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Simultaneous Optimization of Multiple Responses Alternatives to the Taguchi Parameter Design[1)]

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Abstract

In the Taguchi parameter design, the product-array approach using orthogonal arrays is mainly used. However, it often requires an excessive number of experiments. An alternative approach, which is called the combined-array approach, was suggested by Welch et. al. (1990) and studied by Vining and Myers (1990), Box and Jones (1992) and others. In these studies, only single response variable was considered. We propose how to simultaneously optimize multiple responses when there are correlations among responses, and when we use the combined-array approach to assign control and noise factors.

1. Introduction

The Taguchi parameter design is an approach to reducing performance variation of response values in products and processes. Products and their manufacturing processes are influenced both by control factors that can be controlled by designers and by noise factors that are difficult or expensive to control such as environmental conditions. The basic idea of parameter design is to identify, through exploiting interactions between control factors and noise factors, appropriate settings of control factors that make the system's performance robust to changes in the noise factors. Parameter design is a quality improvement technique proposed by the Japanese quality expert Taguchi (1978), which was described by Taguchi (1986, 1987), Kackar (1985), and others.

The control factors are assigned to an "inner array" , which is an orthogonal array. For each row in the inner array, the noise factors are assigned to an "outer array" , also an orthogonal array. Because the outer array is run for every row in the inner array, we call this setup a "product array" . A large number of experimental trials in Taguchi's product array may be required because the noise array is repeated for every row in the control array.

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There have been efforts for integrating Taguchi's important notion of heterogeneous variability the standard experimental design and modeling technology provided by response surface methodology. They combined control and noise factors in a single design matrix, which we call a combined array.

The combined array approach was first proposed by Welch, Yu, Kang, and Sacks (1990). The initial motivation of the combined array is the run-size saving. Related approaches were discussed by Vining and Myers (1990), Box and Jones (1992), Shoemaker, Tsui and Wu (1991), and Myers, Khuri and Vining (1992), etc. Treatment of the mean and variance responses via a constrained optimization was discussed in Vining and Myers (1990).

In many experimental situations, a number of responses are measured for a given setting of design variables. Khuri and Conlon (1981) introduced a procedure for the simultaneous optimization of multiple responses using a distance function.

2. Simultaneous Optimization of Multiple Responses

2.1 Multivariate Linear Model

Suppose the response y, depends on control variables (or factors) and noise variables. Let a set of control variables be denoted by $\underline{x}=(x_1,x_2,\cdots,x_l)'$ and a set of noise variables by $\underline{z}=(z_1,z_2,\cdots,z_m)'$. Suppose that all response functions in a multiresponse system depend on the same set of $\underline{x}$ and $\underline{z}$ and that they can be represented by second order models within a certain region of interest. Let N be the number of experimental runs and r be the number of response functions. The ith second order model is

$$y_i(\underline{x},\underline{z})=\beta_0+\underline{x}'\ \underline{\beta}_i+\underline{x}'B_i\underline{x}+\underline{z}'R_i\underline{z}+\underline{z}'\ \underline{\gamma}_i+\underline{z}'D_i\underline{x}+\varepsilon_i,\qquad i=1,2,\cdots,r, \tag{2.1}$$

where $\underline{\beta}_i$ is $l\times 1$, $\underline{\gamma}_i$ is $m\times 1$, $B_i'=B_i$ is $l\times 1$, $R_i'=R_i$ is $m\times m$, D_i is $l\times m$, which are vectors or matrices of unknown regression parameters, and ε_i is a random error associated with the ith response.

Equation (2.1) can be expressed in matrix notation as

$$\underline{y}_i=X\ \underline{\theta}_i+\ \underline{\varepsilon}_i,\qquad i=1,2,\cdots,r, \tag{2.2}$$

in which $\underline{y}_i$ is an $N\times 1$ vector of observations on the ith response, X is an $N\times p$ full column rank matrix of known constants, $\underline{\theta}_i$ is the $p\times 1$ column vector of unknown regression parameters, and $\underline{\varepsilon}_i$ is a vector of random errors associated with the ith response. We also assume that

$$E(\underline{\varepsilon}_i)=\underline{0},\ Var(\underline{\varepsilon}_i)=\sigma_{ii}I_N,\ Cov(\underline{\varepsilon}_i,\ \underline{\varepsilon}_j)=\sigma_{ij}I_N \qquad i,j=1,2,\cdots,r,\quad i\neq j.$$

The $r\times r$ matrix whose (i,j)th element is σ_{ij} will be denoted by Σ. An unbiased estimator of Σ is given by

$$\hat{\Sigma}=Y'[I_N-X(X'X)^{-1}X']Y/(N-p)\quad ,$$

where $Y=(\underline{y}_1,\underline{y}_2,\cdots,\underline{y}_r)$, and I_N is an identity matrix of order $N\times N$. The r equations given in (2.2) may be written in a compact form

$$\underline{y}=\begin{pmatrix}\underline{y}_1\\ \underline{y}_2\\ \vdots\\ \underline{y}_r\end{pmatrix}=\begin{pmatrix}X & 0 & \cdots & 0\\ 0 & X & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & X\end{pmatrix}\begin{pmatrix}\underline{\theta}_1\\ \underline{\theta}_2\\ \vdots\\ \underline{\theta}_r\end{pmatrix}+\begin{pmatrix}\underline{\varepsilon}_1\\ \underline{\varepsilon}_2\\ \vdots\\ \underline{\varepsilon}_r\end{pmatrix}=Z\underline{\theta}+\underline{\varepsilon}, \qquad (2.3)$$

where $\underline{y}$ is $rN\times 1$, Z is $rN\times rp$, $\underline{\theta}$ is $rp\times 1$, and $\underline{\varepsilon}$ is $rN\times 1$. The variance-covariance matrix of $\underline{\varepsilon}$ is

$$Var(\underline{\varepsilon})=\Sigma\otimes I=\Omega,$$

where $\otimes$ is a symbol for the direct (or Kronecker) product of matrices.

The BLUE (best linear unbiased estimator) of $\underline{\theta}$ in (2.3) is

$$\widehat{\underline{\theta}}=(Z'\Omega^{-1}Z)^{-1}(Z'\Omega^{-1}\underline{y})=(Z'Z)^{-1}Z'\underline{y}.$$

Thus, the BLUE of $\underline{\theta}$ is $\widehat{\underline{\theta}}=(\widehat{\underline{\theta}_1}',\widehat{\underline{\theta}_2}',\cdots,\widehat{\underline{\theta}_r}')'$, where $\widehat{\underline{\theta}_i}=(X'X)^{-1}X'\underline{y}_i$ is the least squares estimator of the $p\times 1$ vector of regression coefficients for the ith response model (see Huang (1970, p.188)). The variance-covariance of $\widehat{\underline{\theta}_i}$

$$Var(\widehat{\underline{\theta}})=(Z'\Omega^{-1}Z)^{-1}=(X'X)^{-1}\Sigma.$$

The prediction equation for the ith response is given by

$$\widehat{y_i}(\underline{x},\underline{z})=\underline{g}'(\underline{x},\underline{z})\,\widehat{\underline{\theta}_i}, \qquad i=1,2,\cdots,r, \qquad (2.4)$$

where $(\underline{x}', \underline{z}')'$ is the vector of coded input variables, $\underline{g}'(\underline{x}, \underline{z})$ is a vector of the same form as a row of the matrix X evaluated at the point $(\underline{x}, \underline{z})$. From (2.4) it follows that

$$Var[\hat{y}_i(\underline{x}, \underline{z})] = \underline{g}'(\underline{x}, \underline{z})(X'X)^{-1}\underline{g}(\underline{x}, \underline{z})\sigma_{ii}, \quad i = 1, 2, \cdots, r,$$

$$Cov[\hat{y}_i(\underline{x}, \underline{z}), \hat{y}_j(\underline{x}, \underline{z})] = \underline{g}'(\underline{x}, \underline{z})(X'X)^{-1}\underline{g}(\underline{x}, \underline{z})\sigma_{ij}, \quad i, j = 1, 2, \cdots, r; i \neq j.$$

Hence,

$$Var[\hat{\underline{y}}(\underline{x}, \underline{z})] = \underline{g}'(\underline{x}, \underline{z})(X'X)^{-1}\underline{g}(\underline{x}, \underline{z})\Sigma, \quad i = 1, 2, \cdots, r,$$

where $\hat{\underline{y}}(\underline{x}, \underline{z}) = (\hat{y}_1(\underline{x}, \underline{z}), \hat{y}_2(\underline{x}, \underline{z}), \cdots, \hat{y}_r(\underline{x}, \underline{z}))'$ is the vector of predicted responses at the point $(\underline{x}, \underline{z})$. An unbiased estimator of $Var[\hat{\underline{y}}(\underline{x}, \underline{z})]$ is given by

$$\widehat{Var}[\hat{\underline{y}}(\underline{x}, \underline{z})] = \underline{g}'(\underline{x}, \underline{z})(X'X)^{-1}\underline{g}(\underline{x}, \underline{z})\hat{\Sigma} \quad .$$

2.2 Estimated Mean and Variance Models in A Multiresponse System

Box and Jones (1992) modeled the mean and variance separately in a single response. But, we are interested in showing the estimated mean and variance response models in multiple responses.

The fitted ith second-order model in (2.4) can be rewritten as

$$\hat{y}_i(\underline{x}, \underline{z}) = b_{i0} + \underline{x}'\underline{b}_i + \underline{x}'\hat{B}_i\underline{x} + \underline{z}'\hat{R}_i\underline{z} + \underline{z}'\underline{r}_i + \underline{z}'\hat{D}_i\underline{x}, \quad i = 1, 2, \cdots, r.$$

The noise variables $\underline{z}$ are not controllable and they are random variables. In the absence of other knowledge, $\underline{z}$ would be usually uniformly distributed over R_z.

Let $\hat{m}_i(\underline{x})$ ith estimated mean response at an $\underline{x}$ averaged over the noise variables

$$\hat{m}_i(\underline{x}) = \int_{R_z} \hat{y}_i(\underline{x}, \underline{z})p(\underline{z})d\underline{z}, \quad i = 1, 2, \cdots, r,$$

where $p(\underline{z})$ is a probability density function of $\underline{z}$, and $\underline{z}$ has a uniform distribution over $R_z(-1 \leq z \leq 1)$. Box and Jones (1992) showed that the ith estimated mean becomes

$$\hat{m}_i(\underline{x}) = b_{i0} + \underline{x}'\underline{b}_i + \underline{x}'\hat{B}_i\underline{x} + \frac{1}{3}tr\,\hat{R}_i, \quad i = 1, 2, \cdots, r, \tag{2.5}$$

where $tr\ \hat{R}_i$ is the trace of the matrix $\hat{R}_i$. Let us write $\hat{v}_i(\underline{x})$ for the ith mean square variation about the ith mean response

$$\hat{v}_i(\underline{x}) = \int_{R_z}(\hat{y}_i(\underline{x}, \underline{z}) - \hat{m}_i(\underline{x}))^2 p(\underline{z})d\underline{z}, \quad i = 1, 2, \cdots, r.$$

Let us call this measure the ith estimated variance, which becomes

$$\hat{v}_i(\underline{x}) = \frac{1}{3}(\underline{r}_i + \hat{D}_i\underline{x})'(\underline{r}_i + \hat{D}_i\underline{x}) + \hat{A}_i, \quad i = 1, 2, \cdots, r, \tag{2.6}$$

where $\hat{A}_i = [4\Sigma_{j=1}^{m}(r^i_{jj})^2 + 5\Sigma_{j=1}^{m-1}\Sigma_{k=j+1}^{m}(r^i_{jk})^2]/45$ and r^i_{jk} is the jth row and kth column element of the matrix $\hat{R}_i$.

From (2.5) the ith estimated mean can be rewritten as

$$\hat{m}_i(x) = \underline{h}'(x)\,\hat{\underline{\theta}}_{0i} \quad i = 1, 2, \cdots, r, \tag{2.7}$$

where $\underline{h}'(\underline{x}) = (1, x_1, \cdots, x_l, x_1^2, \cdots, x_l^2, x_1x_2, \cdots, x_{l-1}x_l, 1/3, \cdots, 1/3)$ and $\hat{\underline{\theta}}_{0i} = (b^i_0, b^i_1, \cdots, b^i_{11}, \cdots, b^i_{ll}, b^i_{12}, \cdots, b^i_{l-1}, r^i_{11}, \cdots, r^i_{mm})'$ is a part of $\hat{\underline{\theta}}_i$. From the fact that $\hat{\underline{\theta}}_{0i}$ is a part of $\hat{\underline{\theta}}_i$, the variance-covariance of $\hat{\underline{\theta}}_0$ is given by

$$Var(\hat{\underline{\theta}}_0) = (X'X)_0^{-1}\Sigma,$$

where $\hat{\theta}_0 = (\hat{\theta}_{01}', \hat{\theta}_{02}', \cdots, \hat{\theta}_{0r}')'$, $(X'X)^{-1}$ is $p \times p$, and $(X'X)_0^{-1}$ is $q \times q$, where $p = (l+m+n)(l+m+2)/2$, and $q = (l+1)(l+2)/2 + m$. Here $(X'X)_0^{-1}$ is the $q \times q$ submatrix of $(X'X)^{-1}$. From (2.7) and above, we then have

$$Var[\hat{\underline{m}}(x)] = \underline{h}'(x)(X'X)_0^{-1}\underline{h}(x)\Sigma,$$

where $\hat{\underline{m}}(x) = (\hat{m}_1(x), \hat{m}_2(x), \cdots, \hat{m}_r(\underline{x}))'$ is the vector of estimated mean responces at the point $\underline{x}$. An unbiased estimator of $Var[\hat{\underline{m}}(x)]$ is given by

$$\widehat{Var}[\hat{\underline{m}}(x)] = \underline{h}'(x)(X'X)_0^{-1}\underline{h}(x)\hat{\Sigma}\ . \tag{2.8}$$

2.3 The Proposed D_M Measure

Let us find conditions on a set of control variables $\underline{x}$ which optimize a set of estimated mean responses $\widehat{\underline{m}}(\underline{x})$ subject to maintaining estimated variance responses $\widehat{\underline{v}}(\underline{x})$ within some specified upper bounds. If all the estimated mean $\widehat{\underline{m}}(\underline{x})$ attain their individual optimum values $\underline{\tau}$ at the same set $\underline{x}$ of operating conditions, then the problem of simultaneous optimization is obviously solved. This ideal optimum rarely occurs. In more general situations we might consider finding compromising conditions on the control variables that are somewhat favorable to all mean responses. Such deviation of the compromising conditions from the ideal optimum condition can be evalulated by means of a distance function which measures the distance of $\widehat{\underline{m}}(\underline{x})$, from $\underline{\tau}$.

Let $\underline{\tau}$ be the optimum (or target) value of $\widehat{m}_i(\underline{x})$ over R_x and let $\underline{\tau}=(\tau_1,\tau_2,\cdots,\tau_r)'$. We shall consider a constrained-optimization procedure for each response according to the Taguchi's three basic situations as follows.

1. "nominal-is-best characteristics" : target value of $\widehat{m}_i(\underline{x})=\tau_i$ subject to $\widehat{v}_i(\underline{x})\le l_i$, where l_i is some upper bound on the variation,
2. "larger-the-better characteristics" : $Max_{\,\underline{x}\in R_x}\ \widehat{m}_i(\underline{x})=\tau_i$ subject to $\widehat{v}_i(\underline{x})\le l_i$,
3. "smaller-the-better characteristics" : $Min_{\,\underline{x}\in R_x}\ \widehat{m}_i(\underline{x})=\tau_i$ subject to $\widehat{v}_i(\underline{x})\le l_i$.

A distance function of $\widehat{\underline{m}}(\underline{x})$ for the target value $\underline{\tau}$ may be expressed as

$$D[\widehat{\underline{m}}(\underline{x}),\underline{\tau}]=[(\widehat{\underline{m}}(\underline{x})-\underline{\tau})'\{Var[\widehat{\underline{m}}(\underline{x})]\}^{-1}(\widehat{\underline{m}}(\underline{x})-\underline{\tau})]^{1/2}$$

Using the estimate given in (2.8) for the variance-covariance matrix of $\widehat{\underline{m}}(\underline{x})$, we get a distance function

$$\left[\frac{(\widehat{\underline{m}}(\underline{x})-\underline{\tau})'\,\widehat{\Sigma}^{-1}(\widehat{\underline{m}}(\underline{x})-\underline{\tau})}{\underline{h}'(\underline{x})(X'X)_0^{-1}\underline{h}(\underline{x})}\right]^{1/2}$$

If the mean response $\widehat{\underline{m}}(\underline{x})$ takes on different degrees of importance, we can imply weights $w_1,w_2,\cdots,w_r$ where $0<w_i<1$ for each i and $\Sigma_{i=1}^{r}w_i=1$. Then the distance function can be written as

$$\left[\frac{\{W(\widehat{\underline{m}}(\underline{x})-\underline{\tau})\}'\ \widehat{\Sigma}^{-1}\{W(\widehat{\underline{m}}(\underline{x})-\underline{\tau})\}}{\underline{h}'(\underline{x})(X'X)_0^{-1}\underline{h}(\underline{x})}\right]^{1/2}, \tag{2.9}$$

where $W=\begin{pmatrix} w_1 & 0 & \cdots & 0 \\ & w_2 & \cdots & 0 \\ & & \ddots & \vdots \\ sym & & & w_r \end{pmatrix}$. From the constrained-optimization procedure and the distance measure of $\widehat{\underline{m}}(\underline{x})$ for the target value $\underline{\tau}$, we propose a simultaneous optimization of $\widehat{\underline{m}}(\underline{x})$ while constraining the estimated variance response $\hat{v}_i(\underline{x})$ over the region of interest R_x. From (2.9), the proposed simultaneous-optimization measure can be written as

$$\underset{\underline{x}\in R_x}{Min}\ D_M(\underline{x}) = \underset{\underline{x}\in R_x}{Min}\ \frac{\{W(\widehat{\underline{m}}(\underline{x})-\underline{\tau})\}'\ \widehat{\Sigma}^{-1}\{W(\widehat{\underline{m}}(\underline{x})-\underline{\tau})\}}{\underline{h}'(\underline{x})(X'X)_0^{-1}\underline{h}(\underline{x})} \tag{2.10}$$

$$\text{subject to}\quad \hat{v}_i(\underline{x})\le l_i,\quad i=1,2,\cdots,r.$$

The D_M measure can be used without a prior knowledge about the estimated mean responses. It takes into consideration the variances and correlations of the estimated mean responses. If a simultaneous optimum value is much different from its corresponding individual optimum value, we may choose a bound on it and then reoptimize D_M. Also we may analyze D_M sequentially as the acceptable values for the estimated variance responses are varied.

3. Numerical Example

In this section we give a numerical example, consisting of a multiresponse system of two response variables, y_1 and y_2, and two control variables, x_1 and x_2, and a noise variable z. The design is somewhat similar to the standard central composite design. The cube portion of the experimental arrangement is chosen to be a 2^3 design and star points are added only for the two control variables. The following Table 3.1 gives the factor levels and a set of hypothetical data.

Each of the two responses was fitted to a second order regression model. The estimated response models by the method of least squares are given by

$$\hat{y}_1(\underline{x}, z) = 76.00 - 12.37x_1 - 8.96x_2 - 7.22x_1^2 - 8.45x_2^2 - 8.11x_1x_2 + 5.38z^2 - 1.44z + 2.96x_1z - 1.86x_2z, \quad (3.1)$$

$$\hat{y}_2(\underline{x}, z) = 103.00 - 12.21x_1 + 6.68x_2 - 13.96x_1^2 - 8.50x_2^2 - 2.93x_1x_2 + 6.23z^2 - 1.38z + 1.75x_1z - 2.95x_2z. \quad (3.2)$$

From (3.1) and (3.2), using the mean and variance response equation (2.5) and (2.6), the estimated mean and variance response models are given by

$$\widehat{m}_1(\underline{x}) = 77.79 - 12.37x_1 - 8.96x_2 - 7.22x_1^2 - 8.45x_2^2 - 8.11x_1x_2,$$

$$\widehat{m}_2(\underline{x}) = 105.08 - 12.21x_1 + 6.68x_2 - 13.96x_1^2 - 8.50x_2^2 - 2.93x_1x_2,$$

$$\hat{v}_1(\underline{x}) = (-1.44 + 2.96x_1 - 1.86x_2)^2/3 + 2.57,$$

$$\hat{v}_2(\underline{x}) = (-1.38 - 1.75x_1 - 2.95x_2)^2/3 + 3.45.$$

The region of interest R_x is given by the inequality $-1 \le x_1, x_2 \le 1$. The ranges for $\widehat{m}_1(\underline{x})$, $\widehat{m}_2(\underline{x})$, $\hat{v}_1(\underline{x})$, and $\hat{v}_2(\underline{x})$ over R_x are, respectively, $32.68 \le \widehat{m}_1(\underline{x}) \le 83.25$, $66.66 \le \widehat{m}_2(\underline{x}) \le 109.65$, $2.57 \le \hat{v}_1(\underline{x}) \le 15.63$, $3.45 \le \hat{v}_2(\underline{x}) \le 15.77$.

Suppose that the quality characteristics for y_1 and y_2 are the nominal-is-best characteristics and the larger-the-better characteristics. Let us assume that the target value of $\widehat{m}_1(\underline{x})$ is taken to be 75.00 and the target value of $\widehat{m}_2(\underline{x})$ is taken to be $Max_{\underline{x} \in R_x} \widehat{m}_2(\underline{x}) = 109.65$. The maximum acceptable values for $\hat{v}_1(\underline{x})$ and $\hat{v}_2(\underline{x})$ are $\hat{v}_1(\underline{x}) \le 4$ and $\hat{v}_2(\underline{x}) \le 4$, respectively. Assume it is of interest to obtain the target condition on the estimated mean response $\widehat{\underline{m}}(\underline{x})$ while constraining the variance response $\widehat{\underline{v}}(\underline{x})$.

We obtained the results of simultaneous optimization based on the minimization of the D_M measure over R_x (see Table 2.10). Table 3.2 indicates that the optimal setting for the constaint that $\hat{v}_1(\underline{x}) \le 4.00$, $\hat{v}_2(\underline{x}) \le 4.00$, $w_1 = 0.1$, and $w_2 = 0.9$ is $x_1 = -0.10$ and $x_2 = 0.18$, which produces a predicted value of 77.21, 107.14, 4.00, and 3.80 for $\widehat{m}_1(\underline{x})$, $\widehat{m}_2(\underline{x})$, $\hat{v}_1(\underline{x})$, and $\hat{v}_2(\underline{x})$, respectively.

Table 3.1 Experimental Design and Response Values

Run Number	x_1	x_2	z	y_1	y_2
1	-1	-1	-1	80.6	81.4
2	-1	-1	1	74.9	95.9
3	-1	1	-1	83.1	105.0
4	-1	1	1	71.2	103.0
5	1	-1	-1	66.8	74.0
6	1	-1	1	74.2	76.8
7	1	1	-1	38.1	81.2
8	1	1	1	36.8	76.9
9	-1.41	0	0	80.9	100.0
10	1.41	0	0	42.4	50.5
11	0	-1.41	0	73.4	71.2
12	0	1.41	0	45.0	101.0
13	0	0	0	77.4	102.0
14	0	0	0	74.6	104.0

Table 3.2 Simultaneous Optimization for D_M

Weight		Location of Optima		Simultaneous Optima			
w_1	w_2	x_1	x_2	$\widehat{m}_1(x)$	$\widehat{m}_2(x)$	$\hat{v}_1(x)$	$\hat{v}_2(x)$
0.1	0.9	-0.10	0.18	77.21	107.14	4.00	3.80
0.3	0.7	-0.03	0.29	74.92	106.68	4.00	3.56
0.5	0.5	-0.03	0.29	74.92	106.68	4.00	3.56
0.7	0.3	-0.03	0.29	74.92	106.68	4.00	3.56
0.9	0.1	-0.01	0.26	75.03	106.37	3.84	3.58

4. A Comparitive Study for Two-Level Orthogonal Array Design

In this section, we will study the simultaneous-optimization measure for two-level orthogonal array design. Also, in the case of interactions between the control factors and the noise factors are exist, we will compare the product-array approach with the combined-array approach.

4.1 Product-Array Approach

Suppose that the objective is to find the simultaneous optimum conditions for increasing the strength of plastic product and reducing the wear on the plastic product.

The control factors are listed in Table 4.1. Suppose there are five control factors A, B, C, D, and F which are assigned to an orthogonal array, $L_{16}(2^{15})$. Also suppose there is a noise factor N with two levels (N_0 : normal condition, N_1 : bad condition). Table 4.2 gives a set of hypothetical strength data y_1 and wear data y_2.

From the results of ANOVA for the data y_1 and y_2, we see that interactions between the control factors and the noise factor, that is, $A\times N$ of y_1 and $B\times N$ of y_2 are significant.

We can calculate SN ratios as follows. (1) In the case of larger-the-better characteristics : $SN_i = -10\log_{10}[\Sigma_{j=1}^{3}(1/3y_{ij}^2)] - 30$, (2) In the case of smaller-the-better characteristics : $SN_i = -10\log_{10}[\Sigma_{j=1}^{3}(y_{ij}^2/3)] + 35$.

From the results of Tables 4.3 - 4.4, we see that the case of y_1, the main effects of A, B and C are very significant and the case of y_2, the main effects of D and F are very significant and B is significant. We can find the simultaneous optimum levels, $A_0B_1C_0D_0F_0$ (120 min, 80℃, -20℃, 5%, 800 rpm) by summarizing the results of all the data as shown in Table 4.5.

4.2 Combined-Array Approach

Suppose that the control variables and noise variables $(\underline{x}, \underline{z})$ can be represented by a first-order model in the control variables and noise variables with, in addition, cross-product terms between the control variables and the noise variables. The fitted ith first-order model with interactions between the control and the noise variables can be written as

$$\hat{y}_i(\underline{x}, \underline{z}) = b_{i0} + \underline{x}'\underline{b}_i + \underline{z}'\underline{r}_i + \underline{z}'\hat{D}_i\underline{x}, \quad i = 1, 2, \cdots, r.$$

By the same method of section 2.2, we can obtain the mean response model and variance model. The ith estimated mean model becomes

$$\widehat{m}_i(\underline{x}) = b_{i0} + \underline{x}'\underline{b}_i, \quad i = 1, 2, \cdots, r. \tag{4.1}$$

The ith estimated variance model becomes

$$\hat{v}_i(\underline{x}) = \frac{1}{3}(\underline{r}_i + \hat{D}_i\underline{x})'(\underline{r}_i + \hat{D}_i\underline{x}), \quad i = 1, 2, \cdots, r. \tag{4.2}$$

The ith estimated mean can be rewritten as

$$\widehat{m}_i(\underline{x}) = \underline{h}'(\underline{x})\ \widehat{\underline{\theta}}_{0i} \qquad i = 1, 2, \cdots, r,$$

where $\underline{h}'(\underline{x}) = (1, x_1, \cdots, x_l)$ and $\widehat{\underline{\theta}}_{0i} = (\ b^i_0,\ b^i_1, \cdots,\ b^i_1\ )'$ is a part of $\widehat{\underline{\theta}}_i$.

The combined array consists of five control variables x_1 (or A), x_2 (or B), x_3 (or C), x_4 (or D), and x_5 (or F) and one noise variable z (or N) which are assigned in the orthogonal array, $L_{16}(2^{15})$. In order to compare the product array approach with the combined array approach, the data come from the combination for each level of factors in a product array (see Tables 4.2). For the case that the noise variable is arranged in column 4, the results of assignment are shown in Table 4.6. In the product array, we saw that interactions between the control factors and the noise factor, that is, $A \times N$ of y_1 and $B \times N$ of y_2 are significant. Therefore, we must consider interactions between the control factors and the noise factor in the combined array.

The estimated response models by the method of least squares are given by

$$\begin{aligned} \hat{y}_1(\underline{x}, z) &= 60.00 - 3.25x_1 + 3.13x_2 - 4.25x_3 + 1.38x_4 - 0.13x_5 - 1.00z \\ &\quad - 2.25zx_1 - 0.63zx_2 - 2.25zx_3 - 0.88zx_4 + 0.13zx_5, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \hat{y}_2(\underline{x}, z) &= 29.31 + 0.94x_1 - 1.06x_2 - 1.19x_3 + 3.06x_4 + 2.94x_5 + 0.06z \\ &\quad - 0.81zx_1 - 1.81zx_2 - 0.44zx_3 + 0.31zx_4 + 0.19zx_5. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), using the equation (4.1) and (4.2), the estimated mean and variance models are given by

$$\widehat{m_1}(\underline{x}) = -3.25x_1 + 3.13x_2 - 4.25x_3 + 1.38x_4 - 0.13x_5 + 60.00,$$

$$\widehat{m_2}(\underline{x}) = 0.94x_1 - 1.06x_2 - 1.19x_3 + 3.06x_4 + 2.94x_5 + 29.31,$$

$$\widehat{v_1}(\underline{x}) = (\ -2.25x_1 - 0.63x_2 - 2.25x_3 - 0.88x_4 + 0.13x_5 - 1.00)^2/3,$$

$$\widehat{v_2}(\underline{x}) = (\ -0.81x_1 - 1.81x_2 - 0.44x_3 + 0.31x_4 + 0.19x_5 + 0.06)^2/3.$$

We assume that the region of interest R_x is given by the inequality $-1 \le x_1, x_2, x_3, x_4, x_5 \le 1$. The ranges for $\widehat{m}_1(\underline{x})$, $\widehat{m}_2(\underline{x})$, $\hat{v}_1(\underline{x})$, and $\hat{v}_2(\underline{x})$ are, respectively, $47.86 \le \widehat{m}_1(\underline{x}) \le$

72.14, $20.12 \le \widehat{m}_2(\underset{\sim}{x}) \le 38.50$, $2.57 \le \widehat{v}_1(\underset{\sim}{x}) \le 15.63$, $3.45 \le \widehat{v}_2(\underset{\sim}{x}) \le 15.77$.

Let us assume that the target value of $\widehat{m}_1(\underset{\sim}{x})$ is taken to be $Max_{\underset{\sim}{x} \in R_x} \widehat{m}_1(\underset{\sim}{x}) = 72.14$, and the target value of $\widehat{m}_2(\underset{\sim}{x})$ is taken to be $Min_{\underset{\sim}{x} \in R_x} \widehat{m}_2(\underset{\sim}{x}) = 20.12$.

For the simultaneous optimization obtained under D_M, the optimal values of x_1, x_2, x_3, x_4, and x_5 and the corresponding simultaneous optimum values of $\widehat{m}_1(\underset{\sim}{x})$ and $\widehat{m}_2(\underset{\sim}{x})$ are given in Table 4.7. Table 4.7 indicates that the optimal setting for $\widehat{v}_1(\underset{\sim}{x}) \le 4$, $\widehat{v}_2(\underset{\sim}{x}) \le$ *above* 0.92, $w_1 = 0.1$, and $w_2 = 0.9$ is $x_1 = -1.0$, $x_2 = 1.0$, $x_3 = 0.5$, $x_4 = -1.0$ and $x_5 = -1.0$ which produces a predicted value of 63.01, 20.72, 0.02, and 0.92 for $\widehat{m}_1(\underset{\sim}{x})$, $\widehat{m}_2(\underset{\sim}{x})$, $\widehat{v}_1(\underset{\sim}{x})$, and $\widehat{v}_2(\underset{\sim}{x})$, respectively.

4.3 Comparison of Results and Conclusion

From the results of Sections 4.1 - 4.2, we compare the results of the product array approach with the combined array approach for the case of $w_1 = 0.5$ (equal weights). This is because the product array approach assumes equal weight for each response. The simultaneous optimum condition of the product array approach is $A_0B_1C_0D_0F_0$ ($x_1 = -1.0$, $x_2 = 1.0$, $x_3 = -1.0$, $x_4 = -1.0$, $x_5 = -1.0$), that is, 120 min, 80℃, -20℃, 5%, and 800 rpm (see Table 4.5). In the case of combined array approach, the simultaneous optimum conditions of the D_M measure is the same as the case of product array approach, and so on (see Table 4.7).

From the above results, we see that the combined array approach to simultaneous optimization using the proposed D_M measure is superior to the product array approach for many respects, such as experimental run-size, compromising condition, sequential experiment, flexibility in modeling design variables, and so on.

Table 4.1 Factors and Levels

Control factor	0 level	1 level
A : plasticity time (min)	120	130
B : plasticity temperature (℃)	70	80
C : cooling temperature (℃)	-20	-15
D : quantity of additive (%)	5	10
F : stir speed (rpm)	800	900

Table 4.2 Assignment of Source and Data in the Product Array

Source	*A*	*B*	e	e	e	e	e	*C*	e	e	*D*	e	e	*F*	e	y_1		y_2	
Run \ Col	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	N_0	N_1	N_0	N_0
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	59	63	23	29
2	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	59	64	30	37
3	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	65	69	27	33
4	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	51	56	24	29
5	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	59	75	36	34
6	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	59	65	21	18
7	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	65	74	32	30
8	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	51	59	29	25
9	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	59	55	30	35
10	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	59	45	31	36
11	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	65	57	33	41
12	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	51	44	20	26
13	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	59	61	31	25
14	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	59	58	32	26
15	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	65	62	24	20
16	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	56	51	34	31

Table 4.3 ANOVA (SN for Strength)

Source	S	f	V	F_0
A	3.7830	1	3.7830	13.27
B	3.6290	1	3.6290	12.73
C	7.1824	1	7.1824	25.19
D	0.6400	1	0.6400	2.24
F	0.0240	1	0.0240	0.08
e	2.8514	10	0.2851	
T	18.1148	15		

Table 4.4 ANOVA (SN for Wear)

Source	S	f	V	F_0
A	0.4658	1	0.4658	1.04
B	2.1830	1	2.1830	4.87
C	1.7490	1	1.7490	3.90
D	13.7456	1	13.7456	30.63
F	13.3043	1	13.3043	29.65
e	4.4805	10	0.4487	
T	34.9342	15		

Table 4.5 Summarized Table of Factorial Effects

Source	Level	Sum of SN for y_1	Sum of SN for y_2	Optimum Level
A	0 (120 min)	47.62	47.81	○
	1 (130 min)	39.84	45.08	
B	0 (70 ℃)	39.92	43.49	
	1 (80 ℃)	47.54	49.40	○
C	0 (-20 ℃)	49.09	43.40	○
	1 (-15 ℃)	38.37	49.49	
D	0 (5 %)	42.13	53.86	○
	1 (10%)	45.33	39.03	
F	0 (800 rpm)	43.42	53.74	○
	1 (900 rpm)	44.04	39.15	

Table 4.6 Assignment of Sources and Data in the Combined Array

Run \ Col	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Data y_1	Data y_2
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	59	23
2	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	59	30
3	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	69	33
4	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	56	29
5	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	69	36
6	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	61	21
7	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	74	30
8	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	59	25
9	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	60	30
10	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	51	31
11	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	57	41
12	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	44	26
13	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	64	31
14	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	65	32
15	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	62	20
16	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	51	31
basic mark	a	b	ab	c	ac	bc	abc	d	ad	bd	abd	cd	acd	bcd	abcd		
Source	x_1	x_2	e	z	x_1z	x_2z	e	x_3	e	e	x_4	e	e	x_5	e		

Table 4.7 Simultaneous Optimization under D_M

Weight w_1	Weight w_2	Location of Optima x_1	x_2	x_3	x_4	x_5	Simultaneous Optimum Value $\widehat{m}_1(x)$	$\widehat{m}_2(x)$	$\hat{v}_1(x)$	$\hat{v}_2(x)$
Subject to $\hat{v}_1(x) \le 4.00$, $\hat{v}_2(x) \le$ *above** 0.92										
0.1	0.9	−1.0	1.0	0.5	−1.0	−1.0	63.01	20.72	0.02	0.92
0.3	0.7	−1.0	1.0	−0.9	−1.0	−1.0	68.96	22.38	3.84	0.36
0.5	0.5	−1.0	1.0	−1.0	−0.8	−1.0	69.66	23.11	3.95	0.29
0.7	0.3	−1.0	1.0	−1.0	0.9	−1.0	72.00	28.31	1.27	0.06
0.9	0.1	−1.0	1.0	−1.0	1.0	−1.0	72.14	28.62	1.15	0.05
Subject to $\hat{v}_1(x) \le$ *above* 4.47, $\hat{v}_2(x) \le$ *above* 0.92 or No Constraint										
0.1	0.9	−1.0	1.0	0.5	−1.0	−1.0	63.01	20.72	0.02	0.92
0.3	0.7	−1.0	1.0	−1.0	−1.0	−1.0	69.38	22.50	4.37	0.33
0.5	0.5	−1.0	1.0	−1.0	−1.0	−1.0	69.38	22.50	4.37	0.33
0.7	0.3	−1.0	1.0	−1.0	0.9	−1.0	72.00	28.31	1.27	0.06
0.9	0.1	−1.0	1.0	−1.0	1.0	−1.0	72.14	28.62	1.15	0.05

* " *above m* " means that " m and any value that is larger than m".

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