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A Comparative Study of the GPAC Method and the 3-pattern Method for Identifying ARMA Processes

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Abstract

The generalized partial autocorrelation (GPAC) method of Woodward and Gray (1981) and the 3-pattern method of Choi (1991) have been used for identifying ARMA processes. The methods are based on the extended Yule-Walker equations. The purpose of this paper is to show the 3-pattern method is superior to the GPAC method through theoretical analysis and computer simulations.

1. Introduction

Consider the autoregressive moving-average (ARMA) model of orders p and q,

$$\phi(B)y_t = \theta(B)v_t, \tag{1}$$

where

$$\begin{aligned}
&\phi(B) = -\phi_0 - \phi_1 B - \cdots - \phi_p B^p,\\
&\theta(B) = -\theta_0 - \theta_1 B - \cdots - \theta_q B^q,\\
&\phi_0 = \theta_0 = -1,\\
&\phi_p \neq 0,\\
&\theta_q \neq 0,
\end{aligned}$$

B is the backshift operator and $\{v_t\}$ is a sequence of independent and identically distributed random variables with means 0 and finite variances $\sigma^2(>0)$. Assume that the model is stationary and invertible, i.e., the equations $\phi(z)=0$ and $\theta(z)=0$ have all the roots outside the unit circle. Also, assume that the two equations have no common root. Define the autocovariance function (ACVF) and the autocorrelation function (ACRF) by

$$\begin{aligned}
\sigma(j) &= cov(y_t, y_{t+j}), \quad j=0, \pm1, \pm2, \cdots,\\
\rho_j &= \sigma(j)/\sigma(0), \quad j=0, \pm1, \pm2, \cdots.
\end{aligned}$$

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Let $\{y_1, y_2, \cdots, y_T\}$ be a T-realization of the ARMA process. Then, we estimate the ACVF and the ACRF by the sample ACVF and the sample ACRF defined as

$$\widehat{\sigma}(j) = \widehat{\sigma}(-j) = \frac{1}{T}\sum_{t=1}^{T-j} y_t y_{t+j}, \qquad j=0,1,\cdots,T-1,$$
$$\widehat{\rho}_j = \widehat{\sigma}(j)/\widehat{\sigma}(0), \qquad j=0,\pm 1,\cdots,\pm(T-1).$$

It is well-known that the ACVF and the ACRF satisfy the extended Yule-Walker (EYW) equations.

Property 1.1 A stationary stochastic process has an ARMA(p, q) representation, if and only if the ACVF satisfies a linear difference equation of minimal order p from the minimal rank $q+1$, i.e., there exist constants $\beta_1, \cdots, \beta_p$ satisfying

$$\sigma(j) = \beta_1 \sigma(j-1) + \cdots + \beta_p \sigma(j-p), \qquad j=q+1, q+2, \cdots,$$

where $p>0$, $\beta_p \neq 0$, $q \geq 0$ and q is the least integer satisfying the difference equations. □

In our model, it is clear that $\phi_1 = \beta_1, \cdots, \phi_p = \beta_p$.

It is an important problem in ARMA modeling to determine the orders p and q, and many solutions have been proposed. (See, e.g., Choi [1992a].) Among them there are some methods utilizing the EYW equations. They are called the pattern identification methods. Here, the terminology, identification, does not mean the estimation of parameters but the determination of orders. In this paper we concentrate on two pattern identification methods. One is the generalized partial autocorrelation (GPAC) method proposed by Woodward and Gray (1981). The other is the 3-pattern method proposed by Choi (1991).

We can estimate the GPAC by solving the sample EYW equations, i.e., the EYW equations substituting the sample ACVF for the ACVF. We call it the EYW estimate of the GPAC. Davies and Petruccelli (1984) have presented a report against the use of the GPAC method in ARMA modeling. They have claimed that the EYW estimate of the GPAC pattern shows unstable behavior even for moderate sample sizes, when an MA term is presented. To demonstrate difficulties with application of the GPAC method, they have provided simulation evidence and some real data analyses. Newbold and Bos (1983) have also shown in simulation studies for ARMA(1,1) processes that the empirical variances of the EYW estimates of the GPAC's with MA lag greater than 1 are large even for 100 observations, and has concluded that the EYW estimate of the GPAC should have a heavy tailed distribution. Choi (1991) has derived the asymptotic distribution of the EYW estimate of the GPAC. In the paper he has presented the three functions called the θ, λ and η functions, which are useful for ARMA model identification.

The purpose of this paper is to show that the 3-pattern method is more useful than the GPAC method based on computer simulation as well as theoretical analysis.

2. The GPAC method and the 3-pattern method

For a formal introduction to the 3-pattern method and the GPAC method, we first define two k-dimensional Toeplitz matrices and two associated vectors by

$$B(k,i)=\begin{pmatrix} \rho_i & \rho_{i-1} & \cdots & \rho_{i-k+2} & \rho_{i-k+1} \\ \rho_{i+1} & \rho_i & \cdots & \rho_{i-k+3} & \rho_{i-k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_{i+k-1} & \rho_{i+k-2} & \cdots & \rho_{i+1} & \rho_i \end{pmatrix},$$

$$A(k,i)=\begin{pmatrix} \rho_i & \rho_{i-1} & \cdots & \rho_{i-k+2} & \rho_{i+1} \\ \rho_{i+1} & \rho_i & \cdots & \rho_{i-k+3} & \rho_{i+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_{i+k-1} & \rho_{i+k-2} & \cdots & \rho_{i+1} & \rho_{i+k} \end{pmatrix},$$

$$\boldsymbol{\rho}(k,i)=(\rho_{i+1},\ \rho_{i+2},\ \cdots,\ \rho_{i+k})^t,$$
$$\boldsymbol{r}(k,i)=(\rho_{i-k},\ \cdots,\ \rho_{i-2},\ \rho_{i-1})^t.$$

The GPAC is defined by

$$\phi^{(i)}_{k,k}=\begin{cases} \rho_{i+1}/\rho_i, & k=1, \\ |A(k,i)|/|B(k,i)|, & k>1. \end{cases}$$

Jenkins and Alavi (1981) called it the kth order i-conditioned partial correlation. We refer to $\{\phi^{(i)}_{k,k}\}$ as the GPAC function.

We denote $\tilde{\boldsymbol{x}}=(x_n,x_{n-1},\cdots,x_1)^t$ for any vector $\boldsymbol{x}=(x_1,\cdots,x_{n-1},x_n)^t$. Any function below with the tilde sign are to be defined with $\tilde{\boldsymbol{x}}$. For $k=1,2,\cdots$, and $i=0,1,\cdots$, if $B(k,i)$ is non-singular, then we let

$$\boldsymbol{\phi}(k,i)=B^{-1}(k,i)\,\boldsymbol{\rho}(k,i),$$

$$\tilde{\boldsymbol{\pi}}(k,i)=B^{-1}(k,i)\,\boldsymbol{r}(k,i).$$

Cramer's rule yields that $\phi^{(i)}_{k,k}$ is the kth elements of $\boldsymbol{\phi}(k,i)$. We denote the jth elements of $\boldsymbol{\phi}(k,i)$ and $\boldsymbol{\pi}(k,i)$ by $\phi^{(i)}_{k,j}$ and $\pi^{(i)}_{k,j}$ for $j=1,2,\cdots,k$. Also, let $\phi^{(i)}_{k,0}=\pi^{(i)}_{k,0}=-1$ for each pair (k,i).

The three functions are defined by

$$\theta(k,i)=\rho_{i+k+1}-\tilde{\boldsymbol{\phi}}(k,i)^t\boldsymbol{\rho}(k,i),$$
$$\eta(k,i)=\rho_{i-k-1}-\boldsymbol{\pi}(k,i)^t\boldsymbol{r}(k,i),$$
$$\lambda(k,i)=\rho_i-\boldsymbol{\pi}(k,i)^t\boldsymbol{\rho}(k,i).$$

For $i=0,1,\cdots$, let

$$\theta(0,i)=\rho_{i+1},$$
$$\eta(0,i)=\rho_{i-1},$$
$$\lambda(0,i)=\rho_i$$

as initial values. We refer to $\{\theta(k,i)\}$, $\{\lambda(k,i)\}$ and $\{\eta(k,i)\}$ as the θ, λ and η functions, respectively.

It has been known (Choi, [1991]) that the GPAC is represented by a ratio of θ and λ as follows.

Property 2.1

$$\phi_{k,k}^{(i)}=\frac{\theta(k-1,i)}{\lambda(k-1,i)}\,. \qquad \square$$

If $k>p$ and $i>q$, then $B(k,i)$ is singular, and then the three functions are not defined. However, if the inverse of $B(k,i)$ is taken as a generalized inverse, then the three functions can be extended as follows.

$$\begin{aligned}
\theta^{-}(k,i)&=\rho_{i+k+1}-\tilde{\rho}(k,i)^{t}B^{-}(k,i)\,\rho(k,i),\\
\eta^{-}(k,i)&=\rho_{i-k-1}-\tilde{r}(k,i)^{t}B^{-}(k,i)\,r(k,i),\\
\lambda^{-}(k,i)&=\rho_{i}-\tilde{\rho}(k,i)^{t}B^{-}(k,i)\,r(k,i).
\end{aligned}$$

We also define a function and two index sets as

$$\begin{aligned}
&\phi_{k}^{(i)}(z)=-\phi_{k,0}^{(i)}-\phi_{k,1}^{(i)}z-\cdots-\phi_{k,k}^{(i)}z^{k}, \quad \phi_{k,0}^{(i)}=-1,\\
&I_{r,s}=\{(k,s)\,|\,k=r,r+1,\cdots\}\cup\{(r,i)\,|\,i=s,s+1,\cdots\},\\
&J_{r,s}=\{(k,i)\,|\,k=r,r+1,\cdots,\ i=s,s+1,\cdots\}.
\end{aligned}$$

Choi (1991) presented the patterns of the θ, λ and η functions as follows.

Property 2.2 The θ pattern

A stochastic process has the ARMA(p,q) representation (1), if and only if p and q are the smallest integers satisfying

$$\theta(k,i)=0, \qquad (k,i)\in I_{p,q}\,. \qquad \square$$

Property 2.3 The generalized θ pattern

If a stochastic process has the ARMA(p,q) representation (1), then

$$\theta^{-}(k,i)=0, \qquad (k,i)\in J_{p,q}\,. \qquad \square$$

Property 2.4 The λ pattern

A stochastic process has the ARMA(p,q) representation (1), if and only if p and q are the smallest integers satisfying

$$\lambda(p,q+1)=\lambda(p,q+2)=\cdots=0,$$

$$\lambda(p,q)=\lambda(p+1,q)=\cdots=\frac{\theta_0\theta_q\sigma^2}{\sigma(0)}\,. \qquad \square$$

Property 2.5 The generalized λ pattern

If a stochastic process has the ARMA(p, q) representation (1), then the following holds.

1. If $i > q$, then $\lambda^{-}(p, i) = 0$.
2. If $k \geq p$, then $\lambda^{-}(k, q) = \theta_0 \theta_q \sigma^2 / \sigma(0)$.
3. If $s > r \geq 0$, then $\lambda^{-}(p+r, q+s) = 0$. □

Property 2.6 The η pattern

If a stochastic process has the ARMA(p, q) representation (1), then the following holds.

1. $\eta(p-1, i) \neq 0$, $i = q+1, q+2, \cdots$.
2. $\eta(p, q+1) = -\lambda(p, q) / \phi_p$.
3. $\eta(p, q+2) = \eta(p, q+3) = \cdots = 0$. □

Property 2.7 The generalized η pattern

If a stochastic process has the ARMA(p, q) representation (1), then the following holds.

1. If $r \geq 0$, then $\eta^{-}(p+r, q+1+r) = -\lambda(p, q) / \phi_p$.
2. If $r > s \geq 0$, then $\eta^{-}(p+r, q+1+s) = \pm\infty$.
3. If $s > r \geq 0$, then $\eta^{-}(p+r, q+1+s) = 0$. □

Woodward and Gray (1981) derived the pattern of the GPAC as follows.

Property 2.8 The GPAC pattern

If a stochastic process has the ARMA(p, q) representation (1), then the following holds.

1. If $i \geq q$, then $\phi_{p,p}^{(i)} = \phi_p$.
2. If $k > p$, then $\phi_{k,k}^{(q)} = 0$. □

We estimate these quantities by substituting the sample ACVF for the ACVF and denote them by $\widehat{B}(k, i)$, $\widehat{\rho}(k, i)$, $\widehat{r}(k, i)$, $\widehat{\phi}(k, i)$, $\widehat{\pi}(k, i)$, $\widehat{\theta}(k, i)$, $\widehat{\lambda}(k, i)$, $\widehat{\eta}(k, i)$, $\widehat{\theta^{-}}(k, i)$, $\widehat{\lambda^{-}}(k, i)$ and $\widehat{\eta^{-}}(k, i)$, respectively. For fixed T, $\widehat{B}(k, i)$ is nonsingular with probability 1. Thus, $\widehat{\theta^{-}}(k, i)$, $\widehat{\lambda^{-}}(k, i)$ and $\widehat{\eta^{-}}(k, i)$ equal $\widehat{\theta}(k, i)$, $\widehat{\lambda}(k, i)$ and $\widehat{\eta}(k, i)$ respectively with probability 1. Thus, in practice we can apply Properties 2.3, 2.5 and 2.7 to $\widehat{\theta}(k, i)$, $\widehat{\lambda}(k, i)$ and $\widehat{\eta}(k, i)$, respectively.

3. Comparison

To compare the 3-pattern method and the GPAC method analytically, we cite some asymptotic properties from Choi (1992a, Section 5.1; see also Choi, 1994).

Property 3.1 Let $\{y_1, y_2, \cdots, y_T\}$ be a T-realization from the ARMA(p, q) model (1), and let

$$a(k) = \frac{\sum_{j=-q}^{q} \sigma_z(j)\sigma_z(j+k)}{\sigma^2(0)},$$

where

$$\sigma_z(j) = \sum_{r=0}^{p} \sum_{s=0}^{p} \phi_r \phi_s \sigma(j+r-s).$$

Then, the following holds.

1. Let $(k, i) \in I_{p,q}$ and $(l, m) \in I_{p,q}$. Then, $\sqrt{T}\,\hat{\theta}(k, i)$ is asymptotically normally distributed with mean 0. The asymptotic covariance between $\sqrt{T}\,\hat{\theta}(k, i)$ and $\sqrt{T}\,\hat{\theta}(l, m)$ is $a(k-l+i-m)$.
2. Let $i > q$ and $m > q$. Then, $\sqrt{T}\,\hat{\lambda}(p, i)$ is asymptotically normally distributed with mean 0. The asymptotic covariance between $\sqrt{T}\,\hat{\lambda}(p, i)$ and $\sqrt{T}\,\hat{\lambda}(p, m)$ is $a(i-m)/\phi_p^2$.
3. Let $k > p$ and $m > p$. Then, $\sqrt{T}\,\hat{\phi}_{k,k}^{(q)}$ is asymptotically normally distributed with mean 0. The asymptotic covariance between $\sqrt{T}\,\hat{\phi}_{k,k}^{(q)}$ and $\sqrt{T}\,\hat{\phi}_{l,l}^{(q)}$ is $a(k-l)/\lambda^2(p, q)$. □

As conjectured and illustrated by Newbold and Bos (1983) and Davies and Petruccelli (1984), the variation of $\hat{\phi}_{k,k}^{(i)}$ is large for $(k, i) \in I_{p,q}$, when the underlying process is from the ARMA(p, q) model (1). Using Property 2.1 we can explain it as follows. It is known (Choi [1991, p. 200]) that

$$\lambda(k, i) = \rho_i - \phi_{k,1}^{(i)}\rho_{i-1} - \phi_{k,2}^{(i)}\rho_{i-2} - \cdots - \phi_{k,k}^{(i)}\rho_{i-k}.$$

It means $\lambda(k, i)$ is the residual of ρ_i obtained by fitting the EYW equations of the ARMA (k, i) model. Particularly, if the underlying process is a pure AR(k) process, then $\lambda(k, 0)$ is the variance of the white noise process divided by $\sigma(0)$. Therefore, if k and i are appropriate orders for the underlying ARMA process, then $\hat{\lambda}(k-1, i)$ should be near 0. And then, the sample variance of $\hat{\phi}_{k,k}^{(i)}$ would be large. As shown in Property 3.1, the asymptotic variance of $\hat{\phi}_{k,k}^{(q)}$ is larger than that of $\hat{\theta}(k, q)$ for $k > p$. Thus, the θ function is more reliable than the GPAC function for ARMA model identification.

In order to apply a pattern identification method statistically, we should not test if any element of the pattern is 0. But we should test the patterns simultaneously. For that purpose Choi (1992b) proposed the two chi-squared statistics E and J based on the asymptotic distribution of the θ function estimate of Property 3.1. However, it is almost impossible to derive any chi-squared statistic for testing the constant-behavior of the GPAC, because the asymptotic correlation of $\sqrt{T}(\hat{\phi}_{p,p}^{(l)} - \phi_p)$ and $\sqrt{T}(\hat{\phi}_{p,p}^{(m)} - \phi_p)$ is close to 1 for $l \geq q$ and

$m \geq q$. From the part 3 of Property 3.1 we can guess that the empirical distribution of the chi-square statistic for testing the zero-behavior of the GPAC may be far different from the theoretic distribution because $\lambda(p, q)$ is close to 0. Therefore, the θ function is more desirable than the GPAC function.

We know from Property 3.1 that the asymptotic variance of the row-pattern of the λ function estimate, *i.e.*, the zero-behavior of $\hat{\lambda}(p, i)$, $(i > q)$, is larger than that of the θ function estimate. As shown in Choi (1992a), the column-pattern of the λ function estimate, i.e., the constant-behavior of $\hat{\lambda}(k, q)$, $(k \geq p)$, has more tedious asymptotic variance than that of the θ function estimate, i.e., the zero-behavior of $\hat{\theta}(k, q)$, $(k \geq p)$. Additionally, the zero-behavior of the θ pattern is easier to handle statistically than the constant-behavior of the λ pattern, for the latter needs a statistical test on the random variables whose asymptotic correlations are 1. This is not a familiar problem in statistics. Thus, the θ array is more convenient than the λ array to determine the orders of an ARMA process.

The η array is related to the dual ARMA model, which is the backward ARMA process corresponding to the ARMA model (1). Sometimes the η function estimate is far different from the pattern of Properties 2.6 and 2.7. As shown in Choi (1992a), the η function estimate has asymptotic distributions, which are more difficult to handle than those of the θ and λ function estimates.

To calculate the GPAC array in a computationally efficient way, we can use the simplified Trench-Zohar algorithm, i.e., Algorithm 1.2 of Choi (1992a). As a by-product of the algorithm, we can calculate the θ, λ and η arrays. Thus, the 3-pattern method is more informative than the GPAC method without any extra computational cost.

To compare practicability of the 3-pattern method and the GPAC method, we re-examine ARMA models previously analyzed by Woodward and Gray (1981), Davies and Petruccelli (1984), and Newbold and Bos (1983).

Newbold and Bos studied the ARMA(1,1) model,

$$y_t - \phi y_{t-1} = v_t - \theta v_{t-1},$$

with four pairs of parameters: (ϕ, θ) = (0.50, -0.85), (0.50, 0.85), (0,95, -0.85) and (0.95, 0.40). We first generate 1000 replications of time series with length 500 for each pair of the parameters. We estimate the three functions for each replication, and calculate their empirical means and standard deviations. In the cases (0.95, 0.40) and (0.95, -0.85), the estimated arrays match with the large-sample theory of Choi (1991) very well. The latter case shows clearer pattern than the former. Table 1 is the summary of the case (0.95, 0.40). The row $\{\hat{\theta}(1, j) \mid j \geq 1\}$, shows the zero-behavior. The empirical standard deviations are small and around 0.0165. Even though the column $\{\hat{\theta}(j, 1) \mid j \geq 1\}$ shows the zero-behavior, the empirical variances increase drastically as j does. This phenomenon is not so severe for the case (0.95, -0.85). The row $\{\hat{\lambda}(1,j) \mid j>1\}$ shows the zero-behavior, and the empirical standard deviations are near 0.018. The column $\{\hat{\lambda}(j,1) \mid j\geq 1\}$ does not show the constant-behavior so vividly as is expected. Moreover, the empirical variances increase exponentially, as j does. This phenomenon is less severe in the case (0.95, -0.85). The row $\{\hat{\lambda}(0,j) \mid j\geq 1\}$ matches with the true one very well. Since each element is pretty large and is different from zero, we may expect that the constant-behavior of the GPAC will appear vividly. Actually the GPAC shows a clear constant-behavior and the empirical standard deviations are pretty small (near 0.025). Even though the GPAC shows the zero-behavior as expected, the corresponding empirical variances increase exponentially as the lag does in contrast to the asymptotic theory. This phenomenon is not so severe in the case (0.95, -0.85).

Table 1. Empirical means and standard deviations of the patterns of $(1-0.95B)y_t=(1-0.4B)v_t$ with 500 sample size and 1000 replications

j	0	1	2	3	4
$\hat{\theta}(1,j)$	.0907 (.0219) [.0831]	-.0003 (.0168) [.0000]	-.0015 (.0169) [.0000]	-.0007 (.0165) [.0000]	-.0002 (.0160) [.0000]
$\hat{\theta}(j,1)$	.7799 (.0542) [.8099]	-.0003 (.0168) [.0000]	-.0042 (.0176) [.0000]	-.0061 (.0277) [.0000]	-.0295 (.5847) [.0000]
$\hat{\lambda}(1,j)$	.3107 (.0705) [.2732]	-.1108 (.0323) [-.0975]	.0001 (.0180) [.0000]	.0014 (.0182) [.0000]	.0005 (.0177) [.0000]
$\hat{\lambda}(j,1)$	.8291 (.0432) [.8525]	-.1108 (.0323) [-.0975]	-.1151 (.0588) [-.0975]	-.1006 (.4157) [-.0975]	-.5383 (16.28) [-.0975]
$\hat{\lambda}(0,j)$	1.000 (.0000) [1.000]	.8291 (.0432) [.8525]	.7799 (.0542) [.8099]	.7338 (.0637) [.7694]	.6893 (.0729) [.7309]
$\hat{\phi}^{(j)}_{1,1}$	.8291 (.0432) [.8525]	.9399 (.0219) [.9500]	.9398 (.0230) [.9500]	.9378 (.0258) [.9500]	.9371 (.0263) [.9500]
$\hat{\phi}^{(1)}_{j,j}$		.9399 (.0219) [.9500]	-.0122 (.1527) [.0000]	.0086 (.4363) [.0000]	-.1293 (6.006) [.0000]

NOTE : Figures within the parentheses are standard deviations and figures within the brackets are true parameter values.

The cases (0.5, −0.85) and (0.5, 0.85) show similar results. Table 2 is the summary of the case (0.5, −0.85). The row $\{\hat{\theta}(1,j) \mid j\geq 1\}$ shows the zero-behavior, but the asymptotic variances increase drastically as j does. It does happen in the case (0.5, 0.85), but it is not so severe. The column $\{\hat{\theta}(j,1) \mid j\geq 1\}$ shows the zero-behavior, and the empirical standard deviations are near 0.03. The row $\{\hat{\lambda}(1,j) \mid j>1\}$ shows the zero-behavior, but the empirical variances increase exponentially. The column $\{\hat{\lambda}(j,1) \mid j\geq 1\}$ shows the constant-behavior, and the empirical variances do not vary a lot. For $j\geq 1$, the true value of $\lambda(0,j)$ and its empirical mean are similar, and the empirical variance is small. Property 2.8 makes us expect the estimated GPAC pattern will show the constant-behavior. However, it does not. One of the explainable reasons is that both $\hat{\theta}(0,j)$ and $\hat{\lambda}(0,j)$ decrease exponentially in absolute value as j increases, and then, they may be regarded as 0 for large j. Thus, the ratio,

Table 2. Empirical means and standard deviations of the patterns of $(1-0.5B)y_t=(1+0.85B)v_t$ with 500 sample size and 1000 replications

j	0	1	2	3	4
$\hat{\theta}(1,j)$	-.1869 (.0227) [-.1853]	-.0028 (.0290) [.0000]	-.0076 (.0310) [.0000]	-.0206 (.2344) [.0000]	-.0708 (1.945) [.0000]
$\hat{\theta}(j,1)$	.3657 (.0528) [.3739]	-.0028 (.0290) [.0000]	-.0006 (.0289) [.0000]	-.0075 (.0305) [.0000]	.0031 (.0312) [.0000]
$\hat{\lambda}(1,j)$	.4475 (.0340) [.4408]	.2525 (.0368) [.2478]	.0096 (.0636) [.0000]	.0378 (.5330) [.0000]	.1071 (3.185) [.0000]
$\hat{\lambda}(j,1)$	.7430 (.0229) [.7478]	.2525 (.0368) [.2478]	.2508 (.0539) [.2478]	.2664 (.0775) [.2478]	.2535 (.0968) [.2478]
$\hat{\lambda}(0,j)$	1.0000 (.0000) [1.000]	.7430 (.0229) [.7478]	.3657 (.0528) [.3739]	.1796 (.0650) [.1870]	.0359 (.0704) [.0935]
$\hat{\phi}^{(j)}_{1,1}$	.7430 (.0229) [.7478]	.4905 (.0572) [.5000]	.4765 (.1271) [.5000]	.3565 (2.215) [.5000]	-.1128 (18.92) [.5000]
$\hat{\phi}^{(1)}_{j,j}$		.4905 (.0572) [.5000]	-.0005 (.1153) [.0000]	-.0221 (.1244) [.0000]	-.0036 (.1178) [.0000]

NOTE : Figures within the parentheses are standard deviations and figures within the brackets are true parameter values.

$$\hat{\phi}_{1,1}^{(j)} = \frac{\hat{\theta}(0,j)}{\hat{\lambda}(0,j)},$$

could vary erratically for large j. Actually, they have explosively increasing empirical variances. In contrast to the constant-behavior the GPAC shows the zero-behavior very clearly. The empirical standard deviations are near 0.12. This stability results from the nonzero-behavior of $\{\hat{\lambda}(j-1,1) \mid j>1\}$.

Based on the simulation we can make the following summary.

- The appearance of the expected patterns of the three functions and the GPAC function depends heavily on the parameter values of each model.
- The θ pattern shows more stable results than either the λ pattern or the GPAC pattern.
- If the AR coefficient of an ARMA(1,1) process is large in absolute value, then the row-patterns $\{\theta(1,j)\}$, $\{\lambda(1,j)\}$ and $\{\phi_{1,1}^{(j)}\}$ are more useful than the column-patterns $\{\theta(j,1)\}$, $\{\lambda(j,1)\}$ and $\{\phi_{j,j}^{(1)}\}$. The reverse holds if the MA coefficient is large in absolute value.

To investigate small sample properties of the three functions we generate 1000 time series with series length 100. It shows that the empirical confidence interval of $\lambda(j-1,1)$ for $j>1$ contains 0 with 0.05 significance level. It implies the instability of the zero-behavior of the GPAC. However, if the AR coefficient of an ARMA(1,1) process is large in absolute value, then, even for small sample cases, $\hat{\lambda}(0,j)$ for $j\geq 1$ is not near 0, and then the constant-behavior of the GPAC, i.e., $\{\hat{\phi}_{1,1}^{(j)} \mid j\geq 1\}$, appears vividly.

Consider the ARMA(3,2) model,

$$y_t - 1.5y_{t-1} + 1.21y_{t-2} - 0.455y_{t-3} = v_t + 0.2v_{t-1} + 0.9v_{t-2},$$

which has been studied by Woodward and Gray (1981). Based on their simulation result of this model, Davies and Petruccelli (1984) have concluded that the finite sampling distributions of the GPAC array estimate could have large variances, and suggested that the comforting picture created by single realization examples in Woodward and Gray should be viewed with caution. We generate 1000 replications with length T from the model for each T of 50, 100, 200 and 500. From our simulaton we see that the empirical means of $\{\hat{\phi}_{3,3}^{(j)} \mid j=1,2,\cdots\}$ do not show the constant-behavior even for T=100. When T=500, the empirical means are reasonably near a constant but the corresponding standard deviations have the comparatively high variability. The empirical means of $\{\hat{\phi}_{j,j}^{(2)} \mid j=3,4,\cdots\}$ seem reasonably close to zero when T=100, 500, but the standard deviation is large even when T=100.

Table 3 contains the empirical mean and standard deviation of $\hat{\lambda}(2,j)$ for $j>1$. It shows that any empirical confidence interval of $\lambda(2,j)$ for $j>2$, contains 0 even when T=500. Since the denominator of $\hat{\phi}_{3,3}^{(j)}$ is close to 0, the GPAC does not show a vivid constant-behavior. We should note that this failure is due to the structure of the model, i.e., the smallness of $\lambda(2,j)$ for $j>2$.

Table 4 contains the empirical mean and standard deviation of $\hat{\lambda}(j,2)$, $(j>1)$. It shows that the empirical confidence interval of $\lambda(j,2)$, $(j>1)$, does not contain zero when T=200, 500. Additionally, the empirical variance is small and quite stable. Thus, the zero-behavior of the GPAC appears vividly. However, when T=50, 100, the empirical mean of $\lambda(j,2)$, $(j>1)$, is small in absolute value and the empirical variance is relatively large. Thus, it is expected that the GPAC fails to show the zero-behavior clearly. Actually it happens. This failure is not due to the model itself but because of smallness of the sample sizes. Thus we can not utilize the consistency of $\hat{\lambda}(j,2)$ to test the zero-behavior of the GPAC statistically, unless at least 200 observations are provided. When T=200, 500, the empirical means of $\hat{\lambda}(j,2)$, $(j>2)$, are near a constant and the empirical variances are small enough to assert the constant-behavior.

Table 3. Empirical means and standard deviations of $\hat{\lambda}(2,j)$ for various sample size T of ARMA(3,2) model with 1000 replications.

$T \backslash j$	2	3	4	5	6
50	.0380 (1.932)	-.1343 (3.148)	1.527 (59.37)	-.0449 (1.620)	.0319 (.8320)
100	.1513 (.1419)	-.0695 (4.818)	-1.503 (65.53)	.2350 (10.00)	-.0990 (4.082)
200	.1503 (.0593)	-.0953 (11.04)	.0665 (3.218)	2.959 (103.6)	.0534 (.2087)
500	.1517 (.0371)	.1104 (3.309)	.4496 (14.87)	-.1455 (14.93)	.0703 (.1343)

NOTE : Figures within the parentheses are standard deviations.

Table 4. Empirical means and standard deviations of $\hat{\lambda}(j,2)$ for various sample size T of ARMA(3,2) model with 1000 replications.

$T \backslash j$	2	3	4	5	6
50	.0380 (1.932)	.0637 (.6542)	.1281 (1.802)	.2504 (4.766)	-.8028 (24.66)
100	.1513 (.1419)	.0783 (.0326)	.1301 (.7971)	.1326 (1.168)	.0471 (1.068)
200	.1503 (.0593)	.0737 (.0191)	.0826 (.0343)	.0786 (.0315)	.0709 (.0697)
500	.1517 (.0371)	.0716 (.0109)	.0735 (.0164)	.0728 (.0153)	.0702 (.0213)

NOTE : Figures within the parentheses are standard deviations.

4. Conclusion

Based on the theoretical and the numerical analysis, we can conclude that the 3-pattern method is more useful than the GPAC method for ARMA model identification, particularly when elements of λ array are close to 0.

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