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On the Partial Ordering of Hitting times of Bivariate Processes[1)]

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Abstract

In this papèr, a partial ordering of positive quadrant dependence (*PQD*) for bivariate stochastic processes are introduced and basic properties and closure under certain statistical operations are derived. Examples are given to illustrate these concepts

1. Introduction

Lehmann(1966) introduced the concepts of positive(negative) dependence together with some other dependent concepts. Since then, a great many works have been done on the subject and its extensions and numerous multivariate inequalities have been obtained.

In other words, a great many works have been devoted to various generalizations of Lehmann's concepts to finite-dimensional distributions. For a references of available results, see Karlin and Rinott(1980), Ebrahimi and Ghosh(1981) and Shaked(1982,b) and Sampson (1983). Whereas a number of dependence notions exist for multivariate processes (see Friday (1981)), recently, Ebrahimi(1987) and Baek(1996) introduced some new (weakly) positive quadrant dependence (*PQD*) concepts in terms of the finite-dimensional distributions of the hitting times of the components of a vector process. These concepts not only help us to understand structure of functionals such as hitting times of the given vector process but also have the potentional for new and useful inequalities for stochastic processes. Also, these concepts is a form of qualitative bivariate dependence which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis, sequential testing.

Since *PQD* is a qualitative form of dependence, it would seem difficult, or impossible to compare different pairs of stochastic processes as to their "degree of processes". But fortunately, in this paper we develop a partial ordering which permits us to compare pairs of

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PQD bivariate vector processes of interest as to their degree of *PQD*-ness.

In section 2, we develop some definitions and notations of the *PQD* ordering processes. Next in section 3, we derive useful closure properties of the *PQD* ordering. We show that *PQD* ordering is closed under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations of stochastic processes by increasing functions and convex combination.

Finally in section 4, we present several examples of *PQD* ordering.

2. Preliminaries

First, in this section, we present notations and basic facts used throughout the paper. In what follows 'increasing' means non-decreasing and 'decreasing' means non-increasing.
Suppose that we are given four stochastic processes $\{X_{11}(t) \mid t \geq 0\}$, $\{X_{12}(t) \mid t \geq 0\}$, $\{X_{21}(t) \mid t \geq 0\}$, and $\{X_{22}(t) \mid t \geq 0\}$. The state space of $(X_{11}(t), X_{21}(t))$ and $(X_{12}(t), X_{22}(t))$ will be taken to be a subset, $E = E_1 \times E_2$ of the plane R^2, respectively.
For any state $a_i \in E_i$, $i=1,2$, we define the random times as follows

$$T_{ij}(a_i) = \inf\{t \mid X_{ij}(t) \geq a_i,\ t \geq 0\},\quad j=1,2.$$

In other words, $T_{ij}(a_i)$ is the first time that the process $X_{ij}(t)$ reaches or goes above a_i (see Ebrahimi(1987)). The stochastic process $\{T_{ij}(a) \mid a \in E_i\}$ will be referred to as the hitting time process of the process $X_{ij}(t), i,j=1,2$. If we base the dependence between processes on the dependence of their hitting times, we then have the following definition.

Definition 2.1[Ebrahimi(1987)]. The stochastic process $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ is said to be positive quadrant dependent (PQD) if

$$P\left(\bigcap_{i=1}^{2}(T_{i2}(a_i) \leq t_i)\right) \geq \prod_{i=1}^{2} P(T_{i2}(a_i) \leq t_i) \quad \text{for all } t_i \geq 0,\ a_i \in E_i,\ i=1,2. \tag{2.1}$$

Similarly, we can define that $\{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$ is also PQD.

Let $\beta = \beta(F, G)$ denote the class of bivariate distribution function H having specified marginal distribution functions F and G, where F and G are nondegenerate, and we then consider β^+, a subclass of β, defined by
$\beta^+ = \{H(t_1, t_2) \mid H \text{ is } PQD,\ H(t_1, \infty) = F(t_1),\ H(\infty, t_2) = G(t_2)\}$. Let H_1, H_2 belong to β^+ and use the notation $\overline{H}_1(t_1, t_2) = P(T_{11}(a_1) > t_1, T_{21}(a_2) > t_2)$, $\overline{H}_2(t_1, t_2) = P(T_{12}(a_1) > t_1, T_{22}(a_2) > t_2)$.

Definition 2.2. The bivariate distribution H_2 is said to be more *PQD* than H_1 if

$$\overline{H}_2(t_1, t_2) \geq \overline{H}_1(t_1, t_2) \tag{2.2}$$

for all $t_i \geq 0$, $i=1,2$.

We write $H_2 \overset{PQD}{\succ} H_1$.

Remark 1. An equivalent form of (2.2) is $H_2(t_1, t_2) \geq H_1(t_1, t_2)$, $t_i \geq 0$, $i=1,2$.
Since H_1 and H_2 are *PQD*, it is clear that the distribution H_0 belonging to β^+ exhibiting the least degree of *PQD*-ness is given by $H_0(t_1, t_2) = F(t_1)G(t_2)$ for all $t_i \geq 0, i=1,2$.

Definition 2.3[Ebrahimi(1987)]. The stochastic process $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ is said to be associated if

$$Cov(f(T_{12}(a_1)), g(T_{22}(a_2))) \geq 0$$

for all increasing functions f and g for which the covariance exists and a_1 and a_2.

Definition 2.4[Ebrahimi(1982)]. The real valued functions $f, g : R^n \rightarrow R$ are said to concordant for the j^{th} coordinate if, with all other coordinates held fixed, f and g are either both increasing or both decreasing, $j=1, 2, \ldots, n$.

3. Ordered *PQD* stochastic processes.

In this section, we establish preservation of the *PQD* ordering under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations by increasing functions and convex combination. First note that $\overline{H_2}(t_1, t_2) \geq \overline{H_1}(t_1, t_2)$ if and only if $E_{H_2}(f(T_{12}(a_1))g(T_{22}(a_2))) \geq E_{H_1}(f(T_{11}(a_1))g(T_{21}(a_2)))$ for all increasing functions f and g.

(3.1)

In below, we show that the ordering is preserved under convolution. We need the following Lemma 3,1 which is of independent interest.

Lemma 3.1. Let (a) $\{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$ and $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ have distributions, where H_1, H_2 belong to β^+, respectively, (b) $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\} \overset{PQD}{\succ} \{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$, and (c) $\{(Z_1(t), Z_2(t)) \mid t \geq 0\}$ with an arbitrary *PQD* distribution function H independent of both of $\{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$ and $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$.
Then $(X_{12}(t)+Z_1(t), X_{22}(t)+Z_2(t)) \overset{PQD}{\succ} (X_{11}(t)+Z_1(t), X_{21}(t)+Z_2(t))$.

Proof. Observe that

$$
\begin{aligned}
&Cov\ [f(X_{12}(t)+Z_1(t)), g(X_{22}(t)+Z_2(t))] \\
&= Cov[E\{f(X_{12}(t)+Z_1(t)) \mid (Z_1(t), Z_2(t))\}, E\{g(X_{22}(t)+Z_2(t)) \mid (Z_1(t), Z_2(t))\}] \qquad (3.2) \\
&\qquad + E[\,Cov\{f(X_{12}(t)+Z_1(t)), g(X_{22}(t)+Z_2(t))\} \mid (Z_1(t), Z_2(t))].
\end{aligned}
$$

Note that the first and second terms on the right side of (3.2) are greater than or equal to zero for any increasing functions f and g. Since the stochastic process $(X_{12}(t), X_{22}(t))$ is PQD if and only if $Cov(f(T_{12}(a_1)), g(T_{22}(a_2))) \geq 0$ for all increasing functions f and g, $(X_{12}(t)+Z_1(t), X_{22}(t)+Z_2(t))$ is PQD, similarly we can show that $(X_{11}(t)+Z_1(t), X_{21}(t)+Z_2(t))$ is also PQD. For showing $(X_{12}(t)+Z_1(t), X_{22}(t)+Z_2(t)) \overset{PQD}{>} (X_{11}(t)+Z_1(t), X_{21}(t)+Z_2(t))$,

we have to show (3.1), i.e. $E(f(X_{12}(t)+Z_1(t))\, g(X_{22}(t)+Z_2(t))) \geq E(f(X_{11}(t)+Z_1(t)) g(X_{21}(t)+Z_2(t)))$ for any increasing functions f and g.

Now,

$$
\begin{aligned}
E(f(X_{12}(t)+Z_1(t)) g(X_{22}(t)+Z_2(t))) &= E(E(f(X_{12}(t)+Z_1(t)) g(X_{22}(t)+Z_2(t)) \mid (Z_1(t), Z_2(t))) \\
&= E(E(f(X_{12}(t)+Z_1(t)) g(X_{22}(t)+Z_2(t)))) \quad (by\ (c)\ ) \\
&\geq E(E(f(X_{11}(t)+Z_1(t)) g(X_{21}(t)+Z_2(t)))) \quad (by\ (b)\ ) \\
&= E(f(X_{11}(t)+Z_1(t)) g(X_{21}(t)+Z_2(t))).
\end{aligned}
$$

Theorem 3.2. Suppose that the stochastic process (a) $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ is more PQD than $\{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$,(b) $\{(Y_{12}(t), Y_{22}(t)) \mid t \geq 0\}$ is more PQD than $\{(Y_{11}(t), Y_{21}(t)) \mid t \geq 0\}$, and (c) $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ and $\{(Y_{12}(t), Y_{22}(t)) \mid t \geq 0\}$ be independent process, $\{(X_{11}(t), X_{21}(t)) \mid t \geq 0\}$ and $\{(Y_{11}(t), Y_{21}(t)) \mid t \geq 0\}$ be independent process. Then

$$
\{(X_{12}(t)+Y_{12}(t), X_{22}(t)+Y_{22}(t)) \mid t \geq 0\} \overset{PQD}{>} \{(X_{11}(t)+Y_{11}(t), X_{21}(t)+Y_{21}(t)) \mid t \geq 0\}.
$$

Proof. By assumption, $(X_{12}(t), X_{22}(t)) \overset{PQD}{>} (X_{11}(t), X_{21}(t))$.

Specifying $(Z_1(t), Z_2(t))$ to be $(Y_{12}(t), Y_{22}(t))$, we apply lemma 3.1 to obtain

$$
(X_{12}(t)+Y_{12}(t), X_{22}(t)+Y_{22}(t)) \overset{PQD}{>} (X_{11}(t)+Y_{12}(t), X_{21}(t)+Y_{22}(t)) \qquad (3.3)
$$

Next, we use the assumption $(Y_{12}(t), Y_{22}(t)) \overset{PQD}{>} (Y_{11}(t), Y_{21}(t))$,

specifying $(Z_1(t), Z_2(t))$ to be $(X_{11}(t), X_{21}(t))$, and again use lemma 3.1 yielding

$$(X_{11}(t)+Y_{12}(t),\ X_{21}(t)+Y_{22}(t)) \overset{PQD}{>} (X_{11}(t)+Y_{11}(t),\ X_{21}(t)+Y_{21}(t)). \tag{3.4}$$

By combining (3.3) and (3.4),

$$(X_{12}(t)+Y_{12}(t),\ X_{22}(t)+Y_{22}(t)) \overset{PQD}{>} (X_{11}(t)+Y_{12}(t),\ X_{21}(t)+Y_{22}(t))$$

$$\overset{PQD}{>} (X_{11}(t)+Y_{11}(t),\ X_{21}(t)+Y_{21}(t)).$$

Thus $(X_{12}(t)+Y_{12}(t), X_{22}(t)+Y_{22}(t)) \overset{PQD}{>} (X_{11}(t)+Y_{11}(t), X_{21}(t)+Y_{21}(t))$.

This completes the proof.

The next theorem demonstrates that, under suitable conditions, limits of more PQD processes inherit the more PQD.

Theorem 3.3. Let (a) $\{(X_{n1}(t),\ X_{n2}(t)) \mid t\geq 0\}$, $\{(Y_{n1}(t),\ Y_{n2}(t)) \mid t\geq 0\}$ have distribution H_n and H_n', respectively for every n, (b) $(X_{n1}(t),\ X_{n2}(t)) \overset{PQD}{>} (Y_{n1}(t),\ Y_{n2}(t))$ for every n, and (c) $(X_{n1}(t),\ X_{n2}(t)) \xrightarrow{w} (X_1(t),\ X_2(t))$ and $(Y_{n1}(t), Y_{n2}(t)) \xrightarrow{w} (Y_1(t), Y_2(t))$ as $n\to\infty$, respectively. Then $(X_1(t), X_2(t)) \overset{PQD}{>} (Y_1(t),\ Y_2(t))$.

Proof. The proof is straightforward and omitted.

The following theorem is another application of theorem 3.3 which is very important in recognizing more PQD in compound distributions which arise natrally in stochastic processes.

Theorem 3.4. Let (a) $(Y_1,\ S_1), (Y_2,\ S_2), \ldots$ are independent random variables, (b) $(X_1,\ K_1)$, $(X_2,\ K_2), \ldots$ are independent random variables, (c) (Y_i, S_i) and (X_i, K_i), $i=1,2,\ldots$ are PQD random variables, (d) $(Y_i,\ S_i) \overset{PQD}{>} (X_i,\ K_i)$, $i=1, 2, \ldots$, and (e) $N(t)$ be a Poisson process which is independent of $(Y_i,\ S_i)$ and $(X_i,\ K_i)$, $i=1, 2, \ldots$. Then

$$(Z_{12}(t)=\sum_{i=1}^{N(t)} Y_i,\ Z_{22}(t)=\sum_{i=1}^{N(t)} S_i) \overset{PQD}{>} (Z_{11}(t)=\sum_{i=1}^{N(t)} X_i,\ Z_{21}(t)=\sum_{i=1}^{N(t)} K_i).$$

Proof. Let $T_{ij}(a_i)$ be the hitting time of $Z_{ij}(t)$, $i, j=1, 2$. Then

$$P(T_{12}(a_1)\leq t_1,\ T_{22}(a_2)\leq t_2)$$

$$=P(\sum_{i=1}^{N(s)} Y_i > a_1,\ t_1\leq s<\infty,\ \sum_{i=1}^{N(s)} S_i > a_2,\ t_2\leq s<\infty)$$

$$= P(\sum_{i=1}^{N(t_1)} Y_i > a_1,\ \sum_{i=1}^{N(t_2)} S_i > a_2) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} P(N(t_1) = l_1,\ N(t_2) = l_2) P(\sum_{i=1}^{l_1} Y_i > a_1,\ \sum_{i=1}^{l_2} S_i > a_2)$$

$$\geq \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} P(N(t_1) = l_1,\ N(t_2) = l_2) P(\sum_{i=1}^{l_1} X_i > a_1,\ \sum_{i=1}^{l_2} K_i > a_2)$$

$$= P(\sum_{i=1}^{N(t_1)} X_i > a_1,\ \sum_{i=1}^{N(t_1)} K_i > a_2)$$

$$= P(T_{11}(a_1) \leq t_1,\ T_{21}(a_2) \leq t_2).$$

Our next result deals with the preservation of the *PQD* ordering under mixture. In order to motivate our definition of a subclass of β^+ in which the *PQD* ordering is preserved under mixture we need a definition 3.5 and a proposition 3.6.

Definition 3.5[Ebrahimi(1987)]. A stochastic process $\{X_{22}(t) \mid t \geq 0\}$ is stochastically increasing (SI) in $\{X_{12}(t) \mid t \geq 0\}$ if $E(f(T_{22}(a_2)) \mid T_{12}(a_1) = t_1)$ is increasing in t_1 for all $a_i \in E_i$, $i = 1, 2$, and real valued increasing function f.

Proposition 3.6[Ebrahimi(1978)]. Let (a) the stochastic process $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$, given a scalar λ, be conditionally *PQD*, and (b) $\{X_{i2}(t) \mid t \geq 0\}$ be *SI* in λ for each $i = 1, 2$, and (c) λ be associated. Then $\{(X_{12}(t), X_{22}(t)) \mid t \geq 0\}$ is *PQD*.

We may now define the class β_λ^+ by $\beta_\lambda^+ = \{H_\lambda \mid H(t_1, \infty \mid \lambda) = F(t_1 \mid \lambda),\ H(\infty, t_2 \mid \lambda) = G(t_2 \mid \lambda),$ $H_\lambda \mid \lambda$ *is PQD*, and both F and G are *SI* in $\lambda\}$.

Now consider $(\beta_\lambda^+, \overset{PQD}{\longrightarrow})$. The following theorem shows that if two elements of β_λ^+ are ordered according to $\overset{PQD}{\longrightarrow}$, then after mixing λ, the resulting element in β^+ preserves the same order.

Proposition 3.7. Let $(X_{12}(t), X_{22}(t)) \mid \lambda$ and $(X_{11}(t), X_{21}(t)) \mid \lambda$ belong to β_λ^+ and $(X_{12}(t), X_{22}(t)) \mid \lambda \overset{PQD}{\longrightarrow} (X_{11}(t), X_{21}(t)) \mid \lambda$ for all λ. Then, unconditionally, $(X_{12}(t), X_{22}(t))$, $(X_{11}(t), X_{21}(t))$ belong to β^+ and $(X_{12}(t), X_{22}(t)) \overset{PQD}{\longrightarrow} (X_{11}(t), X_{21}(t))$.

Proof. From the proposition 3.6, $(X_{12}(t), X_{22}(t))$ and $(X_{11}(t), X_{21}(t))$ are *PQD*.

For showing $(X_{12}(t), X_{22}(t)) \overset{PQD}{>} (X_{11}(t), X_{21}(t))$, we have to show (3.1), i.e.

$E(f(T_{12}(a_1))\, g(T_{22}(a_2))) \geq E(f(T_{11}(a_1))\, g(T_{21}(a_2)))$ for any increasing functions f and g.

Now,
$$\begin{aligned} E(f(T_{12}(a_1))\, g(T_{22}(a_2))) &= E_\lambda(E(f(T_{12}(a_1))\, g(T_{22}(a_2)) \mid \lambda)) \\ &\geq E_\lambda(E(f(T_{11}(a_1))\, g(T_{21}(a_2)) \mid \lambda)) \\ &= E(f(T_{11}(a_1))\, g(T_{21}(a_2))). \end{aligned}$$

The inequality comes from the fact that $(X_{12}(t), X_{22}(t)) \mid \lambda \overset{PQD}{>} (X_{11}(t), X_{21}(t)) \mid \lambda$ for all λ

Next, we show that PQD ordering is invariant under transformations of stochastic processes by increasing functions.

Result 3.8. Let $\{ (X_{ij}(t),\ Y_{ij}(t))^{H_j} \mid t \geq 0 \}, i = 1, 2, 3, \ldots, n$ be n-independent pairs from a bivariate distribution H_j, $j = 1, 2$. Suppose H_1 and H_2 belong to β^+ such that $H_2 \overset{PQD}{>} H_1$. Then for every pair (f, g) of concordant functions,

$$cov_{H_2}(f(T_{12}(a_1), T_{22}(a_2), \ldots, T_{n2}(a_n)), g(S_{12}(a_1), S_{22}(a_2), \ldots, S_{n2}(a_n)))$$
$$\geq cov_{H_1}(f(T_{11}(a_1), T_{21}(a_2), \ldots, T_{n1}(a_n)), g(S_{11}(a_1), S_{21}(a_2), \ldots, S_{n1}(a_n))).$$

Proof. Using the proposition 3.7, we can prove result 3.8 by argument similar to those used in the proof of theorem 1 by Lehmann(1966).

We now turn our attention to a simple but important property of the class β^+.

Result 3.9. The class $\beta^+ = \{H \mid H(t_1, t_2)$ is PQD, $H(t_1, \infty) = F(t_1), H(\infty, t_2) = G(t_2)\}$ is closed under convex combination.

Proof. We show that H is convex combination of H_1 and H_2.

Let $H_1, H_2 \in \beta^+$ and for $\alpha \in (0,1), H = \alpha H_1 + (1-\alpha) H_2$. Since each of the H_1 and $H_2 \in \beta^+$,

$$\begin{aligned} &P_H(T_{12}(a_1) \leq t_1,\ T_{22}(a_2) \leq t_2) \\ &\quad = \alpha P_{H_1}(T_{12}(a_1) \leq t_1, T_{22}(a_2) \leq t_2) + (1-\alpha) P_{H_2}(T_{12}(a_1) \leq t_1,\ T_{22}(a_2) \leq t_2) \\ &\quad \geq \alpha P_{H_1}(T_{12}(a_1) \leq t_1) P_{H_1}(T_{22}(a_2) \leq t_2) \\ &\qquad + (1-\alpha) P_{H_2}(T_{12}(a_1) \leq t_1) P_{H_2}(T_{22}(a_2) \leq t_2) \qquad (3.5) \\ &\quad = \alpha P_H(T_{12}(a_1) \leq t_1) P_H(T_{22}(a_2) \leq t_2) \\ &\qquad + (1-\alpha) P_H(T_{12}(a_1) \leq t_1) P_H(T_{22}(a_2) \leq t_2) \\ &\quad = P_H(T_{12}(a_{12}) \leq t_1) P_H(T_{22}(a_{22}) \leq t_2). \end{aligned}$$

Hence H is PQD

Moreover,

$$\lim_{t_2 \to \infty} H(t_1,\ t_2) = \alpha F(t_1) + (1-\alpha) F(t_1) = F(t_1), \tag{3.6}$$

and

$$\lim_{t_1 \to \infty} H(t_1,\ t_2) = \alpha G(t_2) + (1-\alpha) G(t_2) = G(t_2). \tag{3.7}$$

It follows from (3.5), (3.6), (3.7) that $H \in \beta^+$.Thus β^+ is closed under convex combination.

4. Examples

Example 4.1. Consider bivariate processes $\{(X_{n1}, Y_{n1}) \mid n \geq 1\}, \{(X_{n2}, Y_{n2}) \mid n \geq 1\}$ such that $(X_{11}, Y_{11}), (X_{21}, Y_{21}), \ldots$ are independent and $(X_{12}, Y_{12}), (X_{22}, Y_{22}), \ldots$ are independent process. Then we have the following that $(X_{n2}, Y_{n2})) \overset{PQD}{\longrightarrow} (X_{n1}, Y_{n1}), n \geq 1$ whenever $(X_{i2}, Y_{i2}) \overset{PQD}{\longrightarrow} (X_{i1}, Y_{i1})$, for each $i = 1, 2, ..$

Example 4.2. Consider a system with four components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and $\{(X_k, S_k) \mid k = 1, 2 \ldots\}$ and $\{(Y_k, L_k) \mid k = 1, 2, \ldots\}$ are sequence of damages to components 1,2, 3 and 4 by shock k, respectively. Define the compound Poisson processes by

$$Z_{11}(t) = \sum_{k=1}^{N(t)} Y_k, Z_{12}(t) = \sum_{k=1}^{N(t)} X_k, Z_{21}(t) = \sum_{k=1}^{N(t)} L_k, Z_{22}(t) = \sum_{k=1}^{N(t)} S_k.$$

This follows by application of theorem 3.4 implies $(Z_{12}(t), Z_{22}(t)) \overset{PQD}{\longrightarrow} (Z_{11}(t), Z_{21}(t))$ for every $t \geq 0$ whenever $(X_i, S_i) \overset{PQD}{\longrightarrow} (Y_i, L_i)$, for each $i = 1, 2, 3, \ldots$

Example 4.3. Consider the following stress-strength model for four systems. Let the $Z_{ij}(t)$, $i, j = 1, 2$, be the strength of systems 1, 2, 3, and 4 at time t. We will assume that the four systems receive shocks from a common source. Using a cumulative damage shock model (see Barlow and Proschan(1975)),we now let $N(t)$ be the number of shocks accuring by time t and U_i are i.i.d. positive random variables denoting the damage to either system due to the i^{th} shock $(i = 1, 2, \ldots)$. Hence, the stress experienced by either system at time t is given by the process $X_{ij}(t)=\sum_{k=1}^{M(t)} U_k$, $i,j=1,2$. Using the example 4.2, it is esy to check that $(X_{12}(t), X_{22}(t)) \overset{PQD}{>} (X_{11}(t), X_{21}(t))$. Assume that $(Z_{12}(t), Z_{22}(t)) \overset{PQD}{>} (Z_{11}(t), Z_{21}(t))$ such that $(X_{11}(t), X_{21}(t))$ and $(Z_{11}(t), Z_{21}(t))$, $(X_{12}(t), X_{22}(t))$ and $(Z_{12}(t), Z_{22}(t))$ are independent processes with decreasing sample paths, respectively. Then we obtain using theorem 3.2 that

$$(X_{12}(t)-Z_{12}(t), X_{22}(t)-Z_{22}(t)) \overset{PQD}{>} (X_{11}(t)-Z_{11}(t), X_{21}(t)-Z_{21}(t)).$$

Consequently, the life times of the four systems, namely

$$(T_{12}(0), T_{22}(0)) = (\inf\{t \mid (X_{12}(t)-Z_{12}(t))\geq 0\}, \inf\{t \mid (X_{22}(t)-Z_{22}(t))\geq 0\})$$
$$\overset{PQD}{>} (T_{11}(0), T_{21}(0)) = (\inf\{t \mid (X_{11}(t)-Z_{11}(t))\geq 0\}, \inf\{t \mid (X_{21}(t)-Z_{21}(t))\geq 0\}).$$

Useful bounds on the joint survival of the four dependent systems are therefore given by

$$P(T_{12}(0)>t_1, T_{22}(0)>t_2) \geq P(T_{11}(0)>t_1, T_{21}(0)>t_2), \quad t_i\geq 0, \ i=1,2.$$

Example 4.4. Let $Z_j=(Z_{1j}, Z_{2j})$ and $W_j=(W_{1j}, W_{2j}), j\geq 0$ be a sequence of i.i.d. bivariate vectors such that $(Z_{10}, Z_{20}) \overset{PQD}{>} (W_{10}, W_{20})$ random variables with marginal uniform distribution on the interval [0, 1], respectively. Consider sequences $(X_{1n}(t), X_{2n}(t))$ and $(Y_{1n}(t), Y_{2n}(t)$ (in $n\geq 1$) of bivariate processes defined by

$$X_n(t) = (X_{1n}(t), X_{2n}(t)) = (\sqrt{n}(F_{1n}(t)-t), \sqrt{n}(F_{2n}(t)-t)),$$
$$Y_n(t) = (Y_{1n}(t), Y_{2n}(t)) = (\sqrt{n}(G_{1n}(t)-t), \sqrt{n}(G_{2n}(t)-t)), \quad t\in[0,1],$$

where for $i=1,2$, $F_{in}(t)=n^{-1}\sum_{j=1}^{n} I(Z_{ij}\leq t), G_{in}(t)=n^{-1}\sum_{j=1}^{n} I(W_{ij}\leq t)$ are usual empirical c.d.f. of the i.i.d random variables $Z_{i1}, Z_{i2}, \ldots, Z_{in}$ and $W_{i1}, W_{i2}, \ldots, W_{in}$, respectively. Note that $X_n(t)$ and $Y_n(t)$ are simply the combination of the two(dependent)one-dimensional empirical processes, respectively. Such processes have been used by Goel and Ramallingam(1987) to study matching problems. Fix $i=1,2$, then for all real a_i, it is easy to verify that the hitting times $T_i(a_i)=\inf\{t \mid X_{in}(t)\geq a_i\}$ and $S_i(a_i)=\inf\{t \mid Y_{in}(t)\geq a_i\}$ are increasing functions of $Z_{i1}, \ldots, Z_{in}, W_{i1}, \ldots, W_{in}$, respectively. In view of this fact, if we fixed $n\geq 1$, then we can argue (see Tong(1980), p. 84) that for all $a_i, i=1,2$, $(T_1(a_1), T_2(a_2)) \overset{PQD}{\longrightarrow} (S_1(a_1), S_2(a_2))$ random variables. We conclude that $(X_{1n}(t), X_{2n}(t)) \overset{PQD}{\longrightarrow} (Y_{1n}(t), Y_{2n}(t))$, for each $n \geq 1$. It is easy to check that $(X_{1n}(t), X_{2n}(t))$ converges weakly to $(X_1(t), X_2(t))$ and $(Y_{1n}(t), Y_{2n}(t))$ converges weakly to $(Y_1(t), Y_2(t))$ as $n \to \infty$ on the time interval [0,1]. Hence, using the theorem 3.3, we can obtain that $(X_1(t), X_2(t)) \overset{PQD}{\longrightarrow} (Y_1(t), Y_2(t))$.

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References

[1] Ahmed, A., Langberg, N., Leon, R. and Proschan, F. (1978). Two concepts of positive dependence, with applications in multivariate analysis, *Technical Report AFOSR* 78-6, Department of Statistics, Florida State University.

[2] Alzaid, A. A. (1990). A weak quadrant dependence concept with applications, *Commu. Stat. Stoch. Models*, 6(2), 353-363.

[3] Baek, J. I. (1996). A weakly dependence concept of bivariate stochastic processes, *Comm. of Korea, Math, Soci.*, 11, No. 3, 831-839.

[4] Barlow, R. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models,* Holt, Rinehart and Winston, Inc., New York.

[5] Bergmann, R. (1978). Some classes of semi-ordering relations for random vectors and their use for comparing covariances, *Math. Nochr,* 82, 103-114.

[6] Ebrahimi, N. (1982). The ordering of negative quadrant dependence, *Comm. Stat. Theor. Meth*, 11(21), 2389-2399.

[7] Ebrahimi, N. (1987). Bivariate processes with positive or negative dependent structures, *J. Appl. Prob.*, 24, 115-122.

[8] Ebrahimi, N. and Ramallingam (1989). On the dependence structure of hitting times of multivariate Processes, *J. Appl. Prob.,* 26, 287-295.

[9] Ebrahimi, N. and Ghosh, M. (1981). Multivariate negative dependence, *Commun. Statist., A*10, 307-337.

[10] Friday, D. S. (1981). Dependence concepts for stochastic processes, *Proc. Nato Advanced Study Instiutes Series* 5, 349-361.

[11] Goel and Ramallingam (1987). Some properties of the maximum likelihood strategy for repairing a broken random sample, *J. Statist. Planning Inf.*, 16, 237-248.

[12] Karlin, S. and Rinott, Y. (1980). Classes of orderings of measures and related correlation inequalities, *J. Multivariate Anal.*, 10, 467-498.

[13] Lehmann, E. (1966). Some concepts of dependence, *Ann. Math. Statist.*, 37, 1137-1153.

[14] Ross, S. M. (1983). *Stochastic processes*, Wiley, New York.

[15] Sampson, A. R.(1983). Positive dependence properties of elliptically symmetric distribution, *J. of Multivariate Analysis*, 13, 375-381.

[16] Shaked, M. (1982b). Dispersive ordering of distributions, *J. of Applied Proba.*, 19, 310-320.

[17] Tong, Y. L. (1980). *Probability inequalities in multivariate distributions*, Academic press, New York.