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An Upper Bound on the Index of the Smoothest Density with Given Moments

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Abstract

For finite discrete distributions with prescribed moments, there is a well-known upper bound on the index of the support. In this paper, we are interested in the smoothest density with prescribed moments among the class of smooth functions. We define an index of continuous distribution through the support and derive an upper bound on the index of the smoothest density. Some examples are given, some of which achieve the upper bound.

1. Introduction

The $i-th$ moment c_i of a probability measure μ on [0,1] is defined by

$$c_i = \int_0^1 x^i \, d\mu(x), \quad i=0,1,2\ldots$$

Given the first n moments $c_1, \ldots c_n$ of a distribution on [0,1] with density f, we are intrested in the smoothest density f_* with these moments. Here the 'smoothest' means that it minimizes the quantity $J(f) = \int_0^1 (f^{(m)}(x))^2 dx$ over the m-th order Sobolev space W_2^m of functions on [0,1] defined as

$$W_2^m = \{f \text{ on } [0,1] \mid f^{(i)} \text{ is absolutely continuous}, i=0,\cdots,m-1 \text{ and } f^{(m)} \in L^2[0,1]\},$$

with inner product < , >,

$$\langle f, g \rangle = \sum_{i=0}^{m-1} f^{(i)}(1) g^{(i)}(1) + \int_0^1 f^{(m)}(x) g^{(m)}(x) \, dx.$$

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(See Adams (1975) for reference.) $J(f)$ is usually used as the penalty functional for the roughness of a function f. The existence and uniqueness of the smoothest density is proved in Hong and Kim (1995). Although it is not possible to derive the closed form of this density, some characterization was obtained in Hong (1992). He showed that f is the unique smoothest density if and only if f is nonnegative, has the first n given moments $c_1, \ldots c_n$, satisfies the boundary conditions $f^{(i)}(0)=f^{(i)}(1)=0$, $i=m, \ldots 2m-1$ and $f^{(m)}$ is of the form

$$f^{(m)}(x)=\phi(x)+(-1)^m I_{m-1}(\xi)(x),$$

where $I_k(g)(x)=\int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} g(x_k)\, dx_k \cdots dx_1$, $\phi(x)$ is a polynomial of degree $\leq m+n$ and ξ is a nondecreasing function which is constant on each interval where $f(x)>0$.

Modern theory of moments can be found in Krĕin and Nudel'man (1977) and Akhiezer (1961). For finite discrete measure μ on [0,1] with mass $p_1, \cdots, p_k$ at distinct points $x_1, \cdots, x_k$, the index $I(\mu)$ of the measure μ is defined through its support $x_1, \cdots, x_k$ by counting 1 for each $x_i \in (0,1)$ and 1/2 for each $x_i \in \{0,1\}$.
Let

$$M_n=\{(c_1, \ldots, c_n) \mid c_i=\int_0^1 x^i d\mu(x),\ i=1, \cdots, n\}$$

denote the convex set of all possible first n moments from probability measures μ on [0,1]. For every $c=(c_1, \ldots, c_n) \in M_n$ there exist at least one finite discrete measures with these moments. Markov (1898) and Stieltjes (1884) showed that $c \in \partial M_n$, boundary of M_n, if and only if c has a unique representing finite discrete measure μ of index $I(\mu) \leq n/2$ and that each c in the interior of M_n has two representing measures $\underline{\mu}$ and $\overline{\mu}$ of index $\frac{n+1}{2}$. In case $n=2k-1$, $\underline{\mu}$ has index k and its support is the k zeros of the k-th orthogonal polynomial $p_k(x)$ defined with respect to $c=(c_1, \cdots, c_n)$.

Using the characterization of the smoothing density mentioned earlier, somewhat analogous results about the upper bound on the index can be obtained for continuous type distributions. This is the purpose of this paper.

2. An upper bound on the index of the smoothest density.

A corner point of a trajectory f is defined as a point at which an interior segment of the trajectory f joins a boundary segment of f. (cf. Berkovitz (1962) and McIntyre and Paiewonsky (1967)). We also define corner points of f_*, the smoothest deisity with prescribed first n moment $c=(c_1,\cdots,c_n)$, in the same way. Actually the corner points of f_* are the end points of the subintervals in the support of f_*. Using the above mentioned characterization of f_*, we can now find an upper bound on the number of the corner points of f_*. Before we state our theorem, we define the index for special type of subsets of [0,1]. Let $E=\cup_{i=1}^{k}E_i$, where $E_i=[\alpha_i,\beta_i]$ with $\alpha_i<\beta_i$ and E_i's are disjoint. Define the index $I(E)$ of the set E as:

$$I(E)=\frac{1}{2}\sum_{i=1}^{k}(\delta_{(0,1)}(\alpha_i)+\delta_{(0,1)}(\beta_i)),$$

where δ_A is indicator function of set A.

Now we consider the problem of finding an upper bound on the number of the corner points of f_*. Note that $I(Supp(f_*))$ defined above is equal to $\frac{1}{2}$(# of corner points of f_*).

We can prove that the following inequalities hold.

Theorem 2.1

$$I(Supp(f_*)) \leq \frac{n}{2(m-1)} \qquad \textit{when} \quad m\geq 2 \tag{2.1}$$

and

$$I(Supp(f_*)) \leq \frac{n}{2} \qquad \textit{when} \quad m=1\ . \tag{2.2}$$

Proof : When m=1, $f_*^{(m)}$ is m−2 times continuously differentiable by the characterizaion of f_* mentioned above. The $Supp(f_*)$ is a union of disjoint intervals, that is, $Supp(f_*))=\bigcup_{i=1}^{k}[\alpha_i,\beta_i]$ for some k. Without loss of generality, suppose that $0\leq\alpha_1<\beta_1<\cdots<\alpha_k<\beta_k\leq 1$, then

$$f_*^{(j)}(\alpha_i)=f_*^{(j)}(\beta_i)=0, \quad \forall j=0,\cdots,2m-2, \quad \forall i=1,\cdots,k,$$

since f_* is 2m-2 times continuously differentiable. Now since $f_*(x)>0$ on (α_i, β_i) and $f_*(\alpha_i)=f_*(\beta_i)=0$, $f_*^{(1)}$ must have both positive and negative signs in this interval, whence $f_*^{(1)}$ has at least one zero in each (α_i, β_i). In this way one can easily show that $f_*^{(j)}$ has at least j zeros in each interval (α_i, β_i), for $j=0,\cdots,2m-1$. Furthermore, one can show that $f_*^{(2m)}$ has at least 2m-2 zeros in each interval (α_i, β_i). From the characterization of f_*,

$$f_*^{(2m)}(x)=\phi^{(m)}(x)\,\delta_{Supp(f_*)}(x), \tag{2.3}$$

where $\phi^{(m)}$ is a polynomial of degree $\le n$. Consider four different cases.

(i) ($0<\alpha_1$ and $\beta_k<1$) In this case, $f_*^{(2m)}$ has at least 2m-2 zeros in each interval (α_i, β_i) for $i=1,\cdots,k$. But Equation (2.3) shows that the total number of zeros of $f_*^{(2m)}$ in $\bigcup_{i=1}^{k}(\alpha_i, \beta_i)$ is at most n. We get the inequality $k(2m-2)\le n$. Thus,

$$I(Supp(f_*))=\frac{1}{2}(2k)=k\le\frac{n}{2(m-1)}\ .$$

(ii) ($\alpha_1=0$ and $\beta_k<1$) Using the end point conditions of f_* and 2m-2 times continuous differentiability of f_*$ from the characterization, one can easily show that $f_*^{(2m)}$ has at least m-1 zeros in (α_1, β_1). Since $f_*^{(2m)}$ has at least 2m-2 zeros in each interval (α_i, β_i) for $i=2,\cdots,k$, $(k-1)(2m-2)+m-1\le n$, and whence

$$I(Supp(f_*))=\frac{1}{2}(1+2(k-1))\le\frac{n}{2(m-1)}\,.$$

(iii) ($0<\alpha_1$ and $\beta_k=1$) The proof is similar to that of (ii).

(iv) ($\alpha_1=0$ and $\beta_k=1$) Since $f_*^{(2m)}$ has at least m-1 zeros in each of the intervals (α_1, β_1) and (α_k, β_k) and has at least 2m-2 zeros in each interval (α_i, β_i) for $i=2,\cdots,k-1$, $(k-1)(2m-2)\le n$. Thus

$$I(Supp(f_*))=\frac{1}{2}(2+2(k-2))\le\frac{n}{2(m-1)}\,.$$

When m=1, Hong (1992) showed that ξ appearing in the characterization of f_* is continuous and thus $f_*^{(1)}$ is continuous. Using this fact and in the same way as in the case when $m\ge 2$, Inequality (2.2) can easily be derived. □

If we consider the problem of finding smoothest density over the restricted Sobolev space

$$W^m_{2,0} = \{f \in W^m_2 \mid f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 0, \cdots, m-1\},$$

with inner product $\langle f, g\rangle = \int_0^1 f^{(m)}(x) g^{(m)}(x)\, dx$ instead of W^m_2, then we get the same bound on the index of the smoothest density.

Corollary 2.1

Let f_{*0} be the smoothest density with the prescribed moments $c = (c_1, \cdots, c_n)$ over the restricted Sobolev space $W^m_{2,0}$, then we have the following inequalities

$$I(Supp(f_{*0})) \le \frac{n}{2(m-1)} \qquad when \quad m \ge 2$$

$$I(Supp(f_{*0})) \le \frac{n}{2} \qquad when \quad m = 1$$

Proof : The proof of this corollary is almost the smae as that of Theorem 2.1. The only difference is that on the intervals of type (α_1, β_1) with $\alpha_1 = 0$ and (α_k, β_k) with $\beta_k = 1$, $f^{(2m)}_{0*}$ has at least m zeros. Using this fact, one can easily show that the above inequalities holds. □

3. Examples

In this section, we will give some examples. The first three achieve the upper bound on the index. In example 4, we will find the smoothest density with the first 10 moments of a distribution, which is a convex combination of two Beta distributions.

Example 1.

(m=1, n=2, minimization is taken over the restricted Sobolev space $W^1_{2,0}$)

By corollary, the upper bound is 1. Let $c_1 = 0.5,\ c_2 = 0.27$. Using the characterization of f_{*0}, It could be shown that

$$f_{*0}(x) = c(x-a)^2(b-x)^2\,\delta_{[a,b]}(x),$$

where $a = 0.5 - (0.14)^{1/2},\ b = 0.5 + (0.14)^{1/2},\ c = \frac{15}{16(0.14)^{5/2}}$.

Since f_{*0} has support [a,b] with a>0, b<1, $I(Supp(f_{*0})) = 1$, which is the upper bound.

Example 2.

(m=1, n=2, minimization is taken over the Sobolev space W_2^1)

By Theorem 2.1, the upper bound is 1. Let $c_1=1/3$, $c_2=2/17$. Again using the characterization of f_*, we can derive that

$$f_*(x)=c(x-a)^2(b-x)^2\delta_{[a,b]}(x),$$

where $a=0.119437$, $b=0.54723$, $c=2093.89$. Obvioulsy this achieves the upper bound on the index.

Example 3.

(m=2, n=2, minimization is taken over the Sobolev space W_2^2)

By Theorem 2.1, the upper bound is also 1. Let $c_1=1/3$, $c_2=13/108$. The characterization gives the smoothest density f_*

$$f_*(x)=c(x^3_{a)}(b-x)^3\delta_{[a,b]}(x),$$

where $a=(2-\sqrt{3})/6$, $b=(2+\sqrt{3})/6$, $c=140\cdot 3^{7/2}$. This achieves the upper bound on the index.

Example 4.

(m=2, n=10, minimization is taken over the Sobolev space W_2^2)

Here we will use the first 10 moments of the density function $f_\lambda(x)$, which is the convex combination of the two Beta densities, $f\sim$ Beta$(2,8)$ and $g\sim$ Beta$(5,2)$, that is, $f_\lambda(x)=\lambda f(x)+(1-\lambda)g(x)$. Two values of λ, $\lambda=0.2$, $\lambda=0.5$, will be considered. The first 10 moments of f_λ are as follows:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
$\lambda=0.2$	$\frac{107}{175}$	$\frac{846}{1925}$	$\frac{278}{825}$	$\frac{115}{429}$	$\frac{219}{1001}$	$\frac{651}{3575}$	$\frac{1101}{7150}$	$\frac{112263}{850850}$	$\frac{3891}{34034}$	$\frac{420}{4199}$
$\lambda=0.5$	$\frac{16}{35}$	$\frac{909}{3080}$	$\frac{287}{1320}$	$\frac{73}{429}$	$\frac{138}{1001}$	$\frac{327}{2860}$	$\frac{69}{715}$	$\frac{28113}{340340}$	$\frac{4869}{68068}$	$\frac{4203}{67184}$

Figure 1(a) shows $f_{0.2}$ and the smoothest density with the first 10 moments of $f_{0.2}$ and figure 1(b) shows $f_{0.5}$ and the smoothest density with the first 10 moments of $f_{0.5}$. By

Theorem 1, the upper bound on the indices is 5, but for both cases the smoothest densities are postive on [0,1], and have the index 0.

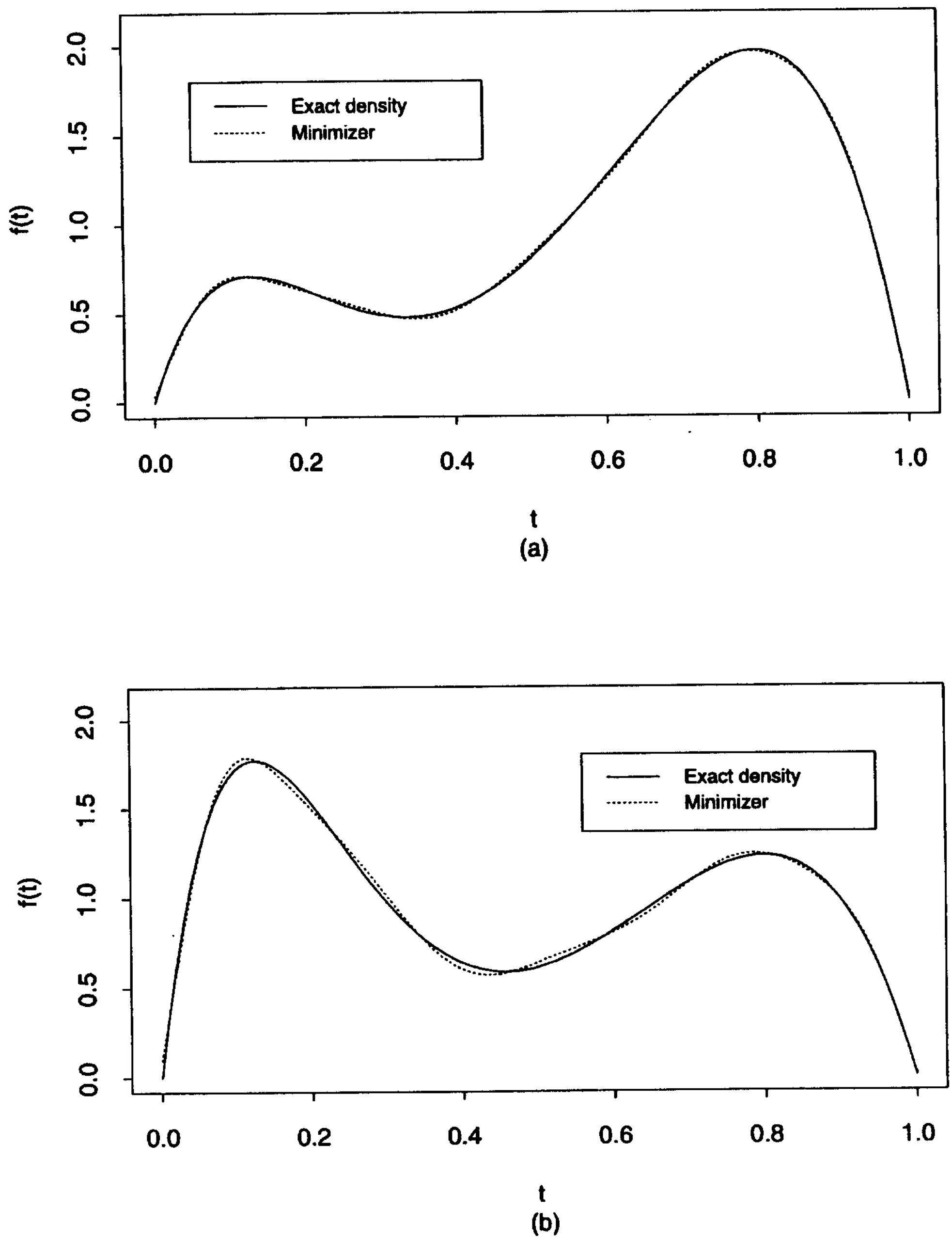

Figure 1. The exact densities f_λ and the minimizers f_*

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