coordinates (r,θ) defined by $x = r\cos\theta$ and $y = r\sin\theta$. It follows from (1) that (3) is further written as

$$\begin{aligned} h(r,\theta) &\sim -\frac{p_0}{\pi K\rho g}\mathrm{Im}\int_{-\pi/2}^{\pi/2}\frac{k_0^2(k_0\cos^2\sigma+K)e^{-k_0H}\cosh k_0He^{ir\psi(\theta,\sigma)}}{1-KH\sec^2\sigma\,\mathrm{sech}^2k_0H} \\ &\quad\times[u(\sigma+\pi/2)-u(\sigma+\sigma_0)+u(\sigma-\sigma_0)-u(\sigma-\pi/2)]d\sigma \qquad \text{for large } r \quad (5) \end{aligned}$$

where $u(\sigma)$ is the unit step function such that $u(\sigma) = 1$ if $\sigma > 0$, $u(\sigma) = 0$ if $\sigma < 0$, and $\psi(\theta, \sigma) = k_0(\sigma)\cos(\theta + \sigma)$. We can readily see that $h(r, \theta)$ in (5) satisfies $h(r, \theta) = h(r, -\theta)$. Thus, waves are symmetrical about the x-axis. Hence, we consider waves only in the upper half plane from now on.

We further approximate (5) for $h(r, \theta)$. In this connection, we apply the principle of stationary phase to (5) for large r. The equation $\psi_\sigma(\theta, \sigma) = 0$ yields the following quadratic equation for stationary points:

$$2\tanh k_0H \tan\theta \tan^2\sigma - F_1(k_0H)\tan\sigma + F_2(k_0H)\tan\theta = 0 \qquad (6)$$

where $F_1(k_0H) = \tanh k_0H + k_0H\mathrm{sech}^2 k_0H$ and $F_2(k_0H) = \tanh k_0H - k_0H\mathrm{sech}^2 k_0H$. It follows from (6) that two stationary points are determined from

$$\tan\sigma = \frac{F_1(k_0H) \pm \sqrt{F_1^2(k_0H) - 8F_2(k_0H)\tanh k_0H \tan^2\theta}}{4\tanh k_0H \tan\theta} \qquad (7)$$

Eq.(7) is a relation among k_0, θ, and the stationary point σ while (2) is a relation between k_0 and σ. Hence, k_0 and σ are functions of θ, *i.e.*, $k_0 = k_0(\theta)$ and $\sigma = \sigma(\theta)$.

3 Wave Patterns for $F_r > 1$

We first consider waves when the point impulse travels at a supercritical speed. The negative sign in (7) is absurd if $F_r > 1$. Hence, the negative sign is discarded. Let θ^* be the angle for the outer limit for waves. Then waves prevail inside the wedge for $0 < \theta < \theta^*$ in the upper half plane. Let $\sigma^* = \sigma(\theta^*)$ and $k_0^* = k_0(\theta^*)$. Since $k_0 \to 0$ as $\sigma \to \sigma_0$ from (2), we get

$$\tan\sigma_0 = \frac{1}{\tan\theta^*} \qquad (8)$$

from (7) by taking the limit $\sigma \to \sigma_0$. Hence, we have $\sigma_0 = \sigma^*$ from (8). Further, (7) yields $\sigma(0) = \pi/2$. Thus, σ varies in $\sigma_0 < \sigma < \pi/2$ if θ varies in $0 < \theta < \theta^*$ in the upper plane. If $\theta > \theta^*$, there is no real root for (6) and there are no real stationary points. Since $\sigma_0 = \cos^{-1}\sqrt{KH} = \cos^{-1}(1/F_r)$ it follows from (8) that

$$\theta^* = \sin^{-1}\left(\frac{1}{F_r}\right) \qquad (9)$$

There are no cusps. Waves are confined inside the V-shaped region in the upper plane bounded by the outer straight line making an angle θ^* with the x-axis. In the region for $\theta > \theta^*$, there are no stationary points. Waves in this region are small and ignored. The angle θ^* in (9) agrees with that of Havelock[1, p.427, Eq.(118)].

Before we apply the principle of stationary phase to (5) for large r, it is necessary to examine the sign of $\psi_{\sigma\sigma}(\theta, \sigma)$. Differentiating $\psi(\theta, \sigma)$ with respect to σ twice yields

$$\psi_{\sigma\sigma}(\theta, \sigma) = (k_0)_{\sigma\sigma}\cos(\theta + \sigma) - 2(k_0)_\sigma \sin(\theta + \sigma) - k_0\cos(\theta + \sigma) \qquad (10)$$

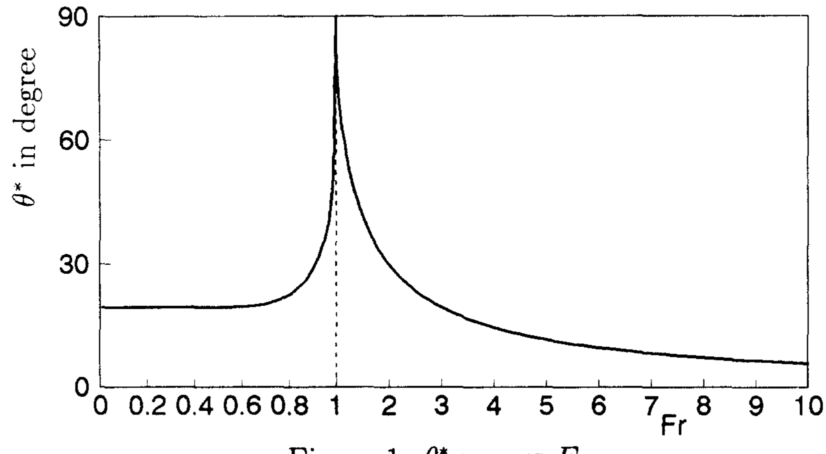

Figure 1: θ^* versus F_r

where $(k_0)_\sigma$ and $(k_0)_{\sigma\sigma}$ are obtained from (2). Direct computations show $\psi_{\sigma\sigma}(\theta,\sigma) < 0$ for θ in $0 < \theta < \theta^*$. We now apply the principle of stationary phase to (5) and get

$$h(r,\theta) \sim A(r,\theta)\sin[r\psi(\theta,\sigma) - \pi/4] \qquad \text{for large } r \tag{11}$$

where

$$A(r,\theta) = -\frac{p_0}{\pi K \rho g}\sqrt{\frac{2\pi}{r|\psi_{\sigma\sigma}(\theta,\sigma)|}}\frac{k_0^2(k_0\cos^2\sigma + K)e^{-k_0H}\cosh k_0H}{1 - KH\sec^2\sigma\,\mathrm{sech}^2 k_0H} \qquad (-p_0 > 0)$$

Wave patterns are now represented by a set of the crestlines from the constant phase

$$r\psi(\theta,\sigma) - \frac{\pi}{4} = 2n\pi + \frac{\pi}{2} \tag{12}$$

where n is a large positive integer. For a given θ, we compute σ and k_0 from (2) and (7). Then, the crestlines are computed from (12).

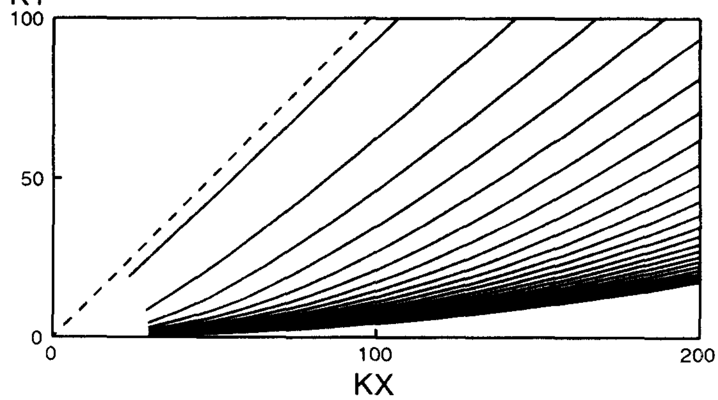

Figure 2: Crestlines of divergent waves for F_r=1.4

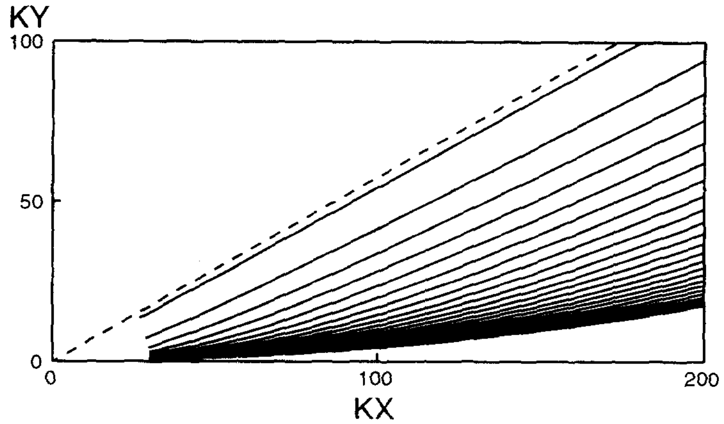

Figure 3: Crestlines of divergent waves for F_r=2.0

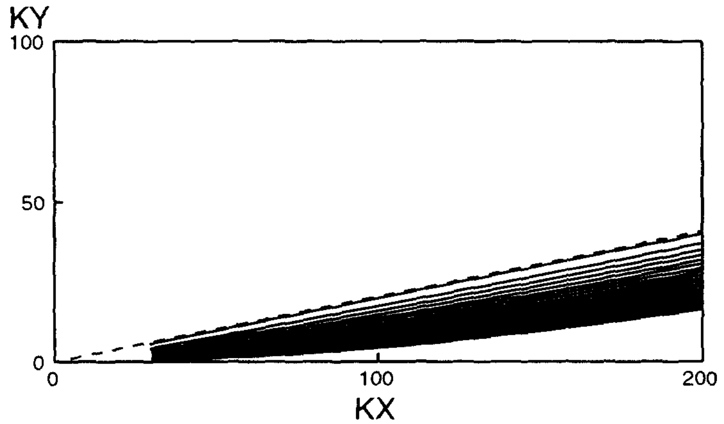

Figure 4: Crestlines of divergent waves for F_r=5.0

4 Wave Patterns for $F_r < 1$

If the point impulse travels at a subcritical speed, (6) yields two stationary points as in the Kelvin wave[1]. The limit angle θ^* becomes the cusp angle. The cusp angle is determined by the vanishing discriminant in (7) as

$$\tan\theta^* = \frac{\tanh k_0^* H + k_0^* H \mathrm{sech}^2 k_0^* H}{\sqrt{8(\tanh k_0^* H - k_0^* H \mathrm{sech}^2 k_0^* H)\tanh k_0^* H}} \tag{13}$$

When the discriminant vanishes, (7) becomes

$$\tan\sigma^* = \frac{\tanh k_0^* H + k_0^* H \mathrm{sech}^2 k_0^* H}{4\tanh k_0^* H \tan\theta^*} \tag{14}$$

It follows from (2), (13), and (14) that

$$k_0^* H = \frac{1}{2F_r^2}(3\tanh k_0^* H - k_0^* H \mathrm{sech}^2 k_0^* H) \tag{15}$$

If $H \to +\infty$, (13) reduces to $\tan\theta^* = 2^{-3/2}$ and (14) reduces to $\tan\sigma^* = 2^{-1/2}$. Those values of θ^* and σ^* agree with the values for the Kelvin wave. The computed values of $k_0^* H, \theta^*$ and σ^* are given in the Table 1.

Table 1: The wave number and Limit angle

F_r	$k_0^* H$	σ^*(deg.)	θ^*(deg.)	θ^*(Havelock)
0.3873	10	35.2644	19.4712	19.47
0.4330	8	35.2644	19.4713	19.47
0.4998	6	35.2624	19.4752	19.48
0.5476	5	35.2521	19.4958	19.50
0.6116	4	35.1917	19.6165	19.62
0.7019	3	34.8576	20.2847	20.30
0.8293	2	33.1539	23.6922	23.70
0.9656	1	25.3414	39.3173	39.32
0.9966	0.5	15.2708	59.4583	59.45
0.9999	0.2	6.5266	76.7469	78.00
1.0000	0	0.0000	90.0000	90.00

Let σ_1 be the stationary point for the negative sign in (7) and σ_2 be the stationary point for the positive sign. Then we see that σ_1 is in $0 < \sigma_1 < \sigma^*$ and σ_2 is in $\sigma^* < \sigma < \pi/2$ for θ in $0 < \theta < \theta^*$. Because k_0 depends on σ from (2), we also write $k_0 = k_0(\sigma)$. Hence, k_0 is written as $k_0 = k_0(\sigma_1)$ when $\sigma = \sigma_1$ in (2). Similarly, k_0 is written as $k_0 = k_0(\sigma_2)$ when $\sigma = \sigma_2$. We find the two first-order stationary points σ_1 and σ_2 from (2) and (7). Then, $k_0(\sigma_1)$ varies in $0 < k_0(\sigma_1) < k_0^*$ whereas $k_0(\sigma_2)$ varies in $k_0^* < k_0(\sigma_2) < \infty$. As $\theta \to \theta^*$, the first-order stationary points σ_1 and σ_2 coalesce into the single second-order stationary point σ^*. We find $\psi_{\sigma\sigma}(\theta, \sigma_1) > 0$ and $\psi_{\sigma\sigma}(\theta, \sigma_2) < 0$. Hence, (5) is approximated asymptotically for large r as

$$h(r,\theta) \sim A_1(r,\theta)\sin[r\psi(\theta,\sigma_1) + \pi/4] + A_2(r,\theta)\sin[r\psi(\theta,\sigma_2) - \pi/4] \quad \text{for large } r \tag{16}$$

where

$$A_i(r,\theta) = -\frac{p_0}{\pi K \rho g}\sqrt{\frac{2\pi}{r|\psi_{\sigma\sigma}(\theta,\sigma_i)|}}\frac{k_0^2(\sigma_i)[k_0(\sigma_i)\cos^2\sigma_i + K]e^{-k_0(\sigma_i)H}\cosh k_0(\sigma_i)H}{1 - KH\sec^2\sigma_i \,\mathrm{sech}^2 k_0(\sigma_i)H} \quad \text{for } i = 1,2 \tag{17}$$

The first wave on the right side of (16) is called the transverse wave and the second is called the divergent wave. The two waves are combined into the single wave as

$$h(r,\theta) \sim \sqrt{(A_2+A_1)^2\cos^2(\omega/2)+(A_2-A_1)^2\sin^2(\omega/2)}\,\sin[r\psi(\theta,\sigma_2)-\beta(r,\theta)-\pi/4] \quad (18)$$

where

$$\begin{aligned}\omega(r,\theta) &= r[\psi(\theta,\sigma_2)-\psi(\theta,\sigma_1)]-\pi/2\\ \beta(r,\theta) &= \tan^{-1}\left[\frac{1}{2}\frac{\alpha\sin\omega}{1-\alpha\sin^2(\omega/2)}\right]\\ \alpha &= \frac{2A_1}{A_2+A_1}\end{aligned}$$

Wave patterns are obtained from the constant phase

$$r\psi(\theta,\sigma_2)-\beta(r,\theta)=2n\pi+\frac{3}{4}\pi \quad (19)$$

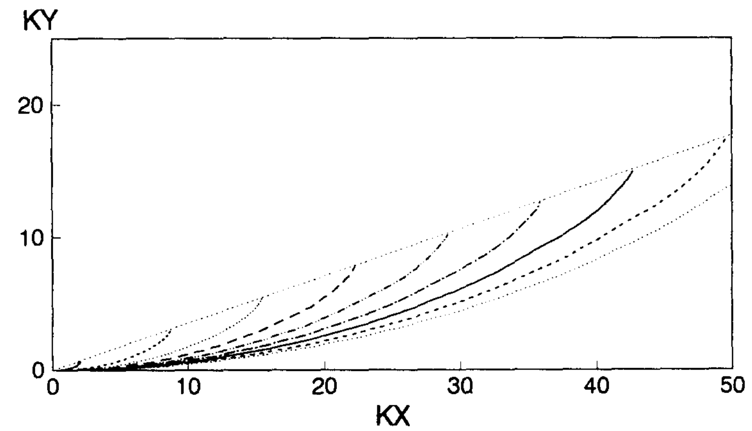

Figure 5: Crestlines of divergent waves for F_r=0.1

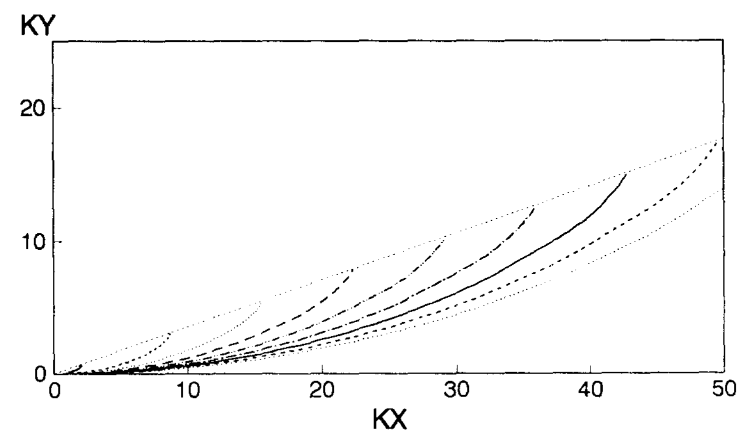

Figure 6: Crestlines of divergent waves for F_r=0.5

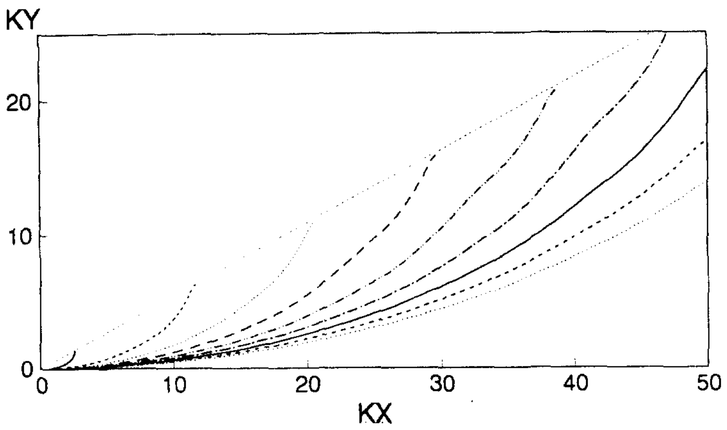

Figure 7: Crestlines of divergent waves for F_r=0.9

5 Discussion and Conclusion

Wave patterns induced by a point impulse in water of finite depth consist of divergent waves whether the impulse travels at a supercritical speed or not. Wave patterns for supercritical speeds consist of divergent waves with infinitely long crestlines whereas wave patterns for subcritical speeds consist of short divergent waves that end on the cusp lines. Interestingly, we often observe that divergent waves with long crestlines follow a high speed boat running at a supercritical speed. As Inui pointed out, the crestlines of the divergent waves for supercritical speeds are not concave to the x-axis as shown in Figs. 2 through 4.

Very recently, Chung and Lim[2] found the new Kelvin wave pattern induced by a point impulse travelling over the free surface in which no transverse wave exists. Hence, the present problem has never been properly studied up to this point.

References

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