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Admissible Hierarchical Bayes Estimators of a Multivariate Normal Mean Shrinking towards a Regression Surface[1]

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Abstract

Consider the problem of estimating a multivariate normal mean with an unknown covarience matrix under a weighted sum of squared error losses. We first provide hierarchical Bayes estimators which shrink the usual (maximum likelihood, uniformly minimum variance unbiased) estimator towards a regression surface and then prove the admissibility of these estimators using Blyth's (1951) method.

1. Introduction

Recently there has been much discussion of the respective merits of bayesian approaches to statistics. Hierarchical Bayes (HB) method is becoming increasingly popular in Bayesian, especailly in the context of simultaneous estimation of several parameters. The HB procedure models the prior distribution in stages. In the first stage, conditional on $\Lambda=\lambda$, θ_i's are independently identically distributed with a prior $\Pi_0(\lambda)$. In the second stage, a prior distribution (often improper) is assigned to Λ. This is an examples of a two stage prior. The idea can be generalized to multistage priors. The term hierarchical Bayes was first used by Good (1965). Lindley and Smith (1972) called such prior multistage priors. Also, Lindley and Smith (1972) developed Bayesian alternative to least squares estimates for the linear model within the framework of a hierarchical prior structure. Smith (1973) extened the use of the general Bayesian linear Model to estimation of parameters in third stage of the hierarchy, as well as the second stage. Tiao and Tan (1965) utilize Bayesian methods to analyze random effect models in the analysis of variance. Strawderman (1971) proposed the HB procedure in estimating the mean vector $\boldsymbol{\theta}$ of multivariate normal distribution with covariance matrix,

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which is a multiple of the identity matrix under the sum of squared error loss. Goel and De Groot (1981) illustrated the Bayesian analysis of covariance components for univariate normal model and general linear hierarchical model. Faith (1978) generalized the HB approach of strawderman (1971). Berger and Robert (1990) obtained subjective hierarchical Bayes estimator of a multivariate normal mean. George, et al (1994) proposed Fully Bayesian hierarchical analysis for exponential familes via Monte carlo computation. Nagi and stroud (1994) obtained Hierarchical Bayes simultaneous estimator of poisson means.

Let conditional on $\boldsymbol{\theta}, r, \boldsymbol{b}$, and a, $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, r^{-1}I_p)$, and let the conditional proper prior distribution of $\boldsymbol{\theta}$ given $r, \boldsymbol{b}$, and a be $N_p(Z\boldsymbol{b}, (ar)^{-1}I_p)$. Let $\boldsymbol{b}$ have marginally the improper uniform distribution over R^q and r have the density r^{α}, $\alpha > 0$, over the interval $(0, \infty)$.

The purpose of this thesis is to produce HB estimators of $\boldsymbol{\theta}$ under a weighted sum of squared error losses $L(\boldsymbol{\theta}, r, d) = r(\boldsymbol{\theta} - \boldsymbol{d})'(\boldsymbol{\theta} - \boldsymbol{d})$ and to verify the admissibility of these estimators.

2. Hierarchical Bayes Estimators

In this Section, we derive HB estimators when a is known and a is unknown, respectively.

2.1 Hierarchical Bayes estimator for known a

We now proceed to find the HB estimator of $\boldsymbol{\theta}$, in the model

(i) conditional on $\boldsymbol{\theta}, r, \boldsymbol{b}$, $\boldsymbol{X} \sim N_p(\boldsymbol{\theta}, r^{-1}I_p)$;

(ii) conditional on $r, \boldsymbol{b}$, $\boldsymbol{\theta} \sim N_p(Z\boldsymbol{b}, (ar)^{-1}I_p)$, where Z is an $p \times q$ known regression matrix and $\boldsymbol{b}$ is an $q \times 1$ unknown vector;

(iii) marginally $\boldsymbol{b}$, and r are independently distributed with $\boldsymbol{b}$ uniform over R^q, and $r \sim r^{\alpha}$, $\alpha > 0$, over $(0, \infty)$.

Then joint (improper) density of $\boldsymbol{X}$, $\boldsymbol{\theta}$, $\boldsymbol{b}$, and r

$$f(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{b}, r)$$

$$\propto r^{(p+\alpha)} \exp[-\frac{r}{2}(\boldsymbol{x} - \boldsymbol{\theta})'(\boldsymbol{x} - \boldsymbol{\theta})]$$

$$\cdot \exp[-\frac{ar}{2}\{[b-(ZZ)^{-1}Z\theta]' ZZ[b-(ZZ)^{-1}Z\theta]\}]$$

$$\cdot \exp[-\frac{ar}{2}\{\theta'\theta - \theta' Z(ZZ)^{-1}Z\theta]\}] \tag{2.1}$$

Also, the joint (improper) density of X, θ, and r becomes

$$f(x, \theta, r)$$

$$\propto r^{(p+\alpha+\frac{q}{2})} \exp[-\frac{r}{2}\{(x-\theta)'(x-\theta)+a[\theta'\theta - Z(ZZ)^{-1}Z\theta]\}]$$

$$= r^{(p+\alpha+\frac{q}{2})} \exp[-\frac{r}{2}\{x'x-2x'\theta+\theta'\theta+a[\theta'(I-Z(ZZ)^{-1}Z)\theta]\}]$$

$$\propto \exp[-\frac{r}{2}[\{\theta-[a(I-Z(ZZ)^{-1}Z)+I]^{-1}x\}'$$

$$\cdot\{a(I-Z(ZZ)^{-1}Z)+I\}$$

$$\cdot\{\theta-[a(I-Z(ZZ)^{-1}Z)+I]^{-1}x\}] \quad . \tag{2.2}$$

Hence, conditional on r and x, the distribution of θ is

$$N([a(I_p-Z(ZZ)^{-1}Z)+I_p]^{-1}x,\ r[a(I_p-Z(ZZ)^{-1}Z)+I_p]^{-1}) \quad .$$

Therefore, the posterior mean of θ given r and x is given by

$$E(\theta|r, x)=\{a[I-Z(ZZ)^{-1}Z]+I\}^{-1}x \quad . \tag{2.3}$$

By using the double expectations, we get

$$\begin{aligned} E(\theta|x) &= E[E(\theta|r, x)|x] \\ &= E\{[a(I-Z(ZZ)^{-1}Z)+I]^{-1}x \mid x\} \\ &= [a(I-Z(ZZ)^{-1}Z)+I]^{-1}x \quad . \end{aligned} \tag{2.4}$$

Since,

$$[I+a(I-Z(ZZ)^{-1}Z)]^{-1} = \frac{1}{a+1}I+\frac{a}{a+1}Z(ZZ)^{-1}Z. \tag{2.5}$$

Hence, from (2.4) and (2.5), HB estimator of θ is given by

$$\begin{aligned} \hat{\theta}_{HB} &= E(\theta|x) \\ &= \left\{\frac{1}{a+1}I+\frac{a}{a+1}Z(ZZ)^{-1}Z\right\}x. \end{aligned} \tag{2.6}$$

2.2 Hierarchical Bayes estimator for unknown a

In this section, we derive a HB estimator for estimating multivariate normal mean using the loss $L(\boldsymbol{\theta}, r, \boldsymbol{d}) = r(\boldsymbol{\theta}-\boldsymbol{d})'(\boldsymbol{\theta}-\boldsymbol{d})$ in the model

(i) conditional on $\boldsymbol{\theta}$, r, $\boldsymbol{b}$, and a, $X \sim N_p(\boldsymbol{\theta}, r^{-1}I_p)$;

(ii) conditional on r, $\boldsymbol{b}$, and a, $\boldsymbol{\theta} \sim N_p(Z\boldsymbol{b}, (ar)^{-1}I_p)$, where Z is a known $p\times q$ regression matrix, $\boldsymbol{b}$ is a $q\times 1$ unknown vector and a is unknown;

(iii) $\boldsymbol{b}$, r, and a are marginally independently distributed with B uniform over R^q, $r \sim r^{\alpha}$, over $(0,\infty)$, and a has the Type II beta distribution with the density $g_0(a) = a^{q-1}(1+a)^{-(p+q)}$, $p>0$, $q>0$, $\alpha < q-1$.

Theorem 2.1 For the hierarchical model given in the above, the hierarchical Bayes estimator $\hat{\boldsymbol{\theta}}_{HB}$ is

$$\hat{\boldsymbol{\theta}}_{HB} = E(\boldsymbol{\theta}|\boldsymbol{x})$$

$$= \frac{p}{p+q-\alpha-1}\boldsymbol{x} + \frac{q-\alpha-1}{p+q-\alpha-1} Z(Z'Z)^{-1}Z'\boldsymbol{x}.$$

Proof. The joint (improper) density of X, $\boldsymbol{\theta}$, $\boldsymbol{b}$, a, and r is given by

$$f(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{b}, a, r)$$

$$\propto r^{p}\exp[-\frac{r}{2}(\boldsymbol{x}-\boldsymbol{\theta})'(\boldsymbol{x}-\boldsymbol{\theta})](ar)^{\frac{p}{2}}\exp[-\frac{ar}{2}|\boldsymbol{\theta}-Z\boldsymbol{b}|^2]$$

$$\cdot a^{q-1}(1+a)^{-(p+q)}r^{\alpha}$$

$$= r^{(p+\alpha)}a^{(\frac{p}{2}+q-1)}(1+a)^{-(p+q)}\exp[-\frac{r}{2}(\boldsymbol{x}-\boldsymbol{\theta})'(\boldsymbol{x}-\boldsymbol{\theta})]$$

$$\cdot \exp[-\frac{ar}{2}\{(\boldsymbol{b}-(ZZ)^{-1}Z\boldsymbol{\theta})'ZZ(\boldsymbol{b}-(ZZ)^{-1}Z\boldsymbol{\theta})\}]$$

$$\cdot \exp[-\frac{ar}{2}\boldsymbol{b}'\boldsymbol{\theta} + \frac{ar}{2}\boldsymbol{\theta}'Z(ZZ)^{-1}Z\boldsymbol{\theta})] \quad . \tag{2.7}$$

Now, integrating with respect to $\boldsymbol{b}$ in (2.7), we find that the joint (improper) density of X, $\boldsymbol{\theta}$, a, and r is given by

$$f(\boldsymbol{x}, \boldsymbol{\theta}, a, r)$$

$$\propto r^{(p+\alpha-\frac{q}{2})} a^{(\frac{p}{2}+\frac{q}{2}-1)}(1+a)^{-(p+q)} \exp[-\frac{r}{2}\{x'x-2x'\theta+\theta'\theta\}]$$

$$\cdot \exp[-\frac{ar}{2}\{\theta'(I-Z(Z'Z)^{-1}Z')\theta]\}]$$

$$= r^{(p+\alpha-\frac{q}{2})} a^{(\frac{p}{2}+\frac{q}{2}-1)}(1+a)^{-(p+q)} \exp[-\frac{r}{2} x'x]$$

$$\cdot \exp[-\frac{r}{2}[\{\theta-[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x\}'\{a(I-Z(Z'Z)^{-1}Z')+I\}$$

$$\{\theta-[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x\}]]$$

$$\cdot \exp[-\frac{r}{2} x'[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x] \tag{2.8}$$

Therefore, conditional on $r, a,$ and x,

$$\theta \sim N_p([a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x,\ r[a(I-Z(Z'Z)^{-1}Z')+I])$$

Hence, the posterior mean of θ, for given $r, a,$ and x becomes

$$E(\theta|r,a,x)=[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x \tag{2.9}$$

And by using triple expectations, we get the posterior mean of θ given x as

$$\begin{aligned} E(\theta|x) &= E[E\{E(\theta|r,a,x)|a,x\}|x] \\ &= E[E\{[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x\ |a,x\}|x] \\ &= E[[a(I-Z(Z'Z)^{-1}Z')+I]^{-1} x\ |x] \\ &= E[[\frac{1}{a+1} I + \frac{a}{a+1} Z(Z'Z)^{-1}Z']x|\ x] \\ &= E(\frac{1}{a+1}|x)x + E(\frac{a}{a+1}|x)Z(Z'Z)^{-1}Z'x. \end{aligned} \tag{2.10}$$

Also, the joint (improper) density of X, θ, a, and r is given by (2.8), and then integrating (2.8) with respect to θ gives the joint density of X, a, and r

$$f(x,r,a)$$

$$\propto r^{(\frac{p}{2}+\alpha-\frac{q}{2})} a^{(\frac{p}{2}+\frac{q}{2}-1)}(1+a)^{-(p+q)}$$

$$|\{a(I-Z(Z'Z)^{-1}Z')+I\}^{-1}|^{\frac{1}{2}}$$

$$\cdot \exp[-\frac{r}{2} x'\{I-[a(I-Z(Z'Z)^{-1}Z')+I]^{-1}\}x]$$

$$= r^{(\frac{p}{2}+\alpha-\frac{q}{2})} a^{(\frac{p}{2}+\frac{q}{2}-1)} (1+a)^{-(p+q)} (1+a)^{\frac{1}{2}(q-p)}$$

$$\cdot \exp[-\frac{r}{2} x'\{I-[a(I-Z(Z'Z)^{-1}Z')+I]^{-1}\}x] \tag{2.11}$$

If $E=\{a(I-Z(Z'Z)^{-1}Z')+I\}^{-1}$, then

$$|E^{-1}| = |a(I-Z(Z'Z)^{-1}Z')+I|$$

$$= \frac{1}{|\frac{1}{a}(Z'Z)|} \begin{vmatrix} \frac{1}{a}(Z'Z) & Z' \\ Z & (a+1)I \end{vmatrix}$$

$$= \frac{1}{|\frac{1}{a}(Z'Z)|} |(a+1)I| \, |\frac{1}{a}(Z'Z)-Z'(a+1)I^{-1}Z|$$

$$= (1+a)^{q-p}.$$

Now integrating (2.11) with respect to r and using the Gamma distribution, we get the joint density of X and a

$$f(x, a)$$

$$\propto a^{(\frac{1}{2}p+\frac{1}{2}q-1)} (1+a)^{(-\frac{3}{2}p-\frac{1}{2}q)} \Gamma(\frac{1}{2}p+\alpha-\frac{1}{2}q+1)$$

$$\cdot [x'\{I-[a(I-Z(Z'Z)^{-1}Z')+I]^{-1}\}x]^{-(\frac{1}{2}p+\alpha-\frac{1}{2}q+1)}$$

$$= a^{(\frac{1}{2}p+\frac{1}{2}q-1)} (1+a)^{(-\frac{3}{2}p-\frac{1}{2}q)} \Gamma^{(\frac{1}{2}p+\alpha-\frac{1}{2}q+1)}$$

$$\cdot (\frac{a}{a+1})^{-(\frac{1}{2}p+\alpha-\frac{1}{2}q+1)} [x'\{I-Z(Z'Z)^{-1}Z'\}x]^{-(\frac{1}{2}p+\alpha-\frac{1}{2}q+1)} \quad . \tag{2.12}$$

Then, the conditional density of a given x is given by

$$f(a|x) \propto a^{(q-\alpha-2)} (1+a)^{(-p-q+\alpha+1)}. \tag{2.13}$$

From (2.13), it follows that

$$E(\frac{a}{a+1}|x) = \frac{\int_0^\infty (\frac{a}{a+1}) a^{(q-\alpha-2)} (1+a)^{(-p-q+\alpha+1)} da}{\int_0^\infty a^{(q-\alpha-2)} (1+a)^{(-p-q+\alpha+1)} da}$$

$$=\frac{\frac{\Gamma(q-\alpha)\Gamma(p)}{\Gamma(q-\alpha+p)}\cdot\int_0^1\frac{\Gamma(q-\alpha+p)}{\Gamma(q-\alpha)\Gamma(p)}b^{(q-\alpha)-1}(1-b)^{p-1}db}{\frac{\Gamma(q-\alpha-1)\Gamma(p)}{\Gamma(q-\alpha-1+p)}\cdot\int_0^1\frac{\Gamma(q-\alpha-1+p)}{\Gamma(q-\alpha-1)\Gamma(p)}b^{(q-\alpha-1)-1}(1-b)^{p-1}db}$$

$$=\frac{q-\alpha-1}{p+q-\alpha-1}. \tag{2.14}$$

From (2.10) and (2.14), the posterior mean of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ becomes

$$E(\boldsymbol{\theta}|\boldsymbol{x})=E(\frac{1}{a+1}|\boldsymbol{x})\boldsymbol{x}+E(\frac{a}{a+1}|\boldsymbol{x})Z(Z'Z)^{-1}Z'\boldsymbol{x}$$

$$=\frac{p}{p+q-\alpha-1}\boldsymbol{x}+\frac{q-\alpha-1}{p+q-\alpha-1}Z(Z'Z)^{-1}Z'\boldsymbol{x}. \tag{2.15}$$

Hence, the Hierarchical Bayes estimator $\hat{\hat{\boldsymbol{\theta}}}_{HB}$ is given by

$$\hat{\hat{\boldsymbol{\theta}}}_{HB}=E(\boldsymbol{\theta}|\boldsymbol{x})$$

$$=\frac{p}{p+q-\alpha-1}\boldsymbol{x}+\frac{q-\alpha-1}{p+q-\alpha-1}Z(Z'Z)^{-1}Z'\boldsymbol{x}. \tag{2.16}$$

3. Admissible Hierarchical Bayes Estimators

The following theorem proves admissibility of $\hat{\boldsymbol{\theta}}_{HB}$ in (2.6) under the sum of squared error losses $L_0(\boldsymbol{\theta},d)=|\boldsymbol{\theta}-d|^2$.

Theorem 3.1 Suppose that r is known the estimator

$$\hat{\boldsymbol{\theta}}_{HB}=(\frac{1}{a+1}I_p+\frac{a}{a+1}Z(Z'Z)^{-1}Z')X,\ a>0$$

is admissible under the sum of squared error losses.

Proof. Consider the sequence of priors $\{\pi_n\}$ for $\boldsymbol{\theta}$

where π_n is $N_p(\mathbf{0},r^{-1}B_n)$ with

$$B_n^{-1}=(a+\frac{1}{n})I_p-aZ(Z'Z)^{-1}Z'. \tag{3.1}$$

Then the posterior distribution of $\boldsymbol{\theta}$ given $X=\boldsymbol{x}$ is $N_p(D_n\boldsymbol{x},\ r^{-1}D_n)$, where

$$\begin{aligned} D_n^{-1} &= I_p + B_n^{-1} \\ &= I_p + (a+\frac{1}{n})I_p - aZ(Z'Z)^{-1}Z' \\ &= (1+a)I_p - aZ(Z'Z)^{-1}Z' + \frac{1}{n}I_p \\ &= D^{-1} + \frac{1}{n}I_p, \end{aligned} \tag{3.2}$$

where $D = \frac{1}{a+1}I_p + \frac{a}{a+1}Z(Z'Z)^{-1}Z'$.

It suffices to show that

$$\Delta_n = [B(\pi_n, DX) - B(\pi_n, D_nX)] \to 0 \tag{3.3}$$

as $n \to \infty$, where $B(\pi, \delta)$ is the Bayes risk of an estimator δ of θ with respect to a prior π.

The marginal distribution of X with respect to π_n is

$$N_p(0,\ r^{-1}[I_p - (I_p + B_n^{-1})^{-1}]^{-1})$$

Hence,

$$\begin{aligned} \Delta_n = (2\pi)^{-p} \int & |r^{-1}D_n|^{-\frac{1}{2}} (\theta - Dx)'(\theta - Dx) \\ & \exp[-\frac{r}{2}(\theta - D_nx)'D_n^{-1}(\theta - D_nx)] \\ & |r^{-1}[I_p - (I_p + B_n^{-1})^{-1}]^{-1}|^{-\frac{1}{2}} \\ & \exp[-\frac{r}{2}x'[I_p - (I_p + B_n^{-1})^{-1}]x]d\theta\, dx \\ - (2\pi)^{-p} \int & |r^{-1}D_n|^{-\frac{1}{2}} (\theta - D_nx)'(\theta - D_nx) \\ & \exp[-\frac{r}{2}(\theta - D_nx)'D_n^{-1}(\theta - D_nx)] \\ & |r^{-1}[I_p - (I_p + B_n^{-1})^{-1}]^{-1}|^{-\frac{1}{2}} \\ & \exp[-\frac{r}{2}x'[I_p - (I_p + B_n^{-1})^{-1}]x]d\theta\, dx \end{aligned}$$

$$= (2\pi)^{-p} r^{p} \int |D_n|^{-\frac{1}{2}} y'y \exp[-\frac{r}{2}(y+Dx-D_nx)'D_n^{-1}(y+Dx-D_nx)]$$

$$\cdot\ |I_p-D_n|^{\frac{1}{2}} \exp[-\frac{r}{2} x'(I_p-D_n)x]\, dy\, dx$$

$$-(2\pi)^{-p} r^{p} \int |D_n|^{-\frac{1}{2}} Z'Z \exp[-\frac{r}{2} Z'D_n^{-1}Z]$$

$$|I_p-D_n|^{\frac{1}{2}} \exp[-\frac{r}{2} x'(I_p-D_n)x]\, dZ\, dx$$

$$= (2\pi)^{-p} r^{p} |D_n|^{-\frac{1}{2}} |I_p-D_n|^{\frac{1}{2}}$$

$$\int y'y \Big[\exp[-\frac{r}{2}(y+Dx-D_nx)'D_n^{-1}(y+Dx-D_nx)]$$

$$-\exp[-\frac{r}{2} y'D_n^{-1}y]\Big] \exp[-\frac{r}{2} x'(I_p-D_n)x]\, dy\, dx$$

$$= (2\pi)^{-\frac{p}{2}} r^{\frac{p}{2}} |I_p-D_n|^{\frac{1}{2}}$$

$$\int \{(Dx-D_nx)'(Dx-D_nx) + tr(r^{-1}D_n) - tr(r^{-1}D_n)\}$$

$$\exp[-\frac{r}{2} x'(I_p-D_n)x]dx$$

$$= (2\pi)^{-\frac{p}{2}} \int [x'(D-D_n)'(D-D_n)x] r^{\frac{p}{2}} |I_p-D_n|^{\frac{1}{2}}$$

$$\exp[-\frac{r}{2} x'(I_p-D_n)x]\, dx$$

$$= tr[r^{-1}(I_p-D_n)^{-1}(D-D_n)'(D-D_n)]. \tag{3.4}$$

Now, note that

$$D_n = (D^{-1} + \frac{1}{n} I_p)^{-1}$$

$$= \frac{1}{a+1+\frac{1}{n}} I_p + \frac{a}{(1+\frac{1}{n})(a+1+\frac{1}{n})} Z(Z'Z)^{-1}Z', \tag{3.5}$$

$$I_p - D_n = \frac{a+\frac{1}{n}}{a+1+\frac{1}{n}} I_p - \frac{a}{(1+\frac{1}{n})(a+1+\frac{1}{n})} Z(Z'Z)^{-1}Z', \tag{3.6}$$

and

$$(I_p - D_n)^{-1} = (na+n+1)\left(\frac{1}{na+1} I_p + \frac{n^2 a}{(na+1)(na+n+1)} Z(Z'Z)^{-1}Z'\right). \tag{3.7}$$

Hence, a lengthy calculation gives

$$\Delta_n = r^{-1} tr[(I_p - D_n)^{-1}(D-D_n)'(D-D_n)] \rightarrow 0$$

$$= r^{-1}\left\{\frac{1}{p_2(n)} p + \left(\frac{p_2(n)}{p_4(n)} + \frac{p_2(n)}{p_3(n)} + \frac{p_4(n)}{p_5(n)}\right) q\right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{3.8}$$

where $p_k(n)$ is the polynomial in n with degrees k.

By Blyth's (1951) method (see Berger 1985, p547) $\hat{\theta}_{HB}$ is admissible.

Theorem 3.2 $\hat{\theta}_{HB} = \left(\frac{1}{1+a} I_p + \frac{a}{1+a} Z(Z'Z)^{-1}Z'\right) X$

$$= DX$$

is admissible under the sum of squared error losses $L(\theta, r, d) = r|\theta - d|^2$ when r is unknown.

Proof. First, note that for every fixed r_0, $\hat{\theta}_{HB}$ is admissible under the sum of squared error losses by theorem 3.1.

Suppose $\hat{\theta}_{HB}$ is inadmissible. Then there exists another estimator $\delta'(X)$ such that

$$R(\theta, r, \delta) \le R(\theta, r, \hat{\theta}_{HB}) \quad \text{for all } (\theta, r)$$

with strict inequality for some (θ, r), say (θ_0, r_0) where $R(\theta, r, \delta)$ dominates the risk of δ under the sum of squared error losses. Thus δ' dominates $\hat{\theta}_{HB}$ when the parameter space is (θ, r_0), which contradicts to the admissibility of $\hat{\theta}_{HB}$ for every fixed r_0.

Remark 3.1 $\hat{\theta}_{HB}$ is also admissible for θ under the weighted sum of squared error losses $L(\theta, r, d) = r|\theta - d|^2$ since the weighting factor r does not affect the admissibility or inadmissibility of an estimator.

Theorem 3.3 $\hat{\boldsymbol{\theta}}_{HB}$ is admissible for $\boldsymbol{\theta}$ (2.16) under the sum of squared error losses when r is known.

Proof. Similar with the proof of Theorem 3.1 taking $a = \frac{q-\alpha-1}{p} > 0$.

Theorem 3.4 $\hat{\boldsymbol{\theta}}_{HB}$ is admissible for $\boldsymbol{\theta}$ under the sum of squared error losses when r is unknown.

Proof. Similar with the proof of Theorem 3.2.

Remark 3.2 $\hat{\boldsymbol{\theta}}_{HB}$ is also admissible for $\boldsymbol{\theta}$ under the weighted sum of squared error losses $L(\boldsymbol{\theta}, r, \boldsymbol{d}) = r|\boldsymbol{\theta} - \boldsymbol{d}|^2$ since the weighting factor r does not affect the admissibility or inadmissibility of an estimator.

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