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On Flexible Bayesian Test Criteria for Nested Point Null Hypotheses of Multiple Regression Coefficients

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Abstract

As flexible Bayesian test criteria for nested point null hypotheses of multiple regression coefficients, partial and overall Bayes factors are introduced under a class of intuitively meaningful prior. The criteria lead to a simple method for considering different prior beliefs on the subspaces that constitute a partition of the coefficient parameter space. A couple of tests are suggested based on the criteria. It is shown that they enable us to obtain pairwise comparisons of hypotheses of the partitioned subspaces. Through a Monte Carlo simulation, performance of the tests based on the criteria are compared with the usual Bayesian test (based on Bayes factor) in terms of their respective powers.

1. Introduction

Let $\boldsymbol{y} = (y_1, \cdots, y_n)'$ be $n \times 1$ vector representing n independent observations on a dependent variable. Assume the normal linear regression model

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{u}, \tag{1}$$

where X is an $n \times k$ matrix, with rank k, of observations on k independent variables; $\boldsymbol{\beta}$ is a $k \times 1$ vector of regression coefficients ; and $\boldsymbol{u}$ is an $n \times 1$ error or disturbance vector.

We assume that the elements of $\boldsymbol{u}$ are normally and independently distributed, each with mean zero and common variance σ^2 ; that is $\mathrm{E}\,\boldsymbol{u} = 0$ and $\mathrm{E}\,\boldsymbol{u}\,\boldsymbol{u}' = \sigma^2\,\mathrm{I}_n$, where I_n is an $n \times n$ identity matrix. With respect to the matrix X, if (1) is assumed to have a nonzero intercept, all elements in the first column of X will be ones. Under the above assumptions, the likelihood function for the elements of $\boldsymbol{\beta}$ and σ , given $\boldsymbol{y}$ and X is

$$\ell(\boldsymbol{\beta}, \sigma \mid \widehat{\boldsymbol{\beta}}, s^2) \propto \frac{1}{\sigma^n} \exp\left[-\frac{1}{2\sigma^2}\{\nu s^2 + (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})'X'X(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\}\right],$$

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where $v = n - k$, $\hat{\boldsymbol\beta} = (X'X)^{-1}X'\boldsymbol y$ and $s^2 = \dfrac{(\boldsymbol y - X\hat{\boldsymbol\beta})'(\boldsymbol y - X\hat{\boldsymbol\beta})}{v}$ are sufficient statistics.

For Bayesian inference, Bayes factor is usually used as measure of evidence in favor of H_0 versus H_A, which is complement of H_0(cf. Berger, 1985). Suppose we are interested in testing a linear compound of $\boldsymbol\beta$, i.e. H_0: $\boldsymbol a'\boldsymbol\beta = \omega_0$ versus H_A: $\boldsymbol a'\boldsymbol\beta \neq \omega_0$, where $\boldsymbol a$ is a $k \times 1$ design vector and ω_0 is a given constant. Further suppose that $\pi \in (0, 1)$ is the prior probability of H_0: $\boldsymbol a'\boldsymbol\beta = \omega_0$ and let P_0 and P be the prior densities of (ω, σ) conditional on H_0 and H_A, where $\omega = \boldsymbol a'\boldsymbol\beta$. Then the overall prior distribution function of (ω, σ) is the mixture

$$F(\omega, \sigma) = \pi\ \mathrm{I}_{[\omega_0, \infty)}(\omega)\int_0^\sigma P_0(t_2)\,dt_2 + (1-\pi)\int_{-\infty}^{\omega}\int_0^\sigma P(t_1, t_2)\,dt_2\,dt_1$$

and the Bayes factor(cf. Jeffreys, 1961) B in favor of H_0 is the weighted (by P_0 and P_1) likelihood ratio of H_0 against H_A:

$$B = \frac{\int_0^\infty \ell(\omega_0, \sigma)P_0(\sigma)\,d\sigma}{\int_0^\infty\int_{-\infty}^\infty \ell(\omega, \sigma)P(\omega, \sigma)\,d\omega\,d\sigma} \tag{2}$$

Here $I(\cdot)$ denotes the indicator function.

It can be verified from the sufficiency principal of likelihood(cf. Lee, 1988) that the likelihood of (ω, σ) in (2) is

$$\ell(\omega, \sigma) = \ell(\boldsymbol a'\boldsymbol\beta, \sigma \mid \boldsymbol a'\hat{\boldsymbol\beta}, s) \propto \frac{1}{\sigma^{v+1}}\exp\left[-\frac{1}{2\sigma^2}\left\{vs^2 + \frac{(\boldsymbol a'\hat{\boldsymbol\beta} - \boldsymbol a'\boldsymbol\beta)^2}{\boldsymbol a'(X'X)^{-1}\boldsymbol a}\right\}\right] \tag{3}$$

for $\boldsymbol a'\hat{\boldsymbol\beta} \mid \sigma, \boldsymbol\beta \sim N(\boldsymbol a'\boldsymbol\beta, [\boldsymbol a'(X'X)^{-1}\boldsymbol a]\sigma^2)$ and $\frac{vs^2}{\sigma^2} \mid \sigma \sim \chi^2_{(v)}$ (cf. Sen and Srivastava, 1990).

In constructing the Bayes factor for the point null hypothesis, it has been usually sensible to take the prior of ω, conditional on H_A, to be symmetrical with respect to ω_0 so as to have an equal weight function(cf. Lee, 1988). In an effective Bayesian analysis concerning the real parameter $\omega \in \Omega$, however, it often occurs that prior attitudes with respect to $\omega < \omega_0$ and $\omega > \omega_0$ are different (for example, it is unusual that the effect of an economic policy will be given a priori an equal chance of being positive or negative). When the prior attitudes are different, to test the point null hypothesis effectively, we need a Bayesian test criterion that does not require any constraints of the symmetry. For one parameter case, Zellner(1987) and Bertolino, Piccinato, and Racugno(1995) showed that partial Bayes factors, based on a partition of the parameter space, provide a solution for the problem.

In this paper, the partial Bayes factors are modified to the case of point null hypothesis with nuisance parameters (nested point null hypothesis). Then we suggest a couple of flexible

Bayesian test criteria based on the modified partial Bayes factors that deals simultaneously, and without any constraints of symmetry, with any number of hypotheses of the multiple regression coefficients.

2. Flexible Bayesian Tests

2.1 Partial and Overall Bayes Factors

For the regression model (1), it is desired to test the point null hypothesis $H_0: \omega = \omega_0$ versus $H_0: \omega \neq \omega_0$, where $\omega = \boldsymbol{a}'\boldsymbol{\beta}$. Introducing the partition $(\Omega_0, \Omega_1, \Omega_2)$, where $\Omega_0 = \{\omega_0\}$, $\Omega_1 = \{\omega : \omega < \omega_0\}$ and $\Omega_2 = \{\omega : \omega_0 < \omega\}$, makes it sensible and easier to perform a Bayesian analysis, especially the subsets Ω_1 and Ω_2 play a priori different roles. Let us assign the probabilities $\pi_j\,(\sum \pi_j = 1)$ to the events $\omega \in \Omega_j\,(i = 0, 1, 2)$, and let P_0, P_1 and P_2 be the prior densities of (ω, σ) conditional on $(\omega, \sigma) \in \Omega_0 \times \Psi$, $(\omega, \sigma) \in \Omega_1 \times \Psi$ and $(\omega, \sigma) \in \Omega_2 \times \Psi$, where $\Psi = \{\sigma : \sigma > 0\}$.

Then the overall prior distribution function of (ω, σ) is

$$F(\omega, \sigma) = \pi_o\, I_{[\omega_0, \infty)}(\omega) \int_0^{\sigma} P_0(t_2)\,dt_2 + \pi_1 \int_{-\infty}^{\omega} \int_0^{\sigma} g_1(t_1 \mid t_2) P(t_2)\,dt_2\,dt_1 + \pi_2 \int_{-\infty}^{\omega} \int_0^{\sigma} g_2(t_1 \mid t_2) P(t_2)\,dt_2\,dt_1, \tag{4}$$

where $g_j(\omega \mid \sigma) P(\sigma) = P_j(\omega, \sigma)$, $j = 1, 2$, is joint prior of $(\omega, \sigma) \in \Omega_j \times \Psi$ and $P_0(\sigma)$ is the prior for σ under H_0. Different prior knowledge about ω in Ω_1 and Ω_2 can be represented by a suitable choice of the conditional densities (or weight functions) g_1 and g_2.

Definition 1. Under the partition and the overall prior distribution given above, the Bayes factors of Ω_0 versus Ω_j conditional on $(\omega, \sigma) \in (\Omega_0 \cup \Omega_1) \times \Psi$, $j = 1, 2$, are said to be partial Bayes factors of Ω_0 versus Ω_j, $j = 1, 2$.

Let B_j denote the partial Bayes factors of $\omega \in \Omega_0$ versus Ω_j, $j = 1, 2$, then Definition 1 leads to following lemma.

Lemma 1. Given the partition $\Omega = (\Omega_0, \Omega_1, \Omega_2)$, respective the partial Bayes factors of $\omega \in \Omega_0$ versus $\omega \in \Omega_1$ and $\omega \in \Omega_2$ are

$$B_1 = \frac{\int_0^{\infty} \ell(\omega_0, \sigma) P_0(\sigma)\,d\sigma}{m(g_1)} \quad \text{and} \quad B_2 = \frac{\int_0^{\infty} \ell(\omega_0, \sigma) P_0(\sigma)\,d\sigma}{m(g_2)}, \tag{5}$$

where $m(g_j) = \int_{\Omega_j}\int_0^\infty \ell(\omega, \sigma) g_j(\omega \mid \sigma) P(\sigma) d\sigma\, d\omega$, $j = 1, 2$ and $\ell(\omega, \sigma)$ is the likelihood function.

Proof. For the partition (Ω_0, Ω_1), the overall prior distribution (4) gives the joint prior density

$$f(\omega, \sigma) = \begin{cases} \pi_0 P_0(\sigma) & \text{for } (\omega, \sigma) \in \Omega_0 \times \Psi \\ \pi_1 g_1(\omega \mid \sigma) P(\sigma) & \text{for } (\omega, \sigma) \in \Omega_1 \times \Psi \end{cases}$$

Since the ratio of posterior odds to the prior odds is the Bayes factor, the Bayes factor for testing Ω_0 versus Ω_1 conditional on $(\omega, \sigma) \in (\Omega_0 \cup \Omega_1) \times \Psi$ is given by

$$\frac{\pi_1}{\pi_0} \times \frac{\pi_0 \int_0^\infty f_1(a'\hat{\beta} \mid \sigma, \omega = 0)\, f_2(s^2 \mid \sigma)\, P_0(\sigma) d\sigma}{\pi_1 \int_{\Omega_1}\int_0^\infty f_1(a'\hat{\beta} \mid \sigma, \omega)\, f_2(s^2 \mid \sigma)\, g_1(\omega \mid \sigma) P(\sigma) d\sigma d\omega}$$
$$= \frac{\int_0^\infty \ell(\omega_0, \sigma) p_0(\sigma) d\sigma}{m(g_1)},$$

because $f_1(a'\hat{\beta} \mid \sigma, \omega) f_2(s \mid \sigma) \propto \ell(\omega, \sigma)$ where f_1 and f_2 are respective probability densities of $a'\hat{\beta}$ and s. Similar proof holds for B_2.

It can be seen that the partial Bayes factors B_1 and B_2 do not depend on π_0, π_1 and π_2 but rather depend on the given partition $\Omega = (\Omega_0, \Omega_1, \Omega_2)$. The overall Bayes factor can be found from (5), provided π_1 and π_2 are specified. The overall Bayes factor(OB) of $H_0 : \omega \in \Omega_0\ (\omega = \omega_0)$ versus $H_A : \omega \notin \Omega_0\ (\omega \neq \omega_0)$ is a weighted harmonic mean of B_1 and B_2 :

$$OB = \frac{\int_0^\infty \ell(\omega_0, \sigma) P_0(\sigma)\, d\sigma}{\int_\Omega \int_0^\infty \ell(\omega, \sigma)\, \tilde{g}(\omega \mid \sigma) P(\sigma) d\sigma d\omega} = (1 + q) \Big/ \left(\frac{1}{B_1} + \frac{q}{B_2}\right) \tag{6}$$

where $q = \pi_2 / \pi_1$ and $\tilde{g}(\omega \mid \sigma) = \frac{\pi_1}{\pi_1 + \pi_2} g_1(\omega \mid \sigma) + \frac{\pi_1}{\pi_1 + \pi_2} g_2(\omega \mid \sigma)$ is the prior density of $\omega \mid \sigma$ conditional on H_A.

The extensions will lead us to get partial Bayes factors B_i, $i = 1, 2, \cdots, k$, and OB for the case where the prior attitudes about the parameter space Ω are more diverse. This will be obtained from the partition $(\Omega_0, \Omega_1, \cdots, \Omega_k)$ of the parameter space Ω, corresponding to the diverse prior attitudes.

2.2 Test Criteria

As prior densities for ω, it will be sensible to consider the class of the densities, g_1 and g_2, with monotone branches before and after ω_0 (nondecreasing for $\omega < \omega_0$ and nonincreasing for $\omega > \omega_0$). To deal with the class, we consider the prior densities uniform on intervals (τ_1, ω_0) and (ω_0, τ_2) so that

$$g_j(\omega \mid \sigma) = 1 / \mid \tau_j - \omega_0 \mid, \text{ for } \omega \in \Omega_j, \ j = 1, 2. \tag{7}$$

Regarding to the prior assumption about σ, we assume that our information is diffuse or vague and represent it by taking $\log \sigma$ uniformly distributed. Thus, the priors in (4) become

$$P_j(\omega, \sigma) = g_j(\omega \mid \sigma) P(\sigma) \propto \frac{1}{\sigma \mid \tau_j - \omega_0 \mid}, \text{ for } \sigma > 0, \ \omega \in \Omega_j : j = 1, 2,$$

$$\text{and } P_0(\omega, \sigma) = P_0(\sigma) \text{ for } \omega \in \Omega_0, \text{ where } P_0(\sigma) \propto \frac{1}{\sigma}. \tag{8}$$

This class of prior is relevant for our approach and implies some modifications to standard class in order to avoid any symmetry around ω_0. Different choices for the prior densities $P_j(\omega, \sigma)$ are available from the literature on testing point null hypotheses (see, for example, Berger and Sellke 1987, and Berger 1990).

Theorem 1. Given the conditional priors (8) and the partition $\Omega = \{\Omega_0, \Omega_1, \Omega_2\}$, the partial Bayes factors of $\omega \in \Omega_0$ versus $\omega \in \Omega_1$ and $\omega \in \Omega_2$ are given by

$$B_1 = \frac{K \mid \tau_1 - \omega_0 \mid}{T_v\{Q(\omega_0)\} - T_v\{Q(\tau_1)\}}$$

and

$$B_2 = \frac{K \mid \tau_2 - \omega_0 \mid}{T_v\{Q(\tau_2)\} - T_v\{Q(\omega_0)\}}, \tag{9}$$

where $Q(a) = \dfrac{a - \boldsymbol{a}'\hat{\boldsymbol{\beta}}}{s\sqrt{\boldsymbol{a}'(X'X)^{-1}\boldsymbol{a}}}$, $s^2 = \dfrac{(\boldsymbol{y} - X\hat{\boldsymbol{\beta}})'(\boldsymbol{y} - X\hat{\boldsymbol{\beta}})}{v}$, $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y}$,

$$K = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi}\,(s\sqrt{\boldsymbol{a}'(X'X)^{-1}\boldsymbol{a}})} v^{\frac{v+1}{2}} \{v + Q^2(\omega_0)\}, \tag{10}$$

and $T_v\{\cdot\}$ denotes the cdf of Student t distribution with degrees of freedom $v = n - k$.

Proof. Upon combining (3) and (8), we see that

$$\ell(\omega, \sigma) g_1(\omega \mid \sigma) P(\sigma) \propto \frac{I_{\Omega_1^*}(\omega)}{\mid \tau_1 - \omega_0 \mid \sigma^{v+2}} \exp\left[-\frac{s^2}{2\sigma^2}\{v + Q^2(\omega)\}\right], \tag{11}$$

and

$$\ell(\omega_0, \sigma) P_0(\sigma) \propto \frac{1}{\sigma^{v+2}} \exp\left\{-\frac{s^2}{2\sigma^2}[v + Q^2(\omega_0)]\right\}, \text{where } \Omega_1^* = \{\omega : \tau_1 < \omega < \omega_0\}.$$

Integrating (11) with respect to σ gives

$$\ell^*(\omega) \propto \frac{I_{\Omega_1^*}(\omega)}{|\tau_1 - \omega_0|}[v + Q^2(\omega)]^{-\frac{v+1}{2}}.$$

Therefore, the partial Bayes factor B_1 of $\omega \in \Omega_0$ versus $\omega \in \Omega_1$, defined in Lemma 1, yields

$$B_1 = \frac{\int_0^\infty \ell(\omega_0, \sigma) P_0(\sigma) d\sigma}{\int_{\Omega_1} \ell^*(\omega) d\omega} = \frac{|\tau_1 - \omega_0| \{v + Q^2(\omega_0)\}^{-\frac{v+1}{2}}}{\int_{\tau_1}^{\omega_0} \{v + Q^2(\omega)\}^{-\frac{v+1}{2}} d\omega} \tag{12}$$

The integrand in the denominator of (12) is the kernel of a generalized t-distribution with v degrees of freedom. Completing the density and transforming ω to the standardized t-distribution by the transformation relation $T = Q(w)$, we have the result. The same proof applies to the derivation of B_2.

Corollary 1. Provided that prior probabilities $\pi_1 = \Pr(\omega \in \Omega_1)$ and $\pi_2 = \Pr(\omega \in \Omega_2)$ are specified, the overall Bayes factor for testing $H_0: \omega = \omega_0$ versus $H_A: \omega \neq \omega_0$ is given by

$$OB = \frac{(1+q)K|\tau_1 - \omega_0| \ |\tau_2 - \omega_0|}{|\tau_2 - \omega_0| \{T_v(Q(\omega_0)) - T_v(Q(\tau_1))\} + q|\tau_1 - \omega_0| \{T_v(Q(\tau_2)) - T_v(Q(\omega_0))\}},$$

where $q = \pi_2 / \pi_1$.

Proof. Definition of OB in (6) and Theorem 1 directly give the result.

Note that, if $g_1(\omega | \sigma) = g_2(\omega | \sigma)$, the overall Bayes factor in (6) boils down to the usual Bayes factor in (2). For the purpose of comparison between Ω_1 and Ω_2, we can obtain the partial Bayes factor of Ω_1 versus Ω_2 simply by B_2 / B_1. If $B_2 / B_1 > 1$, our decision may be in favor of Ω_1.

Task of deciding between H_0 and H_A via suggested criteria (the overall Bayes factor and the partial Bayes factors) is conceptually straightforward. By means of the partial Bayes factors and the overall Bayes factor, we construct following tests for multiple decision between $H_0: \omega = \Omega_0$, $H_1: \omega \in \Omega_1$ and $H_2: \omega \in \Omega_2$, where $H_A = H_1 \cup H_2$.

Test I. (Test by the partial Bayes factors) ;

$$\begin{cases} \text{Accept } H_0 & \text{if } (B_1, B_2) \in A, \text{ where } A = \{(B_1, B_2): B_1 \geq 1, B_2 \geq 1\}, \\ \text{Accept } H_1 & \text{if } (B_1, B_2) \notin A \text{ and } B_2 / B_1 \geq 1, \\ \text{Accept } H_2 & \text{if } (B_1, B_2) \notin A \text{ and } B_1 / B_2 \geq 1. \end{cases} \tag{13}$$

Test II. (Test using the overall Bayes factor) ;

$$\begin{cases} \text{Accept } H_0 & \text{if } \text{OB} \geq 1, \\ \text{Accept } H_1 & \text{if } \text{OB} < 1 \text{ and } B_2 / B_1 \geq 1, \\ \text{Accept } H_2 & \text{if } \text{OB} < 1 \text{ and } B_1 / B_2 \geq 1. \end{cases} \tag{14}$$

It is easily seen that the two tests suggested above yield more flexible tests than the usual test by Bayes factor (2) in that they account not only for the different prior attitudes for the partitioned parameter spaces but also for multiple decision of H_0, H_1, and H_2.

3. Simulation Study

Performance of the two tests is examined through a simulation study. The suggested tests and the usual test by Bayes factor (2) are compared in terms of their respective powers. The regression model considered in this study is

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon \ ,$$

where $\beta_1 = 1 + \delta_1$, $\beta_2 = 3 + \delta_2$ and $\beta_3 = 5 + \delta_2$. In each run of the simulation, data set $\{y_i, x_{1i}, x_{2i}, x_{3i} \ ; i = 1, 2, \cdots, n\}$ of size n is generated. The data for the independent variables (x_1, x_2 and x_3) consist of pseudo-random numbers between 0 and 1, and those for dependent variable y is $\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$ plus $N(0, 1)$ pseudo-random number.

Based on the data, hypothesis $H_0: \omega = \omega_0$ is tested against $H_A: \omega \neq \omega_0$ ($H_A = H_1 \cup H_2$; $H_1: \omega > \omega_0$, $H_2: \omega < \omega_0$). In constructing the tests we use two cases of priors in (7):

Case 1. $g_1(\omega \mid \sigma) = \dfrac{1}{\sqrt{d+5}}$ for $\omega > \omega_0$ and $g_2(\omega \mid \sigma) = \dfrac{1}{\sqrt{d}}$ for $\omega < \omega_0$,

Case 2. $g_1(\omega \mid \sigma) = \dfrac{1}{\sqrt{d}}$ for $\omega > \omega_0$ and $g_2(\omega \mid \sigma) = \dfrac{1}{\sqrt{d+5}}$ for $\omega < \omega_0$.

Table 1. Powers of Test Ⅰ, Test Ⅱ(with $q=1$) and BFT for testing $H_0: \omega = 9$, where $\omega = \beta_1 + \beta_2 + \beta_3$.

(δ_1, δ_2)	d	Case 1			Case 2		
		Test Ⅰ	Test Ⅱ	BFT	Test Ⅰ	Test Ⅱ	BFT
(-0.3, -0.6)	10	1.00	1.00	1.00	1.00	1.00	1.00
	15	1.00	1.00	1.00	1.00	1.00	1.00
	20	1.00	1.00	1.00	1.00	1.00	1.00
(-0.2, -0.5)	10	.980	.970	.970	.995	.970	.970
	15	.980	.970	.970	.990	.970	.970
	20	.975	.965	.970	.990	.970	.960
(-0.15, -0.3)	10	.885	.825	.835	.910	.840	.815
	15	.870	.815	.825	.900	.815	.785
	20	.865	.795	.815	.885	.785	.780
(-0.1, -0.2)	10	.560	.425	.455	.650	.545	.510
	15	.540	.380	.425	.620	.510	.475
	20	.515	.350	.380	.605	.475	.450
(-0.05, -0.1)	10	.185	.135	.150	.225	.155	.140
	15	.175	.120	.135	.210	.140	.120
	20	.165	.105	.120	.185	.120	.100
(0, 0)	10	.940	.965	.975	.945	.970	.970
	15	.950	.975	.985	.955	.975	.970
	20	.950	.985	.985	.955	.980	.980
(0.05, 0.1)	10	.235	.135	.105	.205	.120	.135
	15	.200	.105	.095	.175	.120	.120
	20	.170	.095	.085	.165	.105	.120
(0.1, 0.6)	10	.660	.550	.500	.605	.425	.480
	15	.645	.500	.440	.540	.395	.425
	20	.580	.440	.420	.505	.390	.395
(0.15, 0.3)	10	.925	.865	.830	.870	.805	.830
	15	.890	.830	.805	.860	.780	.805
	20	.875	.805	.800	.845	.760	.780
(0.2, 0.4)	10	.990	.970	.970	1.00	.980	.985
	15	.990	.970	.965	1.00	.970	.980
	20	.985	.965	.955	1.00	.965	.970
(0.3, 0.6)	10	1.00	1.00	1.00	1.00	1.00	1.00
	15	1.00	1.00	1.00	1.00	1.00	1.00
	20	1.00	1.00	1.00	1.00	1.00	1.00

Table 2. Powers of Test Ⅰ, Test Ⅱ(with $q=1$) and BFT for testing $H_0: \omega=1$, where $\omega=\beta_1$.

(δ_1, δ_2)	d	Case 1			Case 2		
		Test Ⅰ	Test Ⅱ	BFT	Test Ⅰ	Test Ⅱ	BFT
(-1.0, 0)	3	.990	.980	.990	1.00	.995	.985
	5	.990	.980	.990	1.00	.995	.985
	10	.990	.980	.980	.995	.985	.975
(-0.7, 0)	3	.875	.765	.835	.925	.870	.790
	5	.845	.740	.800	.910	.830	.765
	10	.810	.725	.740	.880	.765	.755
(-0.5, 0)	3	.600	.460	.555	.750	.615	.505
	5	.570	.435	.505	.705	.555	.485
	10	.525	.390	.435	.620	.480	.470
(-0.3, 0)	3	.330	.180	.290	.435	.310	.220
	5	.310	.165	.220	.390	.260	.220
	10	.245	.145	.155	.330	.220	.205
(-0.1, 0)	3	.070	.045	.065	.150	.100	.060
	5	.060	.045	.050	.115	.070	.060
	10	.050	.040	.045	.090	.055	.050
(0, 0)	3	.875	.930	.915	.890	.960	.935
	5	.900	.945	.935	.915	.960	.955
	10	.920	.975	.970	.935	.985	.970
(0.1, 0)	3	.165	.110	.100	.070	.040	.065
	5	.135	.105	.085	.050	.030	.045
	10	.110	.085	.075	.030	.025	.025
(0.3, 0)	3	.510	.345	.260	.295	.210	.280
	5	.445	.305	.250	.280	.205	.240
	10	.355	.250	.210	.255	.180	.205
(0.5, 0)	3	.805	.635	.530	.685	.475	.615
	5	.725	.570	.495	.635	.460	.550
	10	.655	.490	.435	.585	.415	.460
(0.7, 0)	3	.915	.845	.790	.915	.835	.900
	5	.875	.810	.775	.910	.830	.875
	10	.855	.775	.740	.885	.795	.830
(1.0, 0)	3	.990	.980	.975	.995	.985	.995
	5	.985	.980	.975	.995	.975	.990
	10	.985	.975	.975	.990	.965	.975

Table 1 and Table 2 show respective powers of suggested tests(Test Ⅰ, Test Ⅱ) and the test by Bayes Factor(BFT) calculated from 200 runs of the simulation with $n=20$ and various values of δ_1, δ_2 and d. For BFT, we used the conditional prior of ω as $g(\omega|\sigma)=\frac{1}{\sqrt{d+5}}$ for $\omega\neq w_0$. The powers in the tables indicate that our Monte Carlo estimates of the Bayes factors(partial, overall, and usual) test the null hypotheses properly. It is also noted from the tables that, for testing H_0, BFT and Test Ⅱ result in better powers than that of Test I. On the other hand, when $H_{A}: \omega \neq \omega_0$ is true, Test I uniformly dominates BFT and Test II in terms of power. These results are shown to be consistent with different settings of simulation parameter values of n, d, δ_1 and δ_2. Thus, the simulation results suggest that, for the flexible Bayes test for the nested point null hypotheses; (i) one may use Test I for the multiple decision between H_0, H_1 and H_2; (ii) one may use Test II for testing H_0 versus H_A, where $H_A = H_1 \cup H_2$.

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