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Stochastic Comparisons of Order Statistics under Non-standard Conditions

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Abstract

This paper deals with the stochastic comparisons of order statistics from independent but nonidentically distributed (i.n.i.d) variates. And we consider order statistics under positive dependence, negative dependence, and exchangeability.

1. Introduction

Given a random sample, $X_1, X_2, \cdots, X_n$, we can arrange the X's in ascending order of magnitude and then write $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$. We call $X_{r:n}$ the rth order statistics ($r = 1, \cdots, n$).

If $X_1, X_2, \cdots, X_n$ is a sample of size n from a life distribution F, then the order statistics may be interpreted as the successive failure times of the components of a system. With this interpretation, the rth order statistic is the failure time of a k-out-of-n system of identical components, where $k=n-r+1$. (A system of n components is called a k-out-of-n system if it functions if and only if at least k components function.) Hence $P\{X_{r:n}>t\}$ is the reliability of a k-out-of-n system at time t. The special cases $k=n$ and $k=1$ correspond respectively to series and parallel systems.

Consider the situation of two k-out-of-n systems of independently failing identical components whose lifetimes X and Y have c.d.f.'s F and G. Then, it is well known that if the survival probability of an X-component is greater than that of Y-component (i.e., $P\{X>t\} \ge P\{Y>t\}$ for all t), then $P\{X_{r:n}>t\} \ge P\{Y_{r:n}>t\}$ for all t and $r = 1, 2, \cdots, n$. In other words, if X is stochastically larger than Y (i.e., $X \underset{st}{\ge} Y$), then $X_{r:n}$ is stochastically larger than $Y_{r:n}$ for $r = 1, 2, \cdots, n$. This implies $E[X_{r:n}] \ge E[Y_{r:n}]$ for $r = 1, 2, \cdots, n$ if $E[X] < \infty$ and $E[Y] < \infty$. Stochastic ordering is a very strong kind of ordering. Consequently, many other

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weaker orderings have been studied.

Recently, Kim and David (1990) and Kim (1993) reviewed and systematized the dependent structure and stochastic comparisons of order statistics from univariate populations. David (1985) collected and systematized some aspects of the treatment of order statistics when these arise from non-i.i.d. variates $X_1, X_2, \cdots, X_n$, when n is fixed. Here we will consider the stochastic comparisons of order statistics from two samples under non-standard conditions, when they are stochastically related. (*-shaped ordering, majorization ordering, stochastic ordering, peakedness ordering)

In section 2, we first treat the stochastic comparisons of order statistics from indenpendent but nonidentically distributes (i.n.i.d.) variates as an extension of Kim (1993). In section 3, we then consider order statistics under positive dependence, negative dependence, and exchangeability.

2. I.N.I.D. Case

Marshall and Proschan (1970) showed that for i.n.i.d. nonnegative variates $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ with c.d.f.s $F_1, F_2, \cdots, F_n$ and $G_1, G_2, \cdots, G_n$ for which $F_i \underset{*}{<} G_i$ (i.e. $\Leftrightarrow \frac{G_i^{-1}F_i(x)}{x} \uparrow$ w.r.t. for $x \geq 0$; *-shaped ordering) with pairwise common means μ_i, $i = 1, \cdots, n$, one has

$$E[X_{1:n}] \geq E[Y_{1:n}] \tag{2.1}$$

$$\text{and } E[X_{n:n}] \leq E[Y_{n:n}] . \tag{2.2}$$

Since $X_{1:n}$ ($X_{n:n}$) represent the lifetimes of series (parallel) systems with n independent components, the i-th having lifetime X_i ($i=1, \cdots, n$), we see that the X-series has larger expected life than the Y-series system, and that the result is reversed for parallel systems.

Definition 2.1 A vector $\underline{b} = (b_1, \cdots, b_n)$ majorizes the vector $\underline{a} = (a_1, \cdots, a_n)$ if $\sum_{i=k}^{n} b_{(i)} \geq \sum_{i=k}^{n} a_{(i)}$ for $k = 2, \cdots, n$ and $\sum_{i=1}^{n} b_{(i)} = \sum_{i=1}^{n} a_{(i)}$, where the $b_{(i)}$'s and $a_{(i)}$'s are the components of $\underline{b}$ and $\underline{a}$, respectively, in ascending order. Write $\underline{b} \overset{m}{>} \underline{a}$. A real-valued funtion ϕ defined on R^n is said to be Schur-convex(Schur-concave) if $\underline{b} \overset{m}{>} \underline{a} \Rightarrow \phi(\underline{b}) \leq (\geq) \phi(\underline{a})$.

Motivated by this definition, Boland and Proschan (1986) defined the majorization ordering.

Definition 2.2 [Boland and Proschan (1986)] Let X and Y be nonnegative random variables with c.d.f.'s F and G. G $\overset{m}{>}$ F (m for majorization) if

$$\int_x^\infty [1-G(t)]\,dt \geq \int_x^\infty [1-F(t)]\,dt$$

$$\text{and} \int_0^\infty [1-G(t)]\,dt = \int_0^\infty [1-F(t)]\,dt = \mu < \infty.$$

Boland and Proschan (1986) obtained the following inequalities: Assume that $F_i \overset{m}{<} G_i$ for i=1, …, n,

$$\int_x^\infty P[Y_{n:n} + \cdots + Y_{k:n}]\,dt \geq \int_x^\infty P[X_{n:n} + \cdots + X_{k:n}]\,dt \tag{2.3}$$

for all $x \geq 0$ and k=1,…,n,

$$\text{and } (E(Y_{1:n}), \ldots, E(Y_{n:n})) \overset{m}{>} (E(X_{1:n}), \ldots, E(X_{n:n})). \tag{2.4}$$

Hence if $F_i \overset{m}{>} G_i$,i=1, …, n, then (2.1) and (2.2) are satisfied by (2.4).

Let $X \underset{st}{\leq} Y$ be $P[X \geq x] \leq P[Y \geq x]$ for all x (stochastic ordering). Let $\overline{F}_i(x) = P\{X_i > x\}$, $\overline{G}_i(x) = P\{Y_i > x\}$, i = 1, 2, …, n. Marshall and Olkin (1979,p.351) showed that if $(\overline{G}_1(x), \ldots, \overline{G}_n(x)) \overset{m}{>} (\overline{F}_1(x), \ldots, \overline{F}_n(x))$, $-\infty < x < \infty$, then $Y_{n:n} \underset{st}{\geq} X_{n:n}$ and $Y_{1:n} \underset{st}{\leq} X_{1:n}$. (This implies (2.1) and (2.2).)

Sen (1970) obtained the above result in the special case that $G_1(x) = \cdots = G_n(x) = \sum_{i=1}^{n} \frac{F_i(x)}{n}$ for all x. Let $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be i.n.i.d. variates. Then, Marshall and Olkin (1979, p.325) showed that if $E[\phi(\underline{X})] \leq E[\phi(\underline{Y})]$ for all increasing (decreasing) Schur-convex functions $\phi(\underline{x})$, then $Y_{n:n} \underset{st}{\geq} X_{n:n}$ and $Y_{1:n} \underset{st}{\leq} X_{1:n}$.

Lemma 2.1 [Ross (1983, p.356)] If $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ be i.n.i.d. variates with $X_i \underset{st}{\leq} Y_i$ (i = 1, …, n), then one has for any increasing function ϕ, $\phi(X_1, \ldots, X_n) \underset{st}{\leq} \phi(Y_1, \ldots, Y_n)$. It follows that $X_{r:n} \underset{st}{\leq} Y_{r:n}$, r = 1, …, n.

Theorem 2.1 Let $X_i(Y_i)$, the life length of components i, have absolutely continuous distribution $F_i(G_i)$ with $F_i(0) = G_i(0) = 0$ and $f_i(0) \geq g_i(0) > 0$ for each i = 1, ⋯, n. Let X_1, ⋯, X_n (Y_1, ⋯, Y_n) be independent. Then if $F_i \underset{*}{<} G_i$ (i.e. $\Leftrightarrow \frac{G_i^{-1}F_i(x)}{x} \uparrow$ w.r.t. x) for each i = 1, ⋯, n, then $X_{r:n} \underset{st}{\leq} Y_{r:n}$,r=1, ⋯, n. (this implies $E[X_{r:n}] \leq E[Y_{r:n}]$.)

Proof. $F \underset{*}{<} G \Leftrightarrow \frac{G^{-1}F(x)}{x} \uparrow$ w.r.t. x,

$$\Leftrightarrow \frac{f(x)}{g[\, G^{-1}F(x)]} \geq \frac{G^{-1}F(x)}{x} \text{ for } x>0.$$

Since $\lim_{x \to 0^+} \frac{G^{-1}F(x)}{x} = \frac{f(0)}{g(0)} \geq 1$ (by L'Hospital rule),

$$\Rightarrow \frac{f(x)}{g[\, G^{-1}F(x)]} \geq 1 \text{ for } x>0$$

$$\Rightarrow f[\, F^{-1}(\alpha)] \geq g[\, G^{-1}(\alpha)] \quad \text{for } 0 < \alpha < 1 \text{ with } F(x) = \alpha.$$

$$\Leftrightarrow F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \text{ for any } 0 \leq \alpha \leq \beta \leq 1.$$

$\Rightarrow F \underset{st}{\leq} G$, when F and G are distributions of nonnegative random variables.

Hence applying Lemma 2.1, we have $X_{r:n} \underset{st}{\leq} Y_{r:n}$, r = 1, ⋯, n.

Lemma 2.2 For any a_i and b_i such that $0 < b_i \leq a_i < \frac{1}{2}$ for each i = 1, ⋯, n,

$$\left[\prod_{i=1}^{n}\left(\frac{1}{2} - a_i\right) + \prod_{i=1}^{n}\left(\frac{1}{2} + a_i\right) \right] \geq \left[\prod_{i=1}^{n}\left(\frac{1}{2} - b_i\right) + \prod_{i=1}^{n}\left(\frac{1}{2} + b_i\right) \right] \quad \text{for } n=1, \cdots.$$

Proof. The expansion of $\prod_{i=1}^{n}\left(\frac{1}{2} - a_i\right) + \prod_{i=1}^{n}\left(\frac{1}{2} + b_i\right)$ is $2^{-(n-1)}$ plus a sum of positive terms in a_1, ⋯, a_n. Each of these terms is at least as large as the corresponding term in b_1, ⋯, b_n. The result follows.

Theorem 2.2. Let X_i and Y_i be random variables symmetric about 0, with c.d.f.'s F_i and G_i, for each i=1, ⋯, n. Let $X_1, \cdots, X_n$ ($Y_1, \cdots, Y_n$) be indepedent. Then if X_i is more peaked than Y_i for each i = 1, ⋯, n (i.e. $F_i(x) \geq G_i(x)$ for all x>0 and i = 1, ⋯, n), then subject to the existence of expectations

$$E[\ X_{1:n}] \geq E[\ Y_{1:n}] \tag{2.5}$$

$$\text{and}\quad E[\ X_{n:n}] \leq E[\ Y_{n:n}]\ . \tag{2.6}$$

Proof. Since for any r.v.'s X with c.d.f. F and finite expectation,

$$E[\ X] = \int_0^{\infty}[\ 1-F(x)]\ dx - \int_0^{\infty} F(x)\,dx,$$

we have,

$$E[\ X_{1:n}] = \int_0^{\infty} P[\ \min(X_1,\ldots,X_n)>t]\ dt - \int_{-\infty}^{0} P[\ \min(X_1,\ldots,X_n)\leq t]\ dt$$

$$= \int_0^{\infty}\prod_{i=1}^{n}\overline{F_i}(t)\,dt - \int_{-\infty}^{0}[\ 1-\prod_{i=1}^{n}F_i(t)]\ dt$$

$$= \int_0^{\infty}\prod_{i=1}^{n}\overline{F_i}(t) - \int_0^{\infty}[\ 1-\prod_{i=1}^{n}F_i(t)\,dt \text{ (by symmetry about zero)}$$

$$= \int_0^{\infty}[\ \prod_{i=1}^{n}\overline{F_i}(t)+\prod_{i=1}^{n}F_i(t) - 1]\ dt$$

$$\geq \int_0^{\infty}[\ \prod_{i=1}^{n}\overline{G_i}(t) + \prod_{i=1}^{n}G_i(t) - 1]\ dt = E[\ Y_{1:n}]\ ,$$

where the inequality follow from Lemma 2.2 with $a_i = F_i(t) - \frac{1}{2}$ and $b_i = G_i(t) - \frac{1}{2}$.

Remark 2.1. (2.5) and (2.6) imply $E[Y_{n:n} - Y_{1:n}] \geq E[X_{n:n} - X_{1:n}]$. (i.e., $\underline{Y}$ has greater sample range than $\underline{X}$).

3. Dependent Case

Definition 3.1. Random variables $X_1, \cdots, X_n$ (or $\underline{X}$) with finite means and variances are said to be associated if $\mathrm{cov}(\ f(\underline{X})\ ,\ g(\underline{X})) \geq 0$ for all increasing (or decreasing) functions f and g for which the covariance exists.

Esary et al. (1967) showed that associatedness is preserved under (a) taking subsets, (b) forming unions of independent sets, (c) forming sets of nondecreasing functions, (d) taking limits in distribution. Hence if $(X_1, \cdots, X_n)$ are associated, then $\mathrm{cov}(X_{i:n}, X_{j:n}) \geq 0$ for any $i,j=1, \cdots, n$ by (c). Pitt (1977) shows that if $(X_1, \cdots, X_n)$ is multinormally distributed with $\Sigma = \|\sigma_{ij}\|^n_{i,j=1}$ such that $\sigma_{ij} \geq 0$ for any $i \neq j$, then $(X_1, \cdots, X_n)$ is associated. Hence the covariance of any two order statistics from multinormal populations with nonnegative covariances of any two random variables is nonnegative. However, we will give an example which shows that $\mathrm{cov}(X_{i:n}, X_{j:n})$ can be negative if $X_1, \cdots, X_n$ are sufficiently negatively dependent.

Example 3.1 Thigpen (1961) introduced the following transformations to generate equicorrelated standard variates

$$Y_i = (1-\rho)^{\frac{1}{2}} (X_i - \overline{X} - aX_0) \ , \ i = 1, 2, \cdots, n \tag{3.1}$$

for $-\frac{1}{n-1} \leq \rho < 1$, where $a^2 = \frac{[\ 1+(n-1)\rho]}{[\ n(1-\rho)]}$ and $X_0, X_1, \cdots, X_n$ are n+1 independent standard variates. If the X's are normal, then $(Y_1, Y_2, \cdots, Y_n)$ are identically distributed equicorrelated multinormal variates. From (3.1), we have

$$Y_{r:n} = (1-\rho)^{\frac{1}{2}} (X_{r:n} - \overline{X} - aX_0), \ r = 1, 2, \cdots, n . \tag{3.2}$$

Since $(X_{r:n} - \overline{X})$ and $\overline{X}$ are independent and $\mathrm{cov}(X_{r:n}, \overline{X}) = \frac{1}{n}$, from (3.2), we have

$$\mathrm{cov}(Y_{r:n}, Y_{s:n}) = (1-\rho)\,\mathrm{cov}(X_{r:n}, X_{s:n}) + \rho \tag{3.3}$$

For example, when n=5, if $\rho \gtreqless -0.1763903$, then $\mathrm{cov}(Y_{2:5}, Y_{4:5}) \gtreqless 0$. Hence, if $-0.25<\rho<-0.1763903$, then $\mathrm{cov}(Y_{2:5}, Y_{4:5}) < 0$.

Remark 3.1. From (3.3), the value $\rho_c(r, s ; n)$ of ρ for which $\mathrm{cov}(Y_{r:n}, Y_{s:n})=0$, is given by

$$\rho_c = \frac{-\mathrm{cov}(X_{r:n}, X_{s:n})}{1-\mathrm{cov}(X_{r:n}, X_{s:n})} .$$

For r and s fixed and sufficiently large n, one has $\rho_c < \frac{-1}{n-1}$, which means that $\mathrm{cov}(Y_{r:n}, Y_{s:n}) > 0$ for all permissible ρ. This situation is illustrated in Table 3.1. which shows negative covariances of order statistics do not exist below the underlined entry. Let $\mathrm{cov}(X_{r:n}, X_{s:n}) = \sigma_{rs:n}$.

Table 3.1. Cov($X_{r:n}$, $X_{s:n}$) and corresponding values of ρ_c(r, s ; n).

n	$\sigma_{24:n}$	ρ_c	n	$\sigma_{14:n}$	ρ_c	n	$\sigma_{15:n}$	ρ_c
5	.1499427	-.1763913	5	.1057720	-.1182830	5	.0742153	-.0801648
6	.1396641	-.1623367	6	.1024294	-.1141185	6	.0773638	-.0838508
7	.1307299	-.1503904	7	.0984869	-.1092462	7	.0765598	-.0829072
8	.1232633	-.1405933	8	.0947230	-.1046343	8	.0747650	-.0808065
9	.1170057	-.1325101	9	.0913071	-.1004818	9	.0727422	-.0784487
10	.1117016	-.1257478	10	.0882494	-.0967912	10	.0707414	-.0761266
19	.0852931	-.0952464	12	.0830687	-.0905942	16	.0613087	-.0653130
20	.0835758	-.0911977	13	.0808650	-.0879795	17	.0601272	-.0639737

Also, it can be shown that $X_1, \cdots, X_n$ are associated iff

$$P[\, \underline{X} \in A \cap B] \geq P[\, \underline{X} \in A]\ P[\, \underline{X} \in B]\ , \tag{3.4}$$

whenever A and B are open upper sets (U is an upper set if $\underline{x} \in U$ and $\underline{y} \geq \underline{x}$ imply $\underline{y} \in U$). From (3.4), if $X_1, \cdots, X_n$ are associated, then we have

$$P[\, X_1 > x_1, \cdots, X_n > x_n] = \prod_{i=1}^{n} P[\, X_i > x_i] \tag{3.5}$$

$$\text{and } P[\, X_1 \leq x_1, \cdots, X_n \leq x_n] = \prod_{i=1}^{n} P[\, X_i \leq x_i]\ . \tag{3.6}$$

Let $(Y_1, \cdots, Y_n)$ be independent random variables with the same univariate marginal distribution functions as $(X_1, \cdots, X_n)$. Then (3.5) and (3.6) imply that

$$X_{1:n} \underset{st}{\geq} Y_{1:n} \tag{3.7}$$

$$\text{and } X_{n:n} \underset{st}{\geq} Y_{1:n}\ . \tag{3.8}$$

Block et al (1985) show that if $(X_1, \cdots, X_n)$ is NDS (see Block et al (1985)), then the inequalites (3.5) and (3.6) are reversed and consequently also (3.7) and (3.8). From (3.7) and (3.8), the distribution funtions of $X_{1:n}$ and $X_{n:n}$ could be approximated by distribution functions of $Y_{1:n}$ and $Y_{n:n}$. Glaz and Johnson (1984) suggested sharper bounds than

$\prod_{i=1}^{n} P[X_i \le x_i]$,namely, for the latter, $r_2 = P[X_1 \le x_1] \prod_{i=2}^{n} P[X_i \le x_i \mid X_{i-1} \le x_{i-1}]$

and $r_3 = P[X_1 \le x_1, X_2 \le x_2] \prod_{i=3}^{n} P[X_i \le x_i \mid X_j \le x_j; j = i-2, i-1]$. Bhattacharyya (1970) shows that if $X_1, \cdots, X_n$ form an exchangeable sequence, then (3.7) and (3,8) hold for any subset of n X's. Shaked (1977) showed that if $(X_1, \cdots, X_n)$ is PDM (see Shaked (1977)), then $(X_1, \cdots, X_n)$, being exchangeable as well, (3.7) and (3.8) are satisfied. Futhermore, Shaked (1977) proved that if $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ are, respectively, PDM and i.i.d. r.v.'s having the same univariate marginal, then for any $x \in R$,

$$(F_{X_{1:n}}(x), \cdots, F_{X_{n:n}}(x)) \overset{m}{<} (F_{Y_{1:n}}(x), \cdots, F_{Y_{n:n}}(x)). \tag{3.9}$$

Note that (3.9) implies (3.7) and (3.8). Marshall and Olkin (1979, p.350) show that (3.9) is equivalent to

$$(Eg(X_{1:n}), \cdots, Eg(X_{n:n})) \overset{m}{<} (Eg(Y_{1:n}), \cdots, Eg(Y_{n:n}))$$

for all monotone functions g such that the expectation exist. Comparisons so far have made between dependent random vectors and vectors having independent components. Realising that vectors having independent components represent a special case among the class of all dependent random vectors, Shaked and Tong (1985) make a study of comparisons of dependent vectors according to the strength of dependence. Specifically, they discuss the partial orderings of exchangeable random variables by positive dependence.

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