A Linear Theory of MHD Stability in the Geomagnetotail

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(Manuscript received 5 November 1996)

Abstract

A stability analysis in the geomagnetotail is presented within MHD limit with a modified form from ideal Ohm's law. Using the high k_y approximation (ballooning limit), we derive the basic eigenmode equations which can be reduced to the ideal MHD limit. The incompressible limit is numerically solved for a number of model equilibria of tail by *Kan* [1973], and we have found no unstable *Kan* equilibrium. Also, an analytic theory is carried out for the case where $B\kappa_c$ is assumed to be constant along the field line, following the idea by *Lee and Min* [1996]. In that case, it is suggested that the tail stability to the incompressible antisymmetric mode is determined by the ideal MHD.

1. Introduction

Magnetospheric substorms are often considered as the result of some large scale plasma instabilities in the magnetotail. While there have been proposed a number of different diverging scenarios of substorm mechanisms, the problem of tail dynamics, in particular, its stability in association with substorm onset, still remains confusing and unresolved in the community. While the tearing mode has long been most popular as much as controversial, other instabilities like ballooning mode [e.g., *Roux et al.*, 1991] have been also proposed.

There exist several studies on linear stability theories in the geomagnetosphere within MHD approximations in the very high k_y limit, which is often referred to as ballooning limit in fusion devices. In this limit, it has been recognized by *Lee and Wolf* [1992] that the plasma compression is the key reason to stabilize the compressible interchange/balloon mode of ideal MHD in the tail geometry. Also, *Lee and Min* [1996] argued that the tail-like geometry needs to have a sufficient field-aligned portion of a substantial curvature in order to become unstable to the incompressible antisymmetric mode of ideal MHD. In both results, the plasma compressibility in combination with the "hard" or "closed" ionospheric foot boundary condition is the essential reason to make the balloon/interchange type mode become stable in the typical tail geometry.

In this work, we wish to study the effect of non-ideality in, otherwise, ideal Ohm's law on the plasma compression for the high k_y limit stability analysis. We first derive the basic, coupled eigenmode equations (section 2), and perform an extensive numerical calculation for a number of model equilibria for the incompressible mode (section 3). Also, an analytic theory is developed for a model where $B\kappa_c$ is assumed to be constant along the field line, following the idea by *Lee and Min* [1996] .

2. Derivation of Basic Eigenmode Equations

2.1 Equilibrium

We will consider the two-dimensional equilibrium in which nothing depends on the coordinate y and where the magnetic field has no y-component in the usual geomagnetic (x, y, z) coordinate system. Also, we introduce a new orthogonal coordinate system for the convenience of the stability analysis, namely, (A, y, χ): A represents the flux coordinate across the magnetic flux surface in the normal direction, and χ is the field-aligned coordinate in which the field-aligned length element ds is given by $h_\chi d\chi$, h_χ being the corresponding metric factor.

2.2 A Modified Form of Ohm's Law

The derivation is based on the typical set of MHD equations, namely, the low frequency form of Maxwell equations plus the equations that govern the plasma motion in ideal MHD, except one modification. Specifically, we use the following form of Ohm's law.

$$\mathbf{E}+\mathbf{v}\times\mathbf{B} = (\mathbf{J}\times\mathbf{B}-\nabla p_e)/ne \qquad (1)$$

where p_e represents the electron pressure and the other notations refer to the usual physical quantities. This form of Ohm's law can be derived from a viewpoint of single particle motion within the guiding center approximation [*Wolf*, personal communication, 1989] . In fact, the right hand side of (1) is directly related to the usual drift velocities of the particle such that neglecting them in comparison with the $\mathbf{E}\times\mathbf{B}$ plasma motion simply recovers the perfect conductivity approximation in ideal MHD, i.e., $\mathbf{E}+\mathbf{v}\times\mathbf{B}=0$.

In the derivation of basic eigenmode equation of stability here, we maintain the nonzero contribution of the right hand side of (1). We wish to study whether such a modification in Ohm's law can help in reducing or even removing the plasma compression effect in determining high k_y stability.

2.3 Linearization within the high k_y approximation

In this linear theory, the displacement vector has the variation of $\exp(ik_y y)$. One of the major approximation in the procedure of the derivation is to take the limit $k_\perp = k_y \rightarrow \infty$, which corresponds to assuming that the wavelength in the y direction is very short compared to the scale length in the xz plane. This limit is normally referred to as "high-n ballooning" in fusion devices [*Freidberg*, 1987]. Using this limit, the Ohm's law, (1), can be linearized as follows.

$$\delta\mathbf{E} + \delta\mathbf{v}\times\mathbf{B} = -\{j\rho(\omega-\omega_{i\perp})/ne\}\delta\mathbf{v} + \Delta(\nabla\delta p-\delta p/\gamma p\nabla p)/ne - \mathbf{v}\times\delta\mathbf{B} \quad (2)$$

where δ refer to the perturbed quantities, $\omega_{i\perp}=J_{i\perp}k_y/en$, $J_{i\perp}$ the perpendicular ion current density, ρ the mass density, and γ the ratio of specific heats. Also, it was assumed that $\delta p_i = \Delta\delta p$ for simplicity where Δ is an arbitrary constant and that the equilibrium electric field $\mathbf{E} = 0$. Our derivation partly follows the work by *Miura et al.* [1989], in which the simple perfect conductivity approximation was used, however.

In preparation for the calculation of the magnetic induction equation in which the curl of the electric field has to be computed, we carry out the following calculations setting the right hand side of (2) to $\mathbf{N}$.

$$(\nabla\times\mathbf{N})_\Lambda = -\{j\rho(\omega-\omega_{i\perp})/ne\}(ik_y\delta v_\Lambda-\partial\delta v_y/\partial s) + j\omega_{i\perp}\delta B_\Lambda \quad (3)$$

$$\begin{aligned}(\nabla\times\mathbf{N})_y = &-\{j\rho(\omega-\omega_{i\perp})/ne\}(B/h_\chi)\{\partial(\delta v_\Lambda/B)/\partial\chi\} - v_y(B/h_\chi)\{\partial(\delta B_\parallel/B)/\partial\chi\} \\ &+\{j\rho(\omega-\omega_{i\perp})/ne\}\delta v_\parallel(B/h_\chi)\{\partial h_\chi/\partial A\} + B\partial\{j\rho(\omega-\omega_{i\perp})\delta v_\parallel/ne\}/\partial A \\ &-(B/h_\chi)\{\partial h_\chi v_y\delta B_\Lambda/\partial A\}\end{aligned} \quad (4)$$

$$(\nabla\times\mathbf{N})_\parallel = B\partial\{-j\rho(\omega-\omega_{i\perp})\delta v_y/ne\}/\partial A + \{j\rho(\omega-\omega_{i\perp})/ne\}ik_y\delta v_\Lambda + j\omega_{i\perp}\delta B_\parallel \quad (5)$$

where $v_y = \omega_{i\perp}/k_y$ and $\parallel$ refer to the component parallel to the magnetic field.

Our first derivation concerns the relation between $\delta B_\parallel$ and δp in an attempt to see whether the diamagnetic relation $B\delta B_\parallel+\mu_0\delta p = 0$ [*Miura et al.*, 1987] still holds in the present formulation. First, from Ampere's law, one obtains

$$\mathbf{b}\cdot\nabla\times\delta\mathbf{J}_\perp = (B/\mu_0)(\partial/\partial A)\{B\partial/\partial s(\delta B_A/B)-(B/h_\chi)\partial/\partial A(h_\chi\delta B_\parallel)\} + k_y^2\delta B_\parallel/\mu_0 \qquad (6)$$

Then, linearizing the momentum equation gives

$$\mathbf{b}\cdot\nabla\times\delta\mathbf{J}_\perp = -\{j\rho(\omega-\omega_{i\perp})/B\}\{\delta\mathbf{v}_\perp\cdot\kappa_c + \nabla\cdot\delta\mathbf{v}_\perp) - \delta\mathbf{v}_\perp\cdot\nabla\{j\rho(\omega-\omega_{i\perp})/B\} - \mathbf{b}\cdot\nabla\delta B_\parallel\times\mathbf{J}/B + (\delta B_\parallel/B^2)\mathbf{b}\cdot\nabla B\times\mathbf{J} - (\delta B_\parallel/B)\mathbf{b}\cdot\nabla\times\mathbf{J} + (\mu_0/B^3)\nabla_\perp\delta p\cdot\nabla p + (1/B)\nabla_\perp^2\delta p \qquad (7)$$

where $\kappa_c = \mathbf{b}\cdot\nabla\mathbf{b} = (\mathbf{B}/B)\cdot\nabla(\mathbf{B}/B)$. In the derivation, an additional simplifying assumption was made, $\omega_{i\perp}/\omega \ll O(\varepsilon^{-1})$. Also, equation (5) was used for the computation of $\delta B_\parallel$. Equating (6) and (7) verifies that the same diamagnetic relation can still be used in the present formulation. This relation is used in the future calculations.

Now we will derive an equation for δv_Λ. We start from the momentum equation to obtain

$$\nabla\cdot\delta\mathbf{J}_\perp = \{k_y\rho(\omega-\omega_{i\perp})/B^2\}B\delta v_A - ik_yJ(B\delta B_\parallel - \mu_0\delta p)/B^2 + 2(ik_y/B)\delta p\partial B/\partial A \qquad (8)$$

Then, linearizing Ampere's law gives

$$\nabla\cdot\delta\mathbf{J}_\parallel = -(ik_y/\mu_0)(B/h_\chi)\{\partial\delta B_A/B\partial\chi) \qquad (9)$$

Now, using (8) and (9) for the condition $\nabla\cdot\delta\mathbf{J}=0$, one finally obtains the basic eigenmode equation for X where $X=B\delta v_\Lambda$.

$$v_\Lambda^2B(d/ds)\{(1/B)(dX/ds)\} + \omega_*^2X = -(2B\kappa_c/\rho)(Xdp/dA + \gamma p\nabla\cdot\delta\mathbf{v}) + v_\Lambda^2(k_y\omega_*\rho/ne)\nabla\cdot\delta\mathbf{v} + (k_y\omega_*/ne)\{\gamma p\nabla\cdot\delta\mathbf{v} + 2(B^2/\mu_0)(\kappa_c/B)X - (k_y\omega_*\rho/\mu_0ne)X\} \qquad (10)$$

where v_Λ is the Alfven velocity, and $\omega_*=\omega-\omega_{i\perp}$. Equation (10) reduces to the Alfven-ballooning mode equation of ideal MHD in the limit that $\omega_{i\perp}=0$ by setting the k_y-dependent terms to zero. The effect of the non-ideal Ohm's law is maintained through $\omega_{i\perp}$ in several terms of (10). Also, the plasma compressibility $\nabla\cdot\delta\mathbf{v}$ which is related to the equation of δp appears to be coupled to δv_Λ equation (10). Therefore, equation (10) basically describes a "modified Alfven ballooning mode" being coupled to the plasma compressibility effect.

The equation to govern the plasma compressibility which is in turn coupled to (10) can be derived in a similar fashion. By linearizing the adiabatic compression equation of state, one finds

$$-i(\omega-\omega_{i\perp})\delta p = -B\delta v_A dp/dA - \gamma p\nabla\cdot\delta\mathbf{v} \tag{11}$$

Now, $\nabla\cdot\delta\mathbf{v}$ can be explicitly calculated as follows. First, combining Ohm's law with Faraday's law gives $\nabla\cdot\delta\mathbf{v}_\perp = \nabla\cdot(\delta\mathbf{E}\times\mathbf{B}/B^2) + \nabla\cdot(\mathbf{B}\times\mathbf{N}/B^2)$. After some algebra, one arrives at the following expression.

$$\nabla\cdot\delta\mathbf{v}_\perp = i(\omega-\omega_{i\perp})\delta B_\parallel/B + (\mu_0/B^2)\delta\mathbf{v}_\perp\cdot\nabla p - 2\delta\mathbf{v}_\perp\cdot\kappa_c - ik_y\delta v_A i\rho(\omega-\omega_{i\perp})/neB \tag{12}$$

Next, from the parallel component of the momentum equation, one finds

$$\{(\omega-\omega_{i\perp})ne + k_y dp/dA\}\nabla\cdot\delta\mathbf{v}_\parallel = -i(neB/\rho)\partial/\partial s(1/B\partial\delta p/\partial s) + (neB/\rho)(1/(\omega-\omega_{i\perp}))dp/dA\ \partial/\partial s(1/B\partial B\delta v_A/\partial s) \tag{13}$$

Consequently, substituting (12) and (13) into (11), one obtains the equation for δp as follows.

$$v_s^2 B(d/ds)\{(1/B)(d\delta p/ds)\} + \omega_*\{\omega_* + (k_y/ne)dp/dA\}(1+v_s^2/v_A^2)\delta p = -i(v_s^2 dp/dA)(B/\omega_*)(d/ds)\{(1/B)(dX/ds)\} -i\{\omega_* + (k_y/ne)dp/dA\}\{(1+v_s^2/v_A^2)dp/dA - (2\kappa_c\rho/B)v_s^2 + v_s^2/v_A^2(k_y\omega_*\rho/\mu_0 ne)\}X \tag{14}$$

where $v_s^2 = \gamma p/\rho$.

Equation (14) is, of course, coupled to (10) in complicated manner, so the two equations should be solved simultaneously and in most cases numerically for given equilibrium. Also, it is interesting that the wavenumber k_y appears explicitely in both equations as an independent parameter unlike the case of ideal MHD in the limit $k_\perp = k_y \rightarrow \infty$, where k_y trivially goes away in the final equation.

3. Incompressible Limit and Applications

3.1 Incompressible limit

Rather than attempting to solve the full, coupled eigenmode equations (10) and (14), we limit our interest to an incompressible case for which (10) is only relevant. Specifically, we set $\nabla\cdot\delta\mathbf{v}$ to zero in (10), which results in the following.

$$v_A^2 B(d/ds)\{(1/B)(dX/ds)\} + \omega_*^2(1+k_y^2\rho/\mu_0 n^2 e^2)X - \omega_*(2Bk_y\kappa_c/\mu_0 ne)X + (2B\kappa_c/\rho)(dp/dA)X = 0 \tag{15}$$

The first term is associated with the field line bending of the Alfven ballooning mode while the last term represents the free energy source for deriving the instability. Only the second and third terms appear to be modified and added by the non-ideal Ohm's law (1).

In general, it should not be taken for granted that the incompressible assumption always gives the most unstable chance. Instead, it should set some limitation to the flow velocity as discussed by *Lee and Wolf* [1992]. In ideal MHD, it gives the most unstable situation when $\nabla\cdot\delta\mathbf{v}$ is constant along the field line. Although it is not clear if the same condition can give the most unstable situation even in the present formulation, we will limit our interest to such a case for simplicity below.

Now we proceed to compute some useful expression for $\nabla\cdot\delta\mathbf{v}$. From Faraday's law together with Ohm's law, one can first obtain the expression for $\delta B_\parallel$ which, in combination with the adiabatic compression relation, then gives

$$-j\omega_*(B\delta B_\parallel + \mu_0\delta p) = -B^2(2\delta\mathbf{v}_\perp\cdot\kappa_c + \nabla\cdot\delta\mathbf{v}_\perp) - \mu_0\gamma p\nabla\cdot\delta\mathbf{v} - j\omega_*(B\rho/ne)(ik_y\delta v_A) \qquad (16)$$

Then, using the diamagnetic relation, one obtains the expression for $\delta B_\parallel$ below.

$$\nabla\cdot\delta\mathbf{v} = -(B^2/\mu_0\gamma p)(2\delta\mathbf{v}_\perp\cdot\kappa_c + \nabla\cdot\delta\mathbf{v}_\perp) - (B/\mu_0\gamma p)j\omega_*(\rho/ne)(ik_y\delta v_A) \qquad (17)$$

If $\nabla\cdot\delta\mathbf{v}$ is assumed to be constant along the magnetic field, then, after integrating (10) along the field line, one arrives at

$$\nabla\cdot\delta\mathbf{v} = -\int ds/B\ \{2\kappa_c/B - (\omega_*\rho/ne)(k_y/B^2\}X\ /\ \{\int ds/B + \mu_0\gamma p\int ds/B^3\} \qquad (18)$$

Therefore, one notice from (18) that any X that is antisymmetric about the equatorial plane is incompressible. We consider such a mode in the following section.

3.2 Numerical calculation

Equation (15) can be solved numerically for a given equilibrium and for various values of k_y. We employ a shooting method in combination with the Secant method [*Burden and Faires*, 1989] to find the eigenvalues and eigenfunctions in an automatic way.

We adopt a specific equilibrium of *Kan* [1973] which is a self-consistent analytic solution of Grad-Shafranov equation. The model has two physical parameters as boundary conditions, the plasma number density $N(0,0)$ and the equatorial magnetic field $B_{zc}(0,0)$ at the reference point $x^*=z^*=0$. The two parameters as well as the k_y values have been varied over a range of major interest. For all cases tested, we find no instability of antisymmetric ballooning in the Kan model. Table 1 summarizes a few examples of the computation result. As was pointed out for ideal MHD case by *Lee and Min* [1996], the equilibrium field line curvature is too strongly localized near the equatorial plane that the field line bending stabilizes the antisymmetric ballooning mode even in the present nonideal MHD formulation.

$N(0,0)$	0.11		0.44		ky
$B_{ze}(0,0)$	2.7	6.3	2.7	6.3	
ω^2	+0.043	+0.037	+0.052	+0.071	0 (ideal MHD)
ω_{real}	0.219	0.201	0.237	0.279	10^{-7}
ω_{imagi}	0.	0.	0.	0.	
ω_{real}	0.235	0.212	0.255	0.299	3×10^{-7}
ω_{imagi}	0.	0.	0.	0.	

Table 1. ω^2 is in units of 7.5×10^{-3} sec^{-2}, ω_{real} and ω_{imagi} in units of 8.6×10^{-2} sec^{-1}. Also, $N(0,0)$ in cm^{-3}, $B_{ze}(0,0)$ in nT, and k_y in m^{-1}, respectively. The computations are for the field line of $x^{*}=-4$, and terminated at $x^{*}=0$ where eigenfunctions were required to vanish.

3.3 An analytic theory

Recently, *Lee and Min* [1996] argued within the ideal MHD that the high beta tail-like field line model might be unstable to antisymmetric balloon mode if the field line has a sufficient field-aligned portion of a substantial curvature near the equatorial plane. We wish to apply their argument to the present formulation, specifically, to equation (15) which can be rewritten as follows.

$$f_1(d/ds)\{1/B\,dX/ds\} + f_2\omega_*^2X + f_3\omega_*X + f_4X = 0 \quad (19)$$

where $f_1=Bv_A^2$, $f_2=1+k_y^2\rho/\mu_0n^2e^2$, $f_3=-2B\kappa_ck_y/\mu_0ne$, and $f_4=(2B\kappa_c/\rho)(dp/dA)$.

We proceed to expand X in terms of the eigenfunctions of the incompressible ballooning equation of ideal MHD, X_k^{ideal}, with the boundary condition that the perturbation is zero at both ionospheric ends. Namely, $X=\sum c_kX_k^{ideal}$, and this is substituted back into (19) to obtain the following relation.

$$\sum c_k(f_2\omega_*^2 - \omega_k^{2\,ideal} + f_3\omega_*)X_k^{ideal} = 0 \quad (20)$$

Since we are interested in the case where $B\kappa_c$ (and so f_3) is constant along the field line, following the idea by *Lee and Min* [1996], equation (20) implies that for each independent X_k^{ideal}, $c_k(f_2\omega_*^2-\omega_k^{2\,ideal}+f_3\omega_*) = 0$. Since at least one c_k should be nonzero for nontriviality, it follows that $f_2\omega_*^2-\omega_k^{2\,ideal}+f_3\omega_* = 0$ for at least one eigenvalue ω_k^{ideal}. Therefore, it is clear that, for an imaginary ω_* (i.e., instability) to exist, at least one eigenvalue of ideal MHD has to be unstable according to $\omega_k^{2\,ideal} < -f_3^2/4f_2$. In other words, the stability of the antisymmetric mode in equilibrium where $B\kappa_c$ is constant along the field line is determined by that of ideal MHD. Once an unstable eigenvalue of ideal MHD is found, then the appropriate range of k_y values for the unstable mode of the present "nonideal" MHD ballooning can be selected from the relation $\omega_k^{2\,ideal} < -f_3^2/4f_2$.

4. Summary

In this work we have studied the stability of the two dimensional geotail model within MHD approximation with a modified form of Ohm's law in the high k_y limit. By this limit, we have obtained two coupled eigenmode equations which can be solved for each independent field line. The derived equations resemble those of ideal MHD limit, but are modified in some complex manner. The plasma compression effect in a form modified by the nonideal Ohm's law appears to couple the two equations.

The incompressible antisymmetric mode has been extensively tested numerically for a number of equilibria of *Kan* [1973]. We have found no unstable Kan equilibrium in this numerical study. As was pointed out by *Lee and Min* [1996], the magnetic curvature in Kan model is heavily localized only near the equatorial plane. In such a geometry, the huge field line bending is necessarily created suppressing the antisymmetric mode even in this nonideal MHD formulation.

Also, we have considered analytically the case where $B\kappa_c$ is constant along the field line, following the idea of *Lee and Min* [1996]. In this case, the stability to the antisymmetric mode is determined by the ideal MHD. In ideal MHD, the field line needs to have a sufficient field-aligned portion of a large enough curvature in order to become unstable to the antisymmetric mode.

We have not attempted a comprehensive study of the stability of the tail to the compressible mode. Since the compression effect appears to be modified in complicated manner, it requires further intensive study in order to see if there still remains a possibility for reducing the stabilizing contribution from the plasma compression.

Acknowledgements

This work was supported by the Korea Science and Engineering Foundation.

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