한국통계학회논문집 제3권 1호
1996년 4월 pp. 257-265

A Unit Root Test Based on Bootstrapping

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Abstract

We consider nonstationary autoregressive process with infinite variance of error. In the case of infinite variance, the limiting distribution of the estimated coefficient is different from that under the finite variance assumption. In this paper we show that the bootstrap method can be used to approximate the distribution of ordinary least squares estimator of the coefficient in the first order random walk process with infinite variance through some empirical studies and we suggest a test procedure based on bootstrap method for the unit root test.

1. Introduction

Consider AR(1) process defined by the rule

$$X_t = \phi X_{t-1} + \varepsilon_t$$

where ε_t's are independent and identically distributed with $E(\varepsilon_t)=0$, $Var(\varepsilon_t)=\sigma_\varepsilon^2<\infty$. When $|\phi|<1$ the process is stationary, but if $\phi=1$, we call this process a random walk process. In a random walk case, our primary concern is to test if the unknown parameter ϕ is equal to 1. This test is called unit root test and it is closely related to differencing. For estimation, least square estimator is a typical one, which is defined as

$$\hat{\phi}_{OLS} = \frac{\sum_{t=2}^{n} X_t X_{t-1}}{\sum_{t=2}^{n} X_{t-1}^2}.$$

When $\phi=1$ and $\sigma_\varepsilon^2<\infty$, it is known that

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$$n(\hat{\phi}_{OLS}-1) \rightarrow^{L} = \frac{\frac{1}{2}(W(1)^2-1)}{\int_0^1 W(r)^2 dr}.$$

The closed form of the limiting distribution can be found in Abadir(1993), and unit root test has been done mostly by means of the empirical tables in Fuller(1976).

2. OLS with Infinite Variance of Error

In this section we consider the first order random walk process with infinite variance of error terms. This kind of process usually can be found in economic data.

Consider AR(1) process

$$X_t = \phi X_{t-1} + \varepsilon_t, \quad X_0 = 0 \tag{1}$$

where ε_t's are i.i.d., whose distribution is in the domain of attraction of a stable law with index $\alpha \in (1, 2)$. In other words, the distribution of $\{\varepsilon_t\}$ satisfies $P[\,|\varepsilon_1| > x] = x^{-\alpha}L(x)$ where $L(x)$ is a slowly varying function at ∞ and $\lim_{x\to\infty} \frac{P[\,\varepsilon_1 > x]}{P[\,|\varepsilon_1| > x]} = p, \quad 0 \le p \le 1$. See Feller(1971) for more details on the attraction of a stable law. With these assumptions, the variance of error terms is infinite and the mean is finite. Especially, we are interested in the case of $\phi = 1$ known as random walk process.

Typically, ϕ is estimated using the least squares estimator

$$\hat{\phi}_{OLS} = \frac{\sum_{t=2}^{n} X_t X_{t-1}}{\sum_{t=2}^{n} X_{t-1}^2} \tag{2}$$

and Knight(1989) showed that

$$n(\hat{\phi}_{OLS}-1) \rightarrow^{L} \frac{S^2(1) - V(1)}{2\int_0^1 S^2(s)\,ds} \tag{3}$$

where $S(\cdot)$ and $V(\cdot)$ are defined in Knight(1989). Since the properties of $S(\cdot)$ and

$V(\cdot)$ are highly dependent on α, the empirical distribution varies through α. When $\alpha=2$, $V(1)=1$ and $S(\cdot)$ is a standard Brownian motion.

To obtain the empirical distributions of the ordinary least squares estimator of ϕ under the infinite variance assumption, we first generated ε_i's, $1 \le i \le n$, as follows:

Let U and V be independent random variables with $P(U \le t) = t$, $0 \le t \le 1$, and $P(V \le t) = 1 - e^{-t}$, $0 \le t < \infty$. Further, let $h(x)$ be a function defined by

$$h(x) = \left(\frac{\sin(\pi \alpha x)}{\sin(\pi x)} \right)^{1/(1-\alpha)} \left(\frac{\sin(\pi x(1-\alpha))}{\sin(\pi \alpha x)} \right) \text{ for } 0 \le x \le 1$$

Then for $\alpha < 1$, the random variable $S_\alpha = \{h(U)/V\}^{1/\alpha - 1}$ is positive stable of index α.

Let $\varepsilon = N \times (S_{\alpha/2})^{1/2}$ where N is normal with $E(N)=0, E(N^2)=2$ and independent of $S_{\alpha/2}$. Then ε_i's are symmetric stable of index $\alpha \in (1,2)$. Once we obtained ε_i's, using the model defined by $X_t = \phi X_{t-1} + \varepsilon_t$, $1 \le t \le n$, we generated data X_t's with n=50, 100, 200 and $\alpha = 1.2,\ 1.5,\ 1.8$. We repeated the above steps 30,000 times to obtain empirical distributions. Followings are empirical distributions of $n(\hat{\phi}_{OLS} - 1)$, where $\hat{\phi}_{OLS}$ is the ordinary least squares estimator.

<Table 1. Empirical distribution of n($\hat{\phi}_{OLS}$-1) with $\alpha=1.2$>

	Left Tail Probability							
n	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
50	-11.5	-8.43	-6.39	-4.13	0.89	1.35	1.89	2.75
100	-12.3	-8.88	-6.60	-4.47	0.91	1.34	1.85	2.72
200	-11.7	-8.69	-6.50	-4.44	0.86	1.27	1.77	2.57

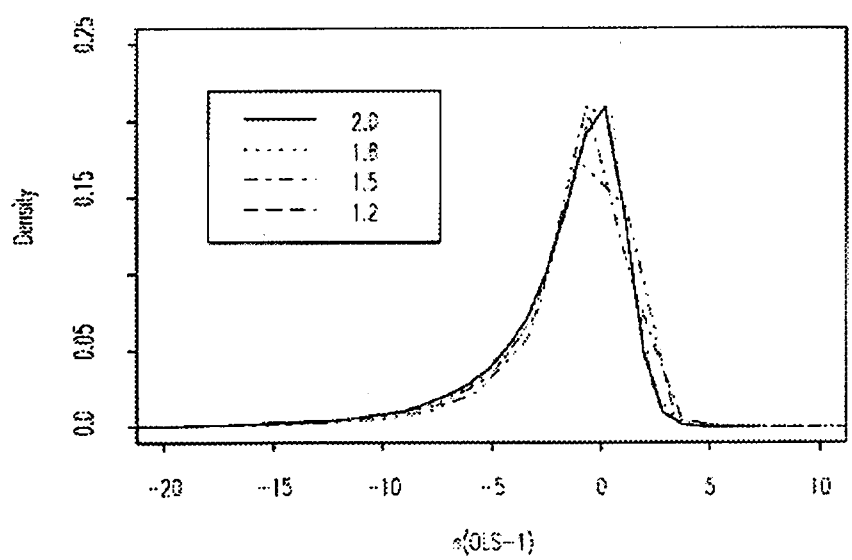
<Table 2. Empirical distribution of n($\hat{\phi}_{OLS}$-1) with $\alpha=1.5$>

	Left Tail Probability							
n	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
50	-12.0	-9.12	-6.95	-4.77	0.94	1.35	1.83	2.53
100	-12.4	-9.31	-7.04	-4.99	0.94	1.35	1.78	2.43
200	-12.6	-9.49	-7.13	-4.97	0.91	1.30	1.73	2.43

<Table 3. Empirical distribution of n($\hat{\phi}_{OLS}$-1) with $\alpha=1.8$>

	Left Tail Probability							
n	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
50	-12.6	-9.62	-7.43	-5.29	0.95	1.36	1.76	2.30
100	-12.8	-9.91	-7.66	-5.38	0.94	1.31	1.69	2.23
200	-13.0	-9.93	-7.58	-5.39	0.95	1.32	1.69	2.17

One can see that as α increases to 2, the empirical distributions approaches to that of finite variance case. See Fuller(1976) for the empirical tables of $\alpha=2$. So one can use the above tadbles for the unit root test under the infinite variance assumption when α is known as 1.2, 1.5 , and 1.8. The following figure illustrates these empirical distributions for various α's with $n=200$.



<Figure 1> Empirical Distribution for Different α's

3. Feasibility of Bootstrap Method with Infinite Variance of Error

However, Table 1-3 hardly can be used because the index of stable law, α, is not known practically. But it is well known that bootstrap method can provide an alternative procedure for studying the distributional properties of various statistics of interest. Basawa *et al.*(1991)

showed that for the AR(1) process with unit root, the standard bootstrapping is invalid to approximate the sampling distribution of the ordinary least squares estimator even if the ε_i's are independently normally distributed. But it was proved in Datta(1993) that under the finite variance assumption, the sampling distribution of the ordinary least squares estimator can be approximated by the bootstrap method when the bootstrap resample size, say, m, is less than the original sample size, say, n.

To show the feasibility of the bootstrap method in approximating the sampling distribution of the least squares estimator for the parameter of the random walk process with infinite variance, we experimented some empirical studies with various m, n and α. First, we would like to begin with the standard bootstrap procedure. Given the original sample $X_1, \cdots, X_n$ as in (1), calculate the residual $\hat{\varepsilon}_1, \cdots, \hat{\varepsilon}_n$ by $\hat{\varepsilon}_t = X_t - \hat{\phi}_{OLS} X_{t-1}$ where $\hat{\phi}_{OLS}$ is given by (2). Define $\hat{F}_n$ the empirical distribution function based on $\{ \hat{\varepsilon}_t : t=1, \cdots, m\}$. Now, pretending that $\hat{F}_n$ is the true distribution, draw a random sample $\{ \hat{\varepsilon}_t : t=1, \cdots, m\}$ from $\hat{F}_n$. Again, pretending that $\hat{\phi}_{OLS}$ is the unknown true parameter, construct the bootstrap sample $X_1^*, \cdots, X_m^*$ by the recursive formula

$$X_t^* = \hat{\phi}_{OLS} X_{t-1}^* + \varepsilon_t^*, \quad X_0^* = 0, \quad t=1, \cdots, m \tag{4}$$

Let $\hat{\phi}_{OLS}{}^*$ be the least squares estimator which is obtained by the following

$$\hat{\phi}_{OLS}{}^* = \frac{\sum_{t=2}^{n} X_t^* X_{t-1}^*}{\sum_{t=2}^{n} X_{t-1}^{*2}} . \tag{5}$$

Here, practically, we use the empirical histogram of the $\hat{\phi}_{OLS}{}^*$ in (5) as the approximation of the bootstrap distribution of the least squares estimator of the parameter of the process, which can be obtained by repeating the whole procedures described in the above for a sufficiently large number of times. To verify the validity of the bootstrap method, we calculated the empirical levels for 90%, 95% and 99% confidence intervals based on the percentiles of the bootstrap distribution of least squares estimator for the parameter of the random walk process with infinite variance. The following is the overall scheme to calculate the empirical level of, for example, 95% confidence interval.

$$
\begin{aligned}
0.95 &= \Pr{}^*[\ m(\ \hat{\phi}_{OLS}^{*,0.025} - \hat{\phi}_{OLS}) \le m(\ \hat{\phi}_{OLS}^{*} - \hat{\phi}_{OLS}) \le m(\ \hat{\phi}_{OLS}^{*,0.975} - \hat{\phi}_{OLS})] \\
&\approx \Pr[\ m(\ \hat{\phi}_{OLS}^{*,0.025} - \hat{\phi}_{OLS}) \le n(\ \hat{\phi}_{OLS} - \phi) \le m(\ \hat{\phi}_{OLS}^{*,0.975} - \hat{\phi}_{OLS})] \\
&= \Pr[\ \hat{\phi}_{OLS} - \frac{m}{n}(\ \hat{\phi}_{OLS}^{*,0.975} - \hat{\phi}_{OLS}) \le \phi \le \hat{\phi}_{OLS} - \frac{m}{n}(\ \hat{\phi}_{OLS}^{*,0.025} - \hat{\phi}_{OLS})]
\end{aligned}
$$

where $\hat{\phi}_{OLS}$ is the least squares estimator based on the original sample and $\hat{\phi}_{OLS}^{*,0.025}$ and $\hat{\phi}_{OLS}^{*,0.975}$ are the 25th and 975th largest value among 1000 bootstrap $\hat{\phi}_{OLS}^{*}$'s, respectively. If the bootstrap distribution of the least squares estimator is approximately equal to the original sampling distribution of the least squares estimator, then, the above empirical level based on the percentiles of the bootstrap distribution should be approximately 95% so that we can claim the validity of the bootstrap method. To calculate the empirical coverage of the confidence intervals we used 1000 iterations. The followings are simulation results with various m, n and $\alpha \in (1,2)$.

< Table 4 : Empirical levels with various m, n and α>

percentiles	n	m	Coverage		
			alpha=1.2	alpha=1.5	alpha=1.8
90%	50	25	90.2	90.7	90.3
		50	91.9	90.6	90.9
	100	50	90.8	90.6	91.6
		100	91.6	90.3	91.3
	200	100	90.5	90.1	92.2
		200	91.0	91.0	91.2
95%	50	25	94.4	94.0	95.2
		50	96.9	94.3	94.9
	100	50	94.5	95.3	95.4
		100	95.5	95.0	95.8
	200	100	94.5	94.6	95.7
		200	94.9	94.2	95.4
99%	50	25	98.6	97.8	98.8
		50	97.4	97.5	98.5
	100	50	98.7	99.1	99.0
		100	99.0	98.6	98.7
	200	100	98.5	99.2	99.0
		200	98.5	98.6	98.5

As one can see, the bootstrap approximation is good enough to reach the nominal 90%, 95% and 99% confidence levels. Therefore, we can say that the bootstrap distribution of the least squares estimator for the parameter of the random walk process with infinite variance is approximately equal to the original sampling distribution of the least squares estimator.

4. Bootstrap Test

In this section, we suggest an alternative test procedure for the unit root test under the infinite variance assumption. When α is known, one can use the empirical tables such as Table 1-3 for the unit test. However, as stated before, we don't know α, practically. Thus, there is no reason to use the empirical tables for the unit root test. But the bootstrapping can play a role to overcome the uncertainty of α. That is, by the fact that, as shown in section 3, the bootstrapping distribution of the least squares estimator can be used to approximate the sampling distribution of the least squares estimator for the parameter of the random walk process with the index $\alpha \in (1,2)$, we can use the bootstrap distribution for the unit root test. The following is the proposed test procedure.

step 1. Generate B samples of size m using recursive formula in (4) where $\{\varepsilon_t^*{'s}\}$ are i.i.d random sample from $\widehat{F_n}$, the empirical distribution of the residuals $\{\hat{\varepsilon}_t{'s}\}$.

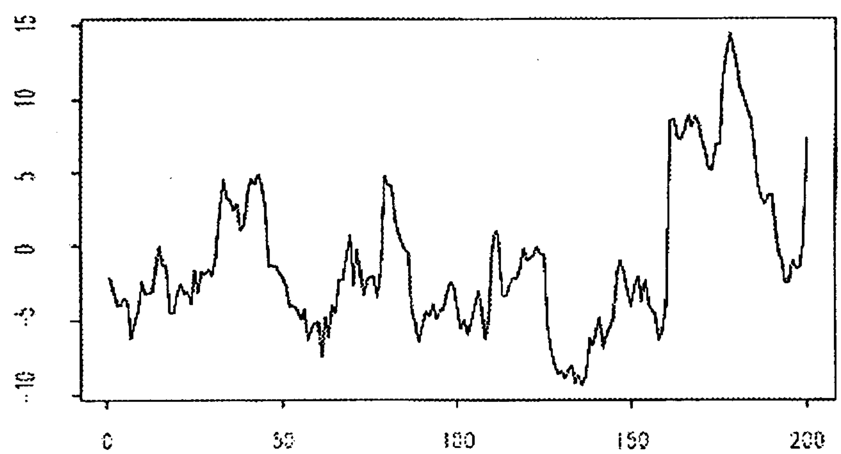
step 2. Evaluate $T_b(X^{*b}) = m(\hat{\phi}_{OLS}{}^{*b} - \hat{\phi}_{OLS})$, $b=1,2,\cdots,B$ on each bootstrap sample where $\hat{\phi}_{OLS}{}^{*b}$ is the least squares estimate based on $\{X^{*b}\}$ and $\hat{\phi}_{OLS}$ is the least squares estimate based on the original sample $\{X\}$.

step 3. Calculate $\widehat{ASL}_{boot} = \sum_{i=1}^{B} I(T_i(X^{*i}) \le T(X))/B$

where $I(\cdot)$ is an indicator function and $T(X) = n(\hat{\phi}_{OLS} - 1)$

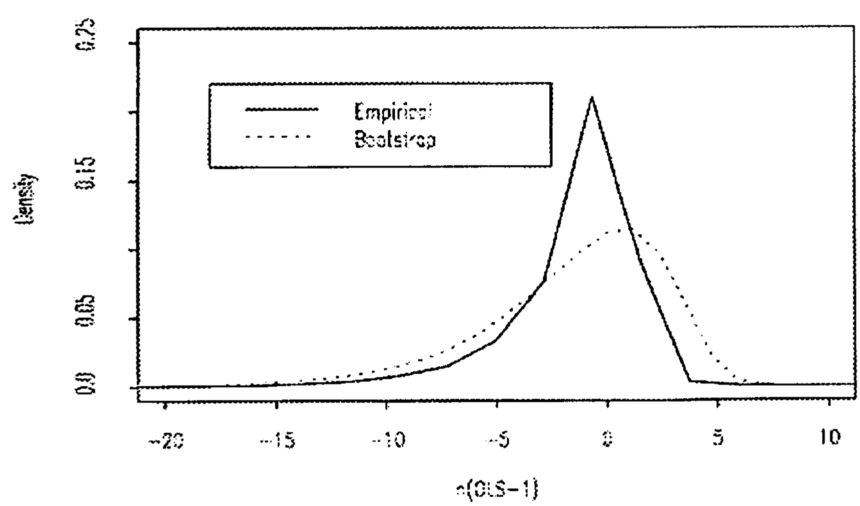
The value $\widehat{ASL}_{boot}$ is called "estimated bootstrap achieved significance level" in Efron and Tibshirani(1993). The bootstrap replication B should be at least 1000. See Efron and Tibshirani(1993) for more details. The smaller the $\widehat{ASL}_{boot}$ is, the stronger the evidence against $H_0 : \phi = 1$ is. So we can reject $H_0 : \phi = 1$ if $\widehat{ASL}_{boot}$ is less than or equal to the desired significance level. If $\widehat{ASL}_{boot}$ is greater than the significance level, then we accept H_0, which amounts to saying that the experimental data do not decisively reject the null hypothesis.

For example, we generated 200 observations by the rule $X_t = X_{t-1} + \varepsilon_t$ with the index of stable law $\alpha = 1.5$. Following is the figure of simulated data.



<Figure 2> Random Walk Process with Infinite Variance

The least squares estimate of $\hat{\phi}_{OLS}$ is 0.941. Then using <Table 2> we approximately obtain p-value 0.015. So we can conclude that this series is not a random walk process. Now for using bootstrap method we follow the proposed test procedure with B=30,000 and m=n/2=100. The resulting p-value is 0.027. Figure of bootstrap distribution and empirical distribution is follows:



<Figure 3> Empirical Distribution and Bootstrapping Distribution

5. Concluding Remarks

In time series analysis, the most commonly used transformation is variance-stabilizing and differencing. In this paper we considered a unit root test closely related differencing. When variance of error term is infinite, the empirical distributions depend on the index of stable law, α. To overcome this dependency of unknown index α, we introduced the bootstrap method. As shown in section 3, the coverage of bootstrapping method turned out to be good enough so we can use the bootstrap method for various infinite variance random walk process. In addition, bootstrap test produces a very close p-value to the empirical distribution. When we take a close look at Figure 3, the left-hand-side tail probabilities of bootstrap distribution approximate the empirical distribution probabilities very well. As unit root test is an one-side test in general, this feature is very important. In this paper, we introduced two unit root tests for infinite variance of errors; empirical distribution and bootstrap method. Simulation results and example provide a good evidence that bootstrap test is a good alternative for infinite variance random walk process.

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