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# Error Rate for the Limiting Poisson-power Function Distribution[1)]

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## Abstract

The number of neutron signals from a neutral particle beam(NPB) at the detector, without any errors, obeys Poisson distribution. Under two assumptions that NPB scattering distribution and aiming errors have a circular Gaussian distribution respectively, an exact probability distribution of signals becomes a Poisson-power function distribution. In this paper, we show that the error rate in simple hypothesis testing for the limiting Poisson-power function distribution is not zero. That is, the limit of $\alpha+\beta$ is zero when Poisson parameter $k \rightarrow \infty$, but this limit is not zero (i.e., $\rho^{\ell} > 0$ ) for the Poisson-power function distribution. We also give optimal decision algorithms for a specified error rate.

## 1. Introduction

A NPB can be used to estimate the density or mass of an object (Feller (1970)). A method of object discrimination proposed here is to use a NPB aimed at the object, and a small number of neutron signals are counted at the detector. The American Physical Society report (1987) include a lot of detailed descriptions for the NPB. Beyer and Qualls (1987) showed that the return neutron particles from an object interrogation for a given dwell time obeys Poisson distribution.

The mean neutron signal $\lambda$ for the Poisson parameter is computed by the *bistatic radar formula*:

$$\lambda = [\ I\tau]\ \cdot \left[\ \frac{A_t}{\pi(R\sqrt{2}\,\sigma_1)^2}\right]\ \cdot K(E,\theta)\cdot\left[\ \frac{A\varepsilon}{4\pi r^2}\right] \tag{1.1}$$

where $I$ is the probe current in amperes divided by $1.602\times 10^{-19}$ coulombs, $\tau$ is the dwell time in seconds, $A_t$ is the object area in $m^2$, $R$ is the probe to object distance in $m$,

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$\sqrt{2}\sigma_1$ is the beam half divergence angle, $K(E,\theta)$ is the mean number of neutrons leaked from the object per incident particle and it depends on the mass of the object, $E$ is the probe particle energy in election volts, $\theta$ is the scattering angle, $A$ is the detector area in $m^2$, $\varepsilon$ is the detector efficiency, and $r$ is the object to detector distance in $m$.

Let

$$S = \frac{A\varepsilon\tau I}{(Rr\sigma_1)^2} \quad \text{and} \quad f(E,\theta) = \frac{K(E,\theta)A_t}{4\pi} \tag{1.2}$$

where $f(E,\theta)$ combines parameters specific to the object design. Note that the mean return signal in (1.1) becomes $\lambda = S\cdot f(E,\theta)/(2\pi)$.

From the result of Beyer and Qualls (1987), we assume that the count of return neutron particles obeys Poisson distribution. The interrogation requires the true value of the parameters of the bistatic radar formula in (1.1) to compute the mean of the Poisson statistics.

One source of errors in measurement is aiming errors (or tracking and pointing errors) which is the uncertainty about the location of the axis of the beam relative to the object. Wehner (1987) studied the aiming error distribution of NPB. Kim (1994b) consider aiming errors of the beam for an object interrogation and make the following two assumptions about aiming errors:

(i) The beam has a circular Gaussian distribution of intensity with standard deviation $\sigma_1$. This distribution is on a plane perpendicular to the beam axis.

(ii) Aiming errors yield a circular Gaussian distribution of the beam axis relative to the object center. The standard deviation of the distribution is $\sigma_2$.

Beckman and Johnson (1987) give evidence from an experiment that the beam has a Pearson Type Ⅶ distribution instead of a circular Gaussian distribution in assumption (i). This distribution is much heavier in the tails than is the Gaussian. Kim (1994b) compared a circular Gaussian distribution with a Pearson Type Ⅶ distribution for scattering distribution of the NPB. He also derived the exact probability distribution of neutron particles in presence of aiming errors.

## 2. Poisson-power Function Distribution and Its Properties

Under the assumption of a Poisson distribution of counts and aiming errors, the probability of exactly $x$ neutron particles, $x = 0, 1, 2, \cdots,$ being counted is

$$P(x|\lambda) = \frac{1}{x!}\int_{\omega_2=-\infty}^{\infty}\int_{\omega_1=-\infty}^{\infty} e^{-\lambda}\lambda^x e^{-(\omega_1^2+\omega_2^2)/(2\sigma_2^2)}\frac{d\omega_1 d\omega_2}{2\pi\sigma_2^2} \tag{2.1}$$

where $\lambda$ is defined by

$$\lambda = k \cdot e^{-(\omega_1^2+\omega_2^2)/(2\sigma_1^2)} \tag{2.2}$$

where $k = (2\pi)^{-1} Sf(E, \theta)$ and it represents the mean return neutron counts without aiming errors, and $S$ and $f$ are defined in (1.2). In (2.2) $\sigma_1$ is a standard deviation of the circular Gaussian intensity distribution of the beam at the object, and $(\omega_1, \omega_2)$ are coordinates of points on beam cross section. $\sigma_2$ in (2.1) is a standard deviation of the circular Gaussian aiming error distribution of the beam relative to the object. We average over the aiming error distribution in (2.1) to modify discrimination for this uncertainty. In repeated sequential interrogation, the probability in (2.1) leads to a reasonable and correct modification.

Using the polar coordinates transformation, and letting

$$\ell = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \tag{2.3}$$

we obtain

$$P(x|\lambda) = \frac{\ell}{k^{\ell} x!} \gamma(x+\ell\,; k), \tag{2.4}$$

where

$$\gamma(\nu; k) = \int_0^k t^{\nu-1} e^{-t}\, dt$$

is the incomplete gamma function.

We have defined in (2.2) that $k$ be the mean number of return neutron signals counted with the assumption that no aiming errors are made in the measurement of the parameters and that the NPB is perfectly centered on the object. In this case, $k=\lambda$.

Consider the probability distribution in (2.4) by $P(x; k, \ell)$

$$\begin{aligned} P(x; k, \ell) &= \frac{\ell}{k^{\ell} x!} \int_0^k e^{-\omega} \omega^{x+\ell-1}\, d\omega \\ &= \frac{1}{x!} \int_0^k e^{-\omega} \omega^{x}\, \ell \left(\frac{1}{k}\right)^{\ell} \omega^{\ell-1}\, d\omega \\ &= \frac{1}{x!} E_{\omega}(e^{-\omega}\omega^{x}) \end{aligned}$$

where $E_{\omega}$ represents expected value of $\omega$, and $\omega$ has a probability distribution

$$f(\omega) = \ell k^{-\ell} \omega^{\ell-1}, \qquad \ell > 1,\ 0 \le \omega \le k \tag{2.5}$$

The distribution in (2.5) is called the power-function distribution. From the above expression, the distribution in (2.4) is a special case of a *compound Poisson distribution* where $\omega$ has a power-function distribution, and $\omega$ is a mean of the Poisson distribution. See Johnson and Kotz (1970) for the definition of compound Poisson distribution. Thus the probability distribution represented by (2.4) may be reasonably called a *Poisson-power function distribution.*

Kim (1995a, 1995b) proved some properties such as unimodality, stochastical ordering, computational recursion formula,, monotone likelihood ratio property, of the distribution. One of results we do better remember is that the Poisson-power function distribution converges to Poisson distribution as $\ell \to \infty$. Here are some properties of the Poisson-power function distribution which can be applied to the distribution of the NPB with aiming errors follow. You can find detailed proofs of each property in Kim (1995a, 1995b).

**Property 1.** The mean and variance of the Poisson-power function random variable are

$$E(X) = k\left(\frac{\ell}{\ell+1}\right)$$

and

$$Var(X) = k\left(\frac{\ell}{\ell+1}\right) + k^2\left\{\frac{\ell}{\ell+2} - \left(\frac{\ell}{\ell+1}\right)^2\right\}.$$

**Property 2.** The m.g.f. corresponding to the Poisson-power function distribution is

$$M_X(t) = E(e^{tX}) = E(E(e^{tX}|\omega)) = E_\omega(e^{\omega(e^t-1)}),$$

which gives

$$M_X(t) = M(\ell,\ \ell+1,\ k(e^t-1)),$$

where M is a Kummer's function (see Abramowitz (1964)) and is defined by

$$M(a,b,x) = 1+\frac{ax}{b}+\frac{(a)_2x^2}{(b)_2 2!}+\cdots+\frac{(a)_n x^n}{(b)_n n!}+\cdots$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ , and $(a)_0 = 1$ .

**Property 3.** The Poisson-power function distribution $P(x;k,\ell)$ converges to Poisson distribution $P(x;k)$ as $\ell \to \infty$.

**Property 4.** The Poisson-power function distribution has the recursion formula:

$$P(x;k,\ell) = \frac{x+\ell}{x+1}P(x;k,\ell) - \frac{1}{x+1}\frac{k^x e^{-k}}{x!}$$

for integer $x \geq 0$ .

**Property 5.** The Poisson-power function distribution is a unimodal probability distribution.

**Property 6.** The Poisson-power function distribution is stochastically ordered in $k$, in fact, stochastically decreasing.

**Property 7.** A useful computation formula for the Poisson-power function c.d.f. is

$$\sum_{x=0}^{c} P(x;k,\ell) = 1-\frac{\gamma(c+1;k)}{\Gamma(c+1)}+\frac{k^{-\ell}\gamma(c+\ell+1;k)}{\Gamma(c+1)}.$$

**Property 8.** (Monotone Likelihood Ratio Property) The random variable of $X$ of the Poisson-power function distribution in (2.4) has a monotone likelihood ratio; and the

Neyman-Pearson test rule for the simple hypotheses of $H_0 : k=t$ vs. $H_1 : k=d$, when $d<t$, is that reject $H_0$ if $X\le c$ which is a left-tail test.

## 3. The Limiting Poisson-power Function Distribution and Its Error Rate

We test the null hypothesis $H_0: \lambda=t$ vs. the alternative hypothesis $H_1: \lambda=\rho t$ assuming $\rho=d/t<1$. For this test, we study the limit of the minimum of $\alpha+\beta$ as the parameter $k\to\infty$. The limit is not zero for the Poisson-power function distribution:

$$P(x;k,\ell)=\frac{\ell}{k^{\ell}x!}\gamma(x+\ell\,;k), \qquad x=0,1,2,\cdots, \tag{3.1}$$

where $\gamma$ is the incomplete gamma function. Note that $k=t$ under $H_0$ and $k=d$ $(d<t)$ under $H_1$, as it is for the Poisson mean counts.

We first find the limiting distribution of the Poisson-power function distribution.

**Theorem 3.1.** Let $X$ be a random variable with Poisson-power function probability distribution $P(x\,;k,\ell)$ in (3.1) and let $Y=X/k$ be a scaled version of $X$. Then $Y\to U$ in distribution as $k\to\infty$, where $U$ has a standard power-function distribution on the interval [0,1] given by (2.5).

**Proof.** It suffices to show, by the continuity theorem, the convergence of the m.g.f. of the $X$ to that of $U$. From the property 2 in section 3, the m.g.f. of $X$ is

$$M_X(t)=\int_0^1 e^{ku(e^t-1)}\,\ell\, u^{\ell-1}du$$

and that of $Y$ is

$$M_Y(t)=M_X(t/k).$$

For each $t\ge 0$, we have

$$k(e^{t/k}-1) = t\int_0^1 e^{(t/k)\,\omega}\,d\omega \le te^{t/k},$$

which is less than $te^t$ for $k>1$. From this, we also have $k(e^{t/k}-1)\to t$ as $k\to\infty$. The Lebesgue Dominated Convergence Theorem then shows

$$M_Y(t)\to M_U(t) \quad \text{as} \quad k\to\infty \qquad ▣$$

From Theorem 3.1, we write

$$\lim_{k\to\infty} X/k \sim U,$$

where $U$ has a standard power-function distribution on the interval [0,1], and we can use the power-function distribution of $kU$ as the asymptotic distribution of $X$. Note that the asymptotic Poisson-power function distribution, under $H_0$, is

$$P(x|H_0) \approx \ell\, t^{-\ell} x^{\ell-1}, \qquad 0 \le x \le t$$

and under $H_1$

$$P(x|H_1) \approx \ell\, d^{-\ell} x^{\ell-1}, \qquad 0 \le x \le d.$$

The graph of the asymptotic Poisson-power function distribution under $H_0$ and under $H_1$ is given in Figure 1.

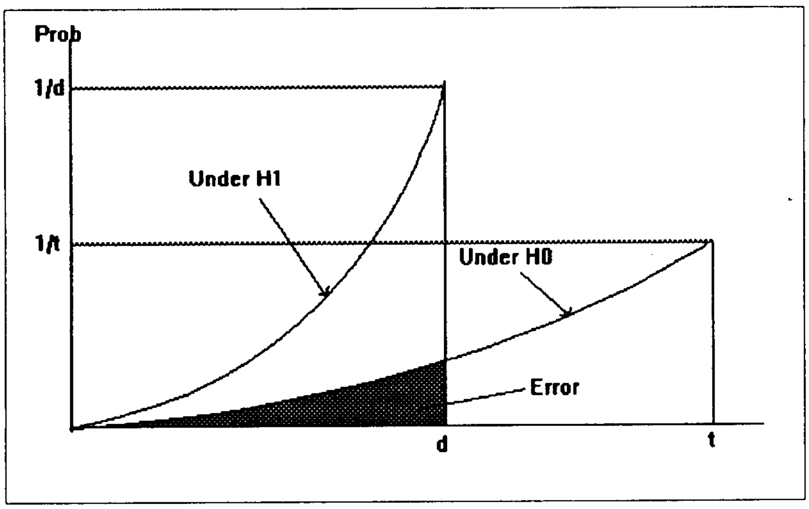


Figure 1. The Asymptotic Poisson-power Function Distribution

For the test $H_0: \lambda = t$ vs. $H_1: \lambda = \rho t$, $(0 \le \rho \le 1)$, one computes the likelihood ratio $L(x) = P(x|H_1)/P(x|H_0)$, and the Neyman-Pearson test that attains minimum $\alpha + \beta$ is $L(x) \ge 1$. Note that the likelihood ratio of the asymptotic Poisson-power function distribution for $H_0$ and $H_1$ is

$$L(x) = \begin{cases} (d/t)^{-\ell}, & \text{if } 0 \le x \le d \\ 0, & \text{if } x > d. \end{cases}$$

By the monotone likelihood ratio property of the Poisson-power function distribution (see property 8 in section 3, see Kim (1995b) for the proof), the Neyman-Pearson test is $X \le \rho t$ which is a left-tail test. Hence for this optimal choice

$$\alpha = P(X \le \rho t | H_0) \approx P(U \le \rho) = \rho^{\ell}$$

and

$$\beta = P(X > \rho t | H_1) = 0.$$

Thus the limiting minimum error rate $\alpha+\beta$ for the asymtotic Poisson-power function distribution is

$$\lim_{k\to\infty}(\alpha+\beta) = (\rho^{\ell}+0) = \rho^{\ell}.$$

The type I error rate $\alpha$ and the type II error rate $\beta$ ($\beta=0$) for the asymptotic Poisson-power function distribution are also shown in Figure 1. Note that the limit of $\alpha+\beta$ is zero when $k\to\infty$ the Poisson return signal, but this limit is not zero (i.e., $\rho^{\ell}>0$ ) for the Poisson-power function distribution.

Using the limiting distribution of the Poisson-power function distribution directly to find the error rate is not justified because the cut-off value for the Neyman-Pearson test also depends on the mean values $t$ and $\rho t$ and need to be controlled in the limit. The following theorem follows to this case.

**Lemma 3.1.**

$$\lim_{k\to\infty}\sum_{x=0}^{pk}\ell\, k^{-\ell}\frac{\gamma(x+\ell\,;k)}{x!} = \begin{cases} p^{\ell} & \text{for } 0\le p<1 \\ 1 & \text{for } p\ge 1. \end{cases} \tag{3.2}$$

**Proof.** First we need a bound on a certain right tail area of the gamma distribution. Write

$$\int_k^{\infty} y^{\alpha-1}e^{-y}dy = \int_k^{\infty} g(y)e^{-(1-q)y}dy \tag{3.3}$$

where $g(y)=y^{\alpha-1}e^{-qy}$. The maximum of $g$ occurs at $y=(\alpha-1)/q$, which is less than $(pk-1)/q$ for all $\alpha\le pk$, and in turn $(pk-1)/q$ is less than $k$ (since $p/q<1$ ). So $g(y)$ is monotone decreasing for all $y\ge k$ and $g(y)\le g(k)$. So it suffices that

$$\int_k^{\infty} y^{\alpha-1}e^{-y}dy \le g(k)\int_k^{\infty} e^{-(1-q)y}dy = \frac{k^{\alpha-1}e^{-k}}{(1-q)}. \tag{3.4}$$

Now first any fixed number of leading terms of the series in equation (3.2) can be neglected, since

$$\lim_{k\to\infty}\ell\, k^{-\ell}\frac{\gamma(x+\ell\,;k)}{x!} \le \lim_{k\to\infty}\ell\, k^{-\ell}\frac{\Gamma(x+\ell)}{x!} = 0.$$

Denoting the limsup of the series in equation (3.2) by $\overline{\alpha}_p$ (and liminf by $\underline{\alpha}_p$ ), we have for any fixed $K>0$,

$$\underline{\alpha}_p = \operatorname{limsup}_{k\to\infty}\ell\, k^{-\ell}\sum_{x=K}^{pk}\frac{\gamma(x+\ell\,;k)}{x!} \le \operatorname{limsup}_{k\to\infty}\ell\, k^{-\ell}\sum_{x=K}^{pk}\frac{\Gamma(x+\ell)}{x!}.$$

$$\overline{\alpha}_p = \operatorname{limsup}_{k\to\infty}\ell\, k^{-\ell}\sum_{x=K}^{pk}\frac{\gamma(x+\ell\,;k)}{x!} \le \operatorname{limsup}_{k\to\infty}\ell\, k^{-\ell}\sum_{x=K}^{pk}\frac{\Gamma(x+\ell)}{x!}$$

Now by stirling's formula (Feller (1970)), we have for any $\varepsilon > 0$, there exists a constant $K_\varepsilon$ such that for all $x \geq K_\varepsilon$,

$$(1-\varepsilon)x^{\ell-1} \leq \frac{\Gamma(x+\ell)}{\Gamma(x+1)} \leq (1+\varepsilon)x^{\ell-1}.$$

So

$$\begin{aligned} \overline{a_p} &\leq \limsup_{k\to\infty}(1+\varepsilon)\,\ell\, k^{-\ell}\int_{K_\varepsilon}^{pk} x^{\ell-1}dx \\ &= \limsup_{k\to\infty}(1+\varepsilon)\left[\, p^{\ell} - \left(\frac{K_\varepsilon}{k}\right)^{\ell}\right] \\ &= (1+\varepsilon)p^{\ell} \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have for $0 < p < 1$, $\overline{a}_p \leq p^\ell$. To show $\underline{a}_p = \overline{a}_p = p^\ell$, consider the remainder $R$. We compute

$$\begin{aligned} 0 \leq R &\equiv p^\ell - \overline{a_p} \\ &\leq p^\ell - \underline{a}_p \\ &= \limsup_{k\to\infty} \ell\, k^{-\ell} \sum_{x=K}^{pk} \frac{\Gamma(x+\ell) - \gamma(x+\ell;k)}{x!} \\ &\leq \limsup_{k\to\infty} \ell\, k^{-\ell} \sum_{x=K}^{pk} \frac{k^{x+\ell+1}e^{-k}}{(1-q)x!} \qquad \text{by (3.4)} \\ &\leq \limsup_{k\to\infty} \frac{\ell}{(1-q)} \sum_{x=0}^{pk} \frac{k^x/x!}{ke^{-k}} \\ &\leq \frac{\ell}{(1-q)} \lim_{k\to\infty}\frac{1}{k} = 0. \end{aligned}$$

This implies the proof of the lemma for $0 < p < 1$.

When $p=0$, the result is obvious. For $p \geq 1$, we note that the series in (3.2) is a c.d.f. of the Poisson-power function distribution and hence we always have $\underline{a}_p \leq \overline{a}_p \leq 1$. For $p \geq 1$, we consider any $0 < \underline{p} < 1 \leq p$ to obtain

$$\underline{a}_p \geq \liminf_{k\to\infty} \ell\, k^{-\ell} \sum_{x=0}^{\underline{p}k} \frac{\gamma(x+\ell;k)}{x!} = \underline{p}^\ell$$

by the part of the lemma already proved. Since $\underline{p} < 1$ was arbitrary, we have

$$\underline{a}_p = \overline{a}_p = 1 \qquad \text{for all } p \geq 1.$$

This completes the proof of the lemma ■

**Theorem 3.2.** Consider a test $H_0 : \lambda = t$ versus $H_1 : \lambda = \rho t$, $(0 \leq \rho \leq 1)$. For the optimal choice of $c$ to the Poisson-power function distribution, we have

$$\lim_{k \to \infty} \min{}_k (\alpha + \beta) = \rho^{\ell}$$

In fact for this $c$, we have

$$\lim_{k \to \infty} \alpha = \rho^{\ell} \qquad \text{and} \qquad \lim_{k \to \infty} \beta = 0 .$$

**Proof.** Since the random variable $X$ of the Poisson-power function distribution has monotone likelihood ratio (see Kim (1995b)), the Neyman-Pearson test that attains $\min(\alpha + \beta)$ is a left-tail test $X \leq c$, where $c$ is the solution of $x$ in $P(x \mid H_0) = P(x \mid H_1)$ or

$$\frac{\gamma(c+\ell ; d)}{\gamma(c+\ell ; t)} = \rho^{\ell} .$$

Now among all possible critical regions, the choices with $c = pt$ and $\rho \leq p < 1$ gives an upper bounds on the minimum $\alpha + \beta$. Among the choices $c = pt$, the choice $p = \rho$ gives the best bound on the limiting minimum $\alpha + \beta$. So

$$\operatorname{limsup}_{k \to \infty} \{\min{}_k (\alpha + \beta)\} \leq \rho^{\ell} \tag{3.5}$$

Now $\gamma(c+\ell ; t)/\Gamma(c+\ell)$ is a c.d.f. of the gamma distribution evaluated at $t$ with shape parameter $c+\ell$ . The gamma distribution has the monotone likelihood ratio property and hence is a stochastically increasing family of distributions indexed with its shape parameter. Suppose the optimal $c < (1-\varepsilon)d$ on some subsequent of $k$'s is tending to infinity for some $\varepsilon > 0$. Then, by the stochastic ordering,

$$\begin{aligned}
\frac{\gamma(c+\ell ; d)}{\Gamma(c+\ell)} &> \frac{\gamma((1-\varepsilon)d+\ell ; d)}{\Gamma((1-\varepsilon)d+\ell)} \\
&= 1 - \frac{\int_d^{\infty} y^{x+\ell-1} e^{-y} dy}{\Gamma(x+\ell)} \qquad \text{for } x = (1-\varepsilon)d \\
&\geq 1 - \frac{d^{x+\ell-1} e^{-d}}{(1-q)\Gamma(x+\ell)} \qquad \text{by (3.4)}
\end{aligned}$$

for $(1-\varepsilon)\rho < q < 1$. Now Stirling's formula gives

$$\frac{d^{x+\ell-1} e^{-d}}{\Gamma(x+\ell)} \leq \frac{d^{x+\ell-1} e^{-d}}{\sqrt{2\pi}(x+\ell)^{x+\ell+1/2} e^{-(x+\ell)}}$$

which tends to zero as $d \to \infty$ 1 since $-(1-\varepsilon)\ln(1-\varepsilon) - \varepsilon < 0$, for $0 < \varepsilon < 1$. So

$$1 > \frac{\gamma(c+\ell ; t)}{\Gamma(c+\ell)} > \frac{\gamma(c+\ell ; d)}{\Gamma(c+\ell)} \to 1$$

on such a subsequence of $k$'s, which contradicts (3.5). We have the optimal $c \geq (1-\varepsilon)d$ for all $\varepsilon > 0$ and for all sufficiently large $k$.

For the type I error rate alpha computed for the optimal $c$, we have

$$\alpha_c = \sum_{x=0}^{c} P(x|H_0) \geq \sum_{x=0}^{(1-\varepsilon)\rho t} \ell\, t^{-\ell} \frac{\gamma(x+\ell\,;t)}{x!} \tag{3.6}$$

which has a limit of $((1-\varepsilon)\rho)^{\ell}$ by the lemma 3.1. And for the type II error rate beta for this $c$, we have

$$\beta_c = 1 - \sum_{x=0}^{c} P(x|H_1) \geq 1 - \sum_{x=0}^{(1-\varepsilon)d} \ell\, t^{-\ell} \frac{\gamma(x+\ell\,;d)}{x!} \tag{3.7}$$

which has a limit of $1-(1-\varepsilon)^{\ell}$. Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{k\to\infty} \beta_c = 0 \quad \text{and} \quad \lim_{k\to\infty} \alpha_c \geq \rho^{\ell} \tag{3.8}$$

Finally by equation (3.8) the limit of $(\alpha_c + \beta_c)$ exists, and

$$\lim_{k\to\infty} \min(\alpha+\beta) = \lim_{k\to\infty} \alpha_c + \lim_{k\to\infty} \beta_c = \rho^{\ell} + 0 \quad ▣$$

The results of $\min(\alpha+\beta)$ in Theorem 3.2 is consistent with the result of Theorem 3.1.

## 4. Computation of Error Rate

In this section we give an example of the application of the Poison-power function distribution in (2.4) which takes account both of the aiming error and the circular Gaussian distribution of intensity across the NPB.

These examples involve two algorithms. In the first algorithm, the type I error rate (or the leakage rate) upper bound $\alpha^*$ and the type II error rate (or the false alarm rate) $\beta^*$ and the smallest $k$ is determined so that the bounds are satisfied. The second algorithm is applied in case the first algorithm fails. The second algorithm determines if there exists an $k$ so that $\min(\alpha+\beta)$ is satisfactory by some standard. If this is not possible, then discrimination at a satisfactory type I error rate and the type II error rate is not possible. It is found that the first algorithm fails if the aiming error variance $\sigma_2^2$ is too large relative to the variance $\sigma_1^2$ of the Gaussian distribution of intensity across the beam.

**<Algorithm I>** Consider the hypothesis of $H_0: k=t$ vs. $H_1: k=\rho t$, where $\rho = d/t < 1$. In a previous section, we showed that the Neyman-Pearson test here is a left-tail test. For this reason, for a given $\alpha^*$ and $\beta^*$ in the interval $(0,1)$, we choose the smallest $k$ that satisfies

$$\alpha = \sum_{x=0}^{c} P(x|H_0) = \frac{\ell}{t^{\ell}} \sum_{x=0}^{c} \frac{\gamma(x+\ell\,;t)}{\Gamma(x+1)} \le \alpha^* \tag{4.1}$$

and

$$\beta = 1-\sum_{x=0}^{c} P(x|H_1) = 1-\frac{\ell}{(\rho t)^{\ell}} \sum_{x=0}^{\ell} \frac{\gamma(x+\ell\,;\rho t)}{\Gamma(x+1)} \le \beta^* \tag{4.2}$$

where for each $k$, the $c$ is the largest critical value which satisfies (4.1).

By a different calculation using an integration by parts, the quantities in (4.1) and (4.2) can be written as

$$\alpha = 1-\frac{\gamma(c+1;t)-t^{-\ell}\,\gamma(\ell+c+1;t)}{\Gamma(c+1)} \tag{4.3}$$

and

$$\beta = \frac{\gamma(c+1;\rho t)-(\rho t)^{-\ell}\,\gamma(\ell+c+1;\rho t)}{\Gamma(c+1)} \tag{4.4}$$

For an example of the results for Algorithm I, we choose the following parameter value:

$$\rho=0.1 \qquad \alpha^*=\beta^*=0.01$$

The following lists some results.

| $\ell$ | $t$ | $\alpha$ | $\beta$ |
|---|---|---|---|
| 4.0 | 27 | 0.00945 | 0.00920 |
| 3.0 | 56 | 0.00977 | 0.00886 |
| 2.5 | 131 | 0.00997 | 0.00974 |
| 2.4 | 182 | 0.00998 | 0.00974 |
| 2.3 | 272 | 0.00995 | 0.00961 |
| 2.2 | 476 | 0.00997 | 0.00927 |
| 2.1 | 1242 | 0.00999 | 0.00973 |

For $\ell=2.0$, there is no solution to the problem attempting to solve. This is the situation when the aiming error variance is too large relative to the variance of the Gaussian distribution of intensity across the NPB.

**<Algorithm 2>** In this algorithm, we successively fix $k$ and find the minimum value of $\alpha+\beta$. This is done by finding the maximum $x^*$ so that

$$P(x^* \mid H_0) \le P(x^* \mid H_1).$$

The following table lists some sample results of this algorithm. We have selected $\rho=0.1$. The quantity $\min(\alpha+\beta)$ decreases monotonically as $t\rightarrow\infty$. In this case, this limitig value approaches $(0.1)^{\ell}$.

| $\ell$ | $t$ | $\alpha$ | $\beta$ | $\alpha+\beta$ |
|---|---|---|---|---|
| 2.0 | 1 | 0.5285 | 0.0642 | 0.5927 |
| 2.0 | 200 | 0.0225 | 0.0030 | 0.0262 |
| 2.0 | 1000 | 0.0152 | 0.0009 | 0.0162 |
| 2.0 | 2000 | 0.0136 | 0.0006 | 0.0142 |
| 2.0 | 3000 | 0.0130 | 0.0004 | 0.0134 |
| 2.0 | 3800 | 0.0126 | 0.0004 | 0.0129 |

## 5. Conclusion and Remarks

We studied that the error rate in simple hypothesis testing for the limiting Poisson-power function distribution. We found the limit of the sum of two types of errors $\alpha+\beta$ is zero when Poisson parameter $k \to \infty$, but this limit is not zero (i.e., $\rho^{\ell} > 0$ ) for the Poisson-power function distribution. We also give two optimal decision algorithms for a specified error rate. It is found that the first algorithm fails if the aiming error variance $\sigma_2^2$ is too large relative to the variance $\sigma_1^2$ of the Gaussian distribution of intensity across the beam.

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