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A Study on the Trend Change Point of NBUE-property

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Abstract

A life distribution F with survival function $\overline{F}=1-F$, finite mean μ and mean residual life $m(t)$ is said to be NBUE(NWUE) if $m(t)\leq(\geq)\mu$ for $t\geq 0$. This NBUE property can equivalently be characterized by the fact that $\varphi(u)\geq(\leq)u$ for $0\leq u\leq 1$, where $\varphi(u)$ is the scaled total-time-on-test transform of F. A generalization of the NBUE properties is that there is a value of p such that $\varphi(u)\geq u$ for $0\leq u\leq p$ and $\varphi(u)\leq u$ for $p\leq u\leq 1$, or vice versa. This means that we have a trend change in the NBUE property.

In this paper we point out an error of Klefsjö's paper (1988). He erroneously takes advantage of trend change point of failure rate to calculate the empirical test size and power in lognormal distribution. We solves the trend change point of mean residual lifetime and recalculate the empirical test size and power of Klefsjö (1988) in nocensoring case.

1. Introduction

For many practical situations where it is more reasonable to assume a certain type of trend change for some parameters, the statistical inference regarding such parameters attracts a great deal of interests among reliability scientists, engineers, or other statisticians recently.

Because of its useful applicability, many authors have considered the testing procedures for non-monotone classes of life distributions such as bathtub-shaped failure rate (BTR), increasing then decreasing mean residual life (IDMRL) (for example, Guess (1984), Matthews, Farewell and Pyke (1985), Guess, Hollander and Proschan (1986), Park (1988), etc.). Furthermore, Mi (1989) shows that the trend change point of failure rate for DIFR life distribution is greater than or equal to the point at which IDMRL distribution changes its mean residual life. Also, Park and Nam (1995) show that change points of mean residual life and failure rate are different in general for various choices of parameters for each parametric model.

Klefsjö (1988) proposes a nonparametric procedure intended for testing exponentiality against

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the situation where the life distribution changes from the new better than used in expectation (NBUE) to new worse than used in expectation (NWUE), assuming knowledge of the proportion p of the population that fail at or before the change-point t_0. We call this life distribution NBUE-NWUE(p). Such a trend change can be used in modelling for several maintenance and replacement policies.

In Section 2, we review a testing procedure for H_0 : F is exponential against H_1 : NBUE-NWUE(p), and is not exponential, (or $H_1{}'$: F is NWUE-NBUE (p), and is not exponential) using complete data.

In Section 3, we solves the trend change point of mean residual lifetime and recalculate the emprical test size and power of Klefsjö (1988) in nocensoring case. In Section 4, Monte Carlo experment is performed to investigate the empirical test size and power of our test procedure of the proposed test statistic and contains conclusion.

2. Review of a test procedure for the NBUE property

We assume knowledge of the proportion p of population that fail at or before the trend change point. Note that $t_0 = F^{-1}(p)$ is assumed to be unknown.

Klefsjö (1988) developed a test of

$$H_0 : F(x) = 1 - \exp(-x/\mu),\ x \geq 0,\ \mu \text{ is unspecified} \tag{2.1}$$

versus

$$H_1 : F \text{ is NBUE-NWUE}(p) \text{ (and is not exponential)}, \tag{2.2}$$

based on random sample $X_1, \cdots, X_n$ from a continuous life distribution F.

His test is motivated by considering the parameter

$$\begin{aligned} T(F) &= \int_0^p (\varphi(u) - u)du + \int_p^1 (u - \varphi(u))du \\ &= \frac{1}{\mu}\Big(\int_0^{t_0} \overline{F}^2(s)ds - \int_{t_0}^{\infty} \overline{F}^2(s)ds - 2\overline{F}(s)\int_0^{t_0} \overline{F}(s)ds\Big) + 1/2 - p^2 \\ &= \frac{1}{\mu}\int_0^{\infty} s(J(F(s))dF(s), \end{aligned} \tag{2.3}$$

where $\varphi(u) = \frac{1}{\mu}\int_0^{F^{-1}(u)} \overline{F}(t)\,dt,\ 0 \leq u \leq 1$ is the scaled total-time-on-test(TTT) transform and

$$J(u)=\begin{cases} -p^2+2p+1/2-2u & \text{for } 0\le u\le p, \\ -p^2-3/2+2u & \text{for } p< u\le 1. \end{cases} \tag{2.4}$$

Let

$$B(t)\equiv \int_t^1 J(u)du$$

$$=\begin{cases} [p(2-p)-(1/2+t)](1-t) & \text{for } 0\le t\le p, \\ [-p^2-1/2+t](1-t) & \text{for } p< t\le 1. \end{cases} \tag{2.5}$$

If F has a finite mean μ, then

$$\int_0^\infty s\, J(F(s))dF(s) = -\int_0^\infty s\, dB(F(s)) = \int_0^\infty B(F(s))ds \tag{2.6}$$

and thus we obtain another expression of $T(F)$ as

$$T(F) = \frac{1}{\mu}\int_0^\infty B(F(s))\,ds. \tag{2.7}$$

Klefsjö (1988) obtains the test statistic for testing H_0 versus H_1 by replacing F by F_n in the expression of (2.7), where F_n is an empirical distribution function of F. $T(F_n)$ can be expressed as

$$T(F_n)=\frac{1}{\mu_n}\Big[\sum_{i=0}^{n-1} B\Big(\frac{i}{n}\Big)(X_{(i+1)}-X_{(i)})\Big]$$

$$=\frac{1}{\mu_n}\sum_{i=1}^{n} X_{(i)}\Big[B\Big(\frac{i-1}{n}\Big)-B\Big(\frac{i}{n}\Big)\Big].$$

Klefsjö (1988) obtains the asymptotic normality of the test statistic $T(F_n)$ by using results from Stisler (1974,1979) and Mason (1981) together with Slutsky's Theorem. Also he consider asymptotic null variance of the test statistic $T(F_n)$ as follows.

$$\sqrt{n}\,T(F_n) \;\to\; N(0,\ \sigma^2(J,\ F_0)/\mu^2) \text{ as } n\to\infty, \tag{2.8}$$

where $\sigma^2(J,\ F_0) = \frac{1}{12} - p^2(1-p)^2$.

NBUE-NWUE(p) test procedure rejects the null hypothesis of exponentiality in favor of the alternative H_1: F is NBUE-NWUE(p) at the approximate level α if

$$\frac{\sqrt{n}\,T_n}{\sqrt{1/12-p^2(1-p)^2}} > z_\alpha \qquad (2.9)$$

where z_α is the upper α-quantile of the standard normal distribution.

3. Corrected trend change points of NBUE-property

It is well known that the lognormal distribution is upside-down bathtub-shaped failure rate (UBR). If a continuous and strictly increasing life distribution function F is UBR with mean μ, then F is NBUE-NWUE distribution(Mitra and Basu (1994)). Using the results by Park (1988) it can be shown that the failure rate of the lognormal distribution with parameters μ and σ^2 changes from increasing to decreasing at $t_0(\mu, \sigma^2)$, where t_0 satisfies

$$\Phi\ ((\log t_0-\mu)/\sigma)$$
$$= 1-(1/\sqrt{2\pi})(\sigma/(\sigma^2+\log t_0-\mu))\exp(-(\log t_0-\mu)^2/2\sigma^2).$$

Thus, the lognormal distribution has the NBUE-NWUE distribution with change-point at $p = F(t_0) = \Phi((\log t_0 - \mu)/\sigma)$, *where* $F(\cdot)$ and $\Phi(\cdot)$ are the lognomal and standard normal cdf's respectively.

We solve trend change t_0 of mean residual lifetime for lognormal distribution. Let $f(x)$ is p.d.f. of lognormal distribution. Thus we find t_0 that mean residual lifetime $m(t)=\int_t^\infty\int_y^\infty f(x)\,dxdy \;/\; \int_t^\infty f(x)dx$ is equal to $m(0)$. Tedious calculations yield

$$t - e^{\frac{\sigma^2}{2}}\frac{[1-\Phi(\frac{\log t}{\sigma}-\sigma)]}{[1-\Phi(\frac{\log t}{\sigma})]}+e^{\frac{\sigma^2}{2}}=0\ . \qquad (3.1)$$

Given various σ, we can solve t_0 by numerical analysis . Table 1 represents trend change points according to various σ. Also Figure 1 presents the trend change and its point of NBUE-NWUE(p) family for σ = 1.1, 1.3.

4. Simulation and conclusions

In this section, Monte Carlo experment is performed to investigate the emprical test size and power of procedure of the proposed test statistic. Table 2 and Table 3 show the simulated empirical test size and power of the NBUE-NWUE(p) at $\alpha = 0.01, 0.05, 0.1$ against the lognormal alternatives, respectively, where the lognormal random numbers are generated for $\mu = 0$ and various choices of σ by the IMSL program. These estimates are based on 1000 replicates each. The sample sizes are $n = 10, 20(20)100$.

Table 2 indicates that convergence to normality under H_0 is somewhat slow and irregular for p = 0.1 and p = 0.9, but faster for p closer to 0.5. Similar results are also found by Park (1988).

In view of Table 3, when p is further away from 0 or 1, the test does not perform very well. However, if p is close to 0 or 1, our test performs reasonably well. Our concern is when p is relatively small. Such situations arise "Burn-in" models. Thus, it is encourging to know that our test procedure performs well when p is close to 0.

Afterward we will study NBUE-property in connection with burn-in problems and replacement optimization. Research on these subjects is under study and will be presented in the near future.

<Table 1> Corrected trend change points

Sigma (σ)	Trend change point (t)
0.9	1.1813
1.1	0.3923
1.2	0.2175
1.3	0.1168
1.5	0.0300
1.7	0.0065
1.9	0.0012
2.1	0.0002
2.3	0.0000

<Table 2>

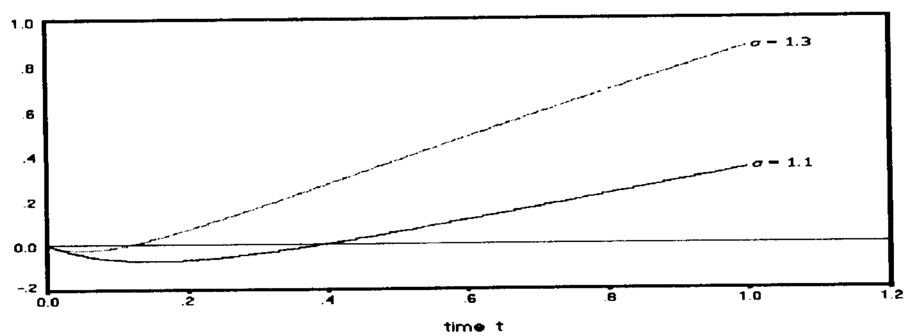
Emprical test size of the NBUE-NWUE(p) test against lognormal alternatives with Parameter $\mu = 0$ and $\sigma^2 > 0$. The observations are simulated from the exponential distribution $F(x) = 1 - \exp(-x),\ x \geq 0$.

P	α	n = 10	n = 20	n = 40	n = 60	n = 80	n = 100
0.1	0.01	0.032	0.020	0.023	0.015	0.018	0.014
	0.05	0.127	0.094	0.081	0.071	0.064	0.077
	0.10	0.216	0.170	0.137	0.136	0.127	0.140
0.3	0.01	0.023	0.023	0.014	0.015	0.018	0.013
	0.05	0.097	0.071	0.067	0.076	0.067	0.068
	0.10	0.156	0.132	0.135	0.129	0.129	0.116
0.5	0.01	0.006	0.005	0.007	0.006	0.011	0.010
	0.05	0.038	0.047	0.043	0.043	0.046	0.048
	0.10	0.092	0.100	0.101	0.085	0.096	0.106
0.7	0.01	0.002	0.007	0.011	0.003	0.008	0.011
	0.05	0.013	0.029	0.042	0.037	0.037	0.053
	0.10	0.038	0.052	0.088	0.076	0.073	0.102
0.9	0.01	0.002	0.005	0.003	0.004	0.006	0.008
	0.05	0.016	0.023	0.029	0.036	0.033	0.033
	0.10	0.031	0.052	0.058	0.073	0.066	0.077

<Table 3>

Emprical power of the NBUE-NWUE test(p) against lognormal alternatives with Parameter $\mu = 0$ and $\sigma^2 > 0$. The observations are simulated from the exponential distribution $F(x) = 1 - \exp(-x),\ x \geq 0$.

σ (p)	α	n = 10	n = 20	n = 40	n = 60	n = 80	n = 100
0.90	0.01	0.043	0.241	0.315	0.624	0.682	0.860
(0.5734)	0.05	0.209	0.541	0.605	0.863	0.888	0.967
	0.10	0.349	0.695	0.745	0.929	0.950	0.987
1.10	0.01	0.124	0.221	0.361	0.486	0.545	0.644
(0.1974)	0.05	0.254	0.365	0.542	0.695	0.729	0.807
	0.10	0.340	0.485	0.631	0.742	0.817	0.868
1.20	0.01	0.231	0.319	0.495	0.627	0.750	0.841
(0.1018)	0.05	0.385	0.505	0.667	0.791	0.862	0.928
	0.10	0.580	0.612	0.759	0.864	0.912	0.955
1.30	0.01	0.255	0.423	0.638	0.796	0.885	0.945
(0.0493)	0.05	0.459	0.586	0.796	0.916	0.959	0.977
	0.10	0.576	0.680	0.865	0.915	0.977	0.990
1.50	0.01	0.424	0.629	0.890	0.970	0.996	0.993
(0.0097)	0.05	0.623	0.797	0.949	0.988	0.999	1.000
	0.10	0.723	0.870	0.977	0.966	1.000	1.000



<Figure 1>

Trend change of the NBUE-NWUE(p) for the lognormal distribution when $\sigma = 1.1, 1.3$

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