Quotient Fuzzy Normed Linear Spaces

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ABSTRACT

The main goal of this paper is to investigate some properties in close connection with the quotient fuzzy norm ρ_q induced by a fuzzy semi-norm ρ on a linear space X and the quotient map $q: X \to X/W$, where W is a subspace of X.

I. Introduction

Since Katsaras and Liu [2] have introduced the notions of fuzzy linear spaces and fuzzy topological linear spaces, the theory of fuzzy topological linear spaces was developed by [3, 5, 7, 8, 9] and so on. In his paper [4], Katsaras defined the fuzzy norm on a linear space and studied its property. Krishna and Sarma [6] introduced the notions of seperation axiom for fuzzy semi-norm and the quotient fuzzy semi-norms and investigated their properties.

Let X be a linear space and W a subspace of X. Let ρ be a semi-norm on X. Krishna and Sarma [6] studied the properties of the quotient fuzzy norm ρ_q, but they restricted the case that the dimension of W is either 1 or equal to that of X.

Our main goal of this paper is to prove some properties of quotient fuzzy norms of fuzzy semi-norms defined on original normed linear spaces because the theory of fuzzy norms is in close connection with the theory of original norms on linear spaces.

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II. Preliminaries

Throughout this paper X is a linear space over the field **K**(**R** or **C**). Fuzzy subsets of X are denoted by Greek letters in general. The set of all fuzzy subsets of X are denoted by I^X. By a fuzzy point μ we mean a fuzzy subset $\mu: X \to [0, 1]$ such that

$$\mu(z) = \begin{cases} \alpha, & \text{if} \quad z = x \\ 0, & \text{otherwise.} \end{cases}$$

We usually denote the fuzzy point with support x and value α by (x, α).

Definition 2.1[2]. For any $\mu, \nu \in I^X$, $\mu + \nu \in I^X$ is defined by $(\mu + \nu)(x) = \sup\{\mu(u) \wedge \nu(v),\ u + v = x\}$. And for a $\mu \in I^X$ and $t \in \mathbf{K}$, $t \neq 0$, $t\mu \in I^X$ is defined by $(t\mu)(x) = \mu(x/t)$,

$$(0 \cdot \mu)(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ \sup_{y \in X} \mu(y) & \text{if} \quad x = 0 \end{cases}.$$

Definition 2.2[2]. If X and Y are any two sets and $f: X \to Y$ is a function, then the fuzzification of f itself, is defined by

$$(a)\ f(\mu)(y)=\begin{cases}\sup_{x\in f^{-1}(y)} & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

for all $y \neq Y$ and for all fuzzy subset μ of X.

(b) $f^{-1}(\mu)(x)=\mu(f(x))$ for all $x\in X$, for all $\mu\in I^Y$.

Let $\mu_1,\ \mu_2\in I^X$. If $\mu_1(x)\leq\mu_2(x)$ for all $x\in X$, we write $\mu_1(x)\subset\mu_2$.

Definition 2.3[2]. $\mu\in I^X$ is said to be:

(a) convex if $t\mu+(1-t)\mu\subset\mu$ for each $t\in[0, 1]$.

(b) balanced if $t\mu\subset\mu$ for each $t\in\mathbf{K}$ with $|t|\leq1$.

(c) absorbing if $\sup_{t>0} t\mu(x)=1$ for all $x\in X$.

Definition 2.4[5]. Let (X, τ), (Y, ψ) be two fuzzy topological spaces. A function $f:(X, \tau)\rightarrow(Y, \psi)$ is said to be fuzzy continuous at the fuzzy point $v=(x, \alpha)$ if and only if for every neighbourhood μ of $f(v)(=(f(x), \alpha))$, $f^{-1}(\mu)$ is a neighbourhood of v. We say that f is continuous if and only if f is continuous at (x, α) for each $x\in\mathbf{X}$ and for each $\alpha\in(0, 1]$.

Definition 2.5[4]. A fuzzy semi-norm ρ on X is a fuzzy subset of X which satisfies the following three conditions.

(a) ρ is convex,

(b) ρ is balanced,

(c) ρ is absorbing,

If in addition a fuzzy semi-norm ρ satisfies the condition

(d) $\inf_{t>0} t\rho(x)=$ for $x\neq0$ in X,

ρ is called a fuzzy norm.

Definition 2.6[6]. Let (X, τ) be a topological space and

$$\omega(\tau)=\{f\mid f:(X, \tau)\rightarrow[0, 1]\text{ is lower semi-continuous}\}.$$

Then $\omega(\tau)$ is a fuzzy topology on X. This topplogy is called the fuzzy topology generated by τ on X.

Definition 2.7[3]. A fuzzy topology on a vector space X is said to be a fuzzy vector topology if vector addition, and scalar multiplication are continuous from $X\times X$ and $\mathbf{K}\times X$, respectively, to X with corresponding product fuzzy topologies and the fuzzy topology generated by the usual topology on $\mathbf{K}$. A vector space with a fuzzy vector topology is called a fuzzy topological vector space.

Theorem 2.8[4]. If ρ is a fuzzy semi-norm on X, then the family

$$\mathbf{B}=\mathbf{B}_\rho=\{\theta\wedge t\rho\mid 0<\theta\leq1,\ t>0\}$$

is a base at zero for a fuzzy vector topology τ_ρ.

Definition 2.9[5]. If ρ is a fuzzy semi-norm on X and $\varepsilon\in(0, 1)$, then the function

$$P_\varepsilon: X\rightarrow\mathbf{R}_+ \text{ defined by } P_\varepsilon=\inf\{t>0\mid t\rho(x)>\varepsilon\},$$

is a semi-norm on X.

Theorem 2.10[5]. A fuzzy semi-norm ρ on a vector space X is a fuzzy norm on X if and only if P_ε is a norm on X for each $\varepsilon\in(0, 1)$.

Ⅲ. Main results

In this section, we prove some properties of the fuzzy semi-norm induced by a fuzzy semi-norm defined on a normed linear space X and the quotient map $q: X\rightarrow X/W$, where W is a subspace of X.

Definition 3.1[6]. Let X be a linear space over the field $\mathbf{K}$($\mathbf{R}$ or $\mathbf{C}$) and W be a subspace of X. If ρ is a fuzzy semi-norm, the quotient fuzzy semi-norm ρ_q is defined by

$\rho_q(x+W)=\sup\{\rho(x+y)\mid y\in W\},\quad x+W\in X/W.$

Firstly we will characterize ρ_q as the image of ρ under q.

Theorem 3.2. The fuzzy semi-norm ρ_q described in Definition 3.1 is the image of ρ under q. That is $\rho_q(x+W)=q(\rho)(x+W)$ for all $x+W\in X/W$.

Proof. For every $x+W\in X/W$,

$$\begin{aligned}\rho_q &= \sup\{\rho(x+y)\mid y\in W\}\\ &=\sup\{\rho(z)\mid z-x\in W\}\\ &=\sup\{\rho(z)\mid z\in x+W\}\\ &=\sup\{\rho(z)\mid z+W=x+W\}\\ &=\sup\{\rho(z)\mid z\in q^{-1}(x+W)\}\\ &=q(\rho)(x+W)\end{aligned}$$

This completes the proof.

Secondly we are to prove that the quotient map is fuzzy continuous.

Theorem 3.3. Let X be a linear space over the field **K** (**R** or **C**) and W a subspace of X. If ρ is a fuzzy semi-norm on X, then the function $q:(X,\rho)\to(X/W,\rho_q)$ is fuzzy continuous at every fuzzy point (x,α) where $x\in X$, $\alpha\in(0,1]$.

In order to prove the theorem, we need the following lemma relative to the original norm X.

Lemma. Let $\varepsilon\in(0,1)$, then $P_\varepsilon(x)\ge P_\varepsilon^q(x+W)$ for all $x\in X$, where P_ε, P_ε^q are induced by ρ, ρ_q respectively as in Definition 2.9.

Proof. Since for $t>0$,

$t\rho(x)>\varepsilon$ implies $t\rho_q(x+W)=\sup\{t\rho(x+y)\mid y\in W\}\ge t\rho(x)>\varepsilon$,

we get

$$\{t>0\mid t\rho(x)>\varepsilon\}\subset\{t>0\mid t\rho_q(x+W)>\varepsilon\}.$$

Therefore $P_\varepsilon(x)=\inf\{t>0\mid t\rho(x)>\varepsilon\}\ge\inf\{t>0\mid t\rho_q(x+W)>\varepsilon\}=P_\varepsilon^q(x+W)$.

This completes the proof of lemma.

Proof of Theorem 3.3. For our goal, it is sufficient to prove that for each $t>0$, there exists $t'>0$ such that for any $y\in X$ and for each $\varepsilon<\alpha$, $P_\varepsilon(y-x)<t'$ implies $P_\varepsilon^q(q(y)-q(x))<t$ by Theorem 3.5 of [5]. Since $P_\varepsilon^q((x-y)+W)\le P_\varepsilon(x-y)$, by the above lemma, putting $t=t'$,

$P_\varepsilon(y-x)<t'$ implies $P_\varepsilon^q(q(y)-q(x))=P_\varepsilon^q((x-y)+W)\le P_\varepsilon(x-y)<t$.

This completes the proof of Theorem 3.3.

Theorem 3.4[1]. Let X be a normed linear space, let W be a closed subspace of X, and let $q:X\to X/W$ be the quotient map. The function $\|\cdot\|:X/W\to[0,\infty)$ defined by $\|x+W\|=inf\{\|x+y\|\mid y\in W\}$ has the following properties:

(a) $\|\cdot\|$ is a norm on X/W.

(b) $\|q(x)\|\le\|x\|$ for all $x\in X$, that is, q is continuous.

(c) If X is a Banach space, then so is X/W.

In Theorem 3.3, we proved a fuzzy version of the property (b). Now we will investigate some fuzzifications which are related to the remainings of the above theorem.

Theorem 3.5. Let X be a normed linear space and W a closed subspace of X. If ρ is a fuzzy semi-norm with bounded support, then the fuzzy semi-norm ρ_q on X/W is a fuzzy norm.

Proof. Let $x+W\in X/W$ and $x\notin W$, that is $x+W\cap W=\phi$. Since ρ has the bounded support, there exists an $M>1$ such that the support of ρ is contained in MB where B is the closed unit ball of X. Let $\alpha=\inf\{\|x+y\|\mid y\in W\}$. Then $\alpha>0$ because W is a closed subspace of X and $x\notin W$. If we set $2s=\frac{\alpha}{M}$, then supp$(s\rho)$

$\subset \frac{\alpha}{2M} MB = \frac{\alpha}{2} B$ and $\|x+y\| > \frac{\alpha}{2}$ for all $y \in W$. Whence $s\rho_q(x+W) = \sup\{s\rho(x+y) \mid y \in W\} = 0$ because $\|(x+y)/s\| = \frac{2M}{\alpha} \|x+y\| > \frac{2M}{\alpha}\frac{\alpha}{2} = M$ for all $y \in W$, and so $\inf_{t>0} t\rho_q(x+W) = 0$. Therefore ρ_q is a fuzzy norm on X/W. This completes the proof.

The above theorem shows that the condition of the boundedness of the support of a fuzzy semi-norm is essential ρ_q to be a fuzzy norm.

Theorem 3.6. Let X be a normed linear space and W a closed subspace. If ρ is a semi-norm with the bounded support on X, then ρ_q is a fuzzy norm on X/W with the bounded support.

Proof. Since ρ_q is a fuzzy norm by the preceeding theorem, we will prove that ρ_q has the bounded support. Let $x+W$ be an element of the suppoert of ρ_q. Since $\rho_q(x+W) > 0$ and $\rho_q(x+W) = \{\rho(z) \mid z \in x+W\}$, there exists z in X such that $\rho(z) > 0$ and $x+W = z+W$. Since ρ has the bounded support, there exists a positive real number M such that $\|x\| \leq W$ for all x in the support of ρ. For this M, $\|x+W\| = \|z+W\| \leq \|z\| < M$. Whence ρ_q has the bounded suport. This completes the proof.

Definition 3.7[8]. Let $\alpha \in (0, 1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a fuzzy α-Cauchy sequence in a fuzzy normed linear space (X, ρ) if for each zero neighbourhood N with $N(0) > \alpha$, there exists a positive integer M such that $n, m \geq M$ implies $\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \subset N$. A fuzzy normed linear space (X, ρ) is said to be fuzzy α-complete if every fuzzy α-Cauchy sequence $\{\mu_n\}$ converges to a fuzzy point $\mu = (x, \alpha)$. (X, ρ) is said to be fuzzy complete if it is fuzzy α-complete for every $\alpha \in (0. 1)$.

By Theorem 3.4, Definition 3.7 and Theorem 3.7 of [8], we get the following theorem.

Theorem 3.8. Let ρ be a semi-norm on a normed linear space X and W a closed subspace of X. If ρ has the bounded support and ρ_q is lower semi-continuous, then $(X/W, \rho_q)$ is a fuzzy complete fuzzy normed linear space.

Definition 3.9[6]. A fuzzy topological space (X, τ) is said to be separated (or T_2) if and only if for each pair (x, α), (y, β), $x \neq y$ of fuzzy points in X, there exist open fuzzy sets μ and ν in X such that $\mu(x) \geq \alpha$, $\nu(y) \geq \beta$ and $(\mu \wedge \nu)(z) = 0$ for every $z \in X$.

With Theorem 3.6 and Theorem 3.2 of [6], and Theorem 3.9 of [6], we obtain the following two corollaries.

Corollary 3.10. Let ρ be a semi-norm on a normed linear space X and W a closed subspace of X. If ρ has the bounded support, then ρ_q is a T_2-semi-norm on X/W.

Corollary 3.11. Under the same hypothesis in Corollary 3.10., the function $P^q{}_o : X/W \to [0, \infty)$ defined by $P^q{}_o(x+W) = \inf\{t > 0 \mid t\rho_q(x+W) > 0\}$ is a norm.

References

1. J. B. Conway, A course in functional analysis, Springer-Verlag INC, New York, 1985.
2. A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. 58 (1977), 135-146.
3. A. K. Katsaras, Fuzzy topological vector spaces I, Fuzzy Sets and Systems 6 (1981), 85-95.
4. A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets and Systems 12 (1984), 143-154.
5. S. V. Krishna and K. K. M. Sarma, Fuzzy continuity of linear maps on vector spaces, Fuzzy Sets and Systems 45 (1992), 341-354.
6. S. V. Krishna and K. K. M. Sarma, Seperation of fuzzy normed linear spaces, Fuzzy Sets and Systems 63 (1994), 207-217.
7. R. Lowen, Fuzzy topological spaces and fuzzy com-

pactness, J. Math. Anal. Appl. 56 (1976), 621-633.
8. Gil Seob Rhie and Dong Sik Kim, On the completeness and fuzzy completeness, Proceedings of KFMS Spring Conference'95, 5 (1995), 114-119.
9. Y. Terao, On a continuous base of fuzzy neighbourhoods of the orign on fuzzy locally convex spaces, Mathematica Japonica 39(1) (1994), 53-57.

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