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Simultaneous Estimation of the Birth and Death Rate of the Linear Growth Birth and Death Process Based on Discrete Time Observation[1]

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Abstract

When the linear growth birth and death process observed at a set of equidistant time points, McNeil and Weiss (1977) present a method for simultaneously estimating the Malthusian parameter and the sum of the two parameters under very restricted assumptions using a diffusion approximation. This article suggests a method, which does not require the restrictions given by McNeil and Weiss, for estimating simultaneously the Malthusian parameter and the sum of the two parameters.

1. Introduction

In the linear growth birth and death process with birth rate λ and death rate μ is considered. When the process is observed only at discrete time points, very little is known about the simultaneous estimation of λ and μ. McNeil and Weiss (1977) present diffusion approximation estimates for $\lambda-\mu$ and $\lambda+\mu$, under the conditions that the time points for observations are equally spaced, $\lambda-\mu$ is close to zero, and both the population size and the number of time points are large. We here give estimators for $\lambda-\mu$ and $\lambda+\mu$ which do not have the restrictions of McNeil and Weiss (1977). These estimators are derived using a combination of a method based on moments, the method suggested by Choi and Severo (1988), and Oh, Severo, and Slivka (1991). In section 2, main results are given. In section 3, an intuitive justification for the suggested estimators is given. In section 4, Monte Carlo simulation results are presented to compare these various estimators.

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2. Estimators for the linear growth birth and death process

Let $Y(t)$ denote the population size at time t of the linear growth process with birth rate λ and death rate $\mu \geq 0$ which satisfy the infinitesimal conditions, for $i=1,2,\cdots$,

$$P\{Y(t+h)=j | Y(t)=i\} = \begin{cases} i\lambda h+o(h) & \text{if } j=i+1 \\ i\mu h+o(h) & \text{if } j=i-1 \\ 1-i(\lambda+\mu)h+o(h) & \text{if } j=i \\ o(h) & \text{if } otherwise \end{cases}$$

where $Y(0)$ = a.

Suppose that we observe the process only at a discrete set of time points resulting in a sample of the form

$$\{(t_0, Y_0)=(0,\text{a}),(t_1, Y_1),\cdots,(t_n, Y_n)=(t^*, \text{a}+B-D)\},$$

in which observation of the process is restricted to the finite number of times t_i with $t_0 < t_1 < \cdots < t_n$, Y_i is the population size at time t_i, and B and D are random variables representing the number of births and deaths in $[0, t^*]$, respectively, with $a+B-D \geq 0$. In what follows, we let $\Delta_i = t_i - t_{i-1}$.

Keiding (1975) proves that when $\Delta_i = t^*/n$, $i=1,\cdots,n$, the so-called equidistant sampling scheme, the maximum likelihood estimator of the Malthusian parameter $\delta=\lambda-\mu$ is

$$\hat{\delta}_D = \frac{n}{t^*} \log\left(\frac{Y_1+\cdots+Y_n}{Y_0+\cdots+Y_{n-1}}\right), \qquad (1)$$

whose properties are investigated by Darwin (1956).

Under the equidistant sampling scheme, McNeil and Weiss (1977) introduce the following simultaneous estimators for $\delta=\lambda-\mu$ and $\phi=\lambda+\mu$, respectively:

$$\hat{\delta}_W = \frac{B-D}{(t^*/n)\sum_{i=1}^{n} Y_i} \qquad (2)$$

and

$$\hat{\phi}_W = \frac{1}{t^*} \sum_{i=1}^{n} \frac{\{Y_i - Y_{i-1} - Y_i(\frac{t^*}{n})\,\hat{\delta}_W\}^2}{Y_{i-1}} \qquad (3)$$

In deriving these estimators they used a diffusion process approximation, which assumed that $\delta=\lambda-\mu$ is close to zero, both the observed value of Y_i and n are large, and the time points are equally spaced.

Following Choi and Severo (1988) and Oh, Severo, and Slivka (1991), we present here two estimators for δ which can be used without the restrictions given above, namely,

$$\hat{\delta}_L = \frac{B-D}{\sum_{i=1}^{n} \Delta_i (Y_i + Y_{i-1})/2} \tag{4}$$

and

$$\hat{\delta}_E = \frac{\bar{\delta}(B-D)}{\sum_{i=1}^{n} Y_{i-1}[\, e^{\bar{\delta}\Delta_i} - 1]} \tag{5}$$

where $\bar{\delta}$ is the median of $\delta_i = (\log Y_i - \log Y_{i-1})/\Delta_i, \quad i=1,\cdots,n.$

For ϕ, we suggest an estimator using a method based on moments, namely,

$$\hat{\phi}_E = \begin{cases} \text{median of } \dfrac{\hat{\delta}_E\{Y_i - Y_{i-1}\exp(\hat{\delta}_E \Delta_i)\}^2}{Y_{i-1}\exp(\hat{\delta}_E\Delta_i)\{\exp(\hat{\delta}_E\Delta_i)-1\}} & \text{if } \hat{\delta}_E \neq 0 \\ \text{median of } \dfrac{(Y_i - Y_{i-1})^2}{Y_{i-1}\Delta_i} & \text{if } \hat{\delta}_E = 0 \end{cases} \tag{6}$$

Note that estimators in (4), (5) and (6) do not assume the equidistant sampling scheme, while (1), (2) and (3) were developed under the equidistant sampling scheme.

3. Justification of the estimators

Suppose the process is observed continuously over the fixed time interval $[0, t^*]$ and $y(t^*) > 0$ where $y(t^*)$ is the observed population size at t^*. Then this gives data of the form $B = b$, $D = d$, $Z_i = z_i$, $i=1,\cdots,b+d$, $z_{b+d+1} > t^* - \sum_{i=1}^{b+d} z_i$ and $N_{j-1} = n_{j-1} > 0$, $j=1,\cdots,b+d+1$ where Z_i is the time between the $(i$-1)-st and i-th transitions, i.e., births or deaths, and N_j is the population size just after the j-th transition. Darwin (1956) showed that the log likelihood is given by

$$\log L(\lambda,\mu) = \log(\prod_{i=1}^{b+d} n_{i-1}) + b\log\lambda - d\log\mu - (\lambda+\mu)s$$

$$s = \int_0^{t^*} y(t)dt = \sum_{i=1}^{b+d} n_{i-1}z_i + n_{b+d}(t^* - \sum_{i=1}^{b+d} z_i) \tag{7}$$

where y(t) is the observed value of Y(t) for $0 \le t \le t^*$. The maximum likelihood estimates of λ and μ are then given by

$$\hat{\lambda}_C = \frac{b}{s} \text{ and } \hat{\mu}_C = \frac{d}{s} \tag{8}$$

When y(t^*) = 0, equation (8) is still valid. Thus for either y(t^*) > 0 or y(t^*) = 0, the maximum likelihood estimates of $\delta = \lambda - \mu$ and $\phi = \lambda + \mu$ are given by

$$\hat{\delta}_C = \frac{b-d}{s} \text{ and } \hat{\phi}_C = \frac{b+d}{s}$$

which is a result of Darwin (1956).

Under the discrete sampling scheme, $b - d$ is known but the denominator s is not known. To approximate s, we follow the method of Choi and Severo (1988) and Oh, Severo, and Slivka (1991). We can rewrite (7) as

$$s = \sum_{i=1}^{n} A_i$$

$$\text{where } A_i = \int_{t_{i-1}}^{t_i} y(t)dt \quad \text{for } i = 1, \cdots, n. \tag{9}$$

Since A_i is not known in this sampling scheme, we employ its trapezoidal approximation $\Delta_i(y_{i-1}+y_i)/2$. This gives the estimator $\hat{\delta}_L$, namely (4).

Another method by which A_i can be approximated is to replace $Y(t)$ in (9) by its conditional expectation given $Y_{i-1} = y_{i-1}$, namely

$$E\{Y(t)|Y_{i-1} = y_{i-1}\} = y_{i-1}\exp\{\delta(t - t_{i-1})\} \tag{10}$$

for $t_{i-1} \le t \le t_i$ (See e.g., Chiang (1980, p. 275)). Since δ is not known, we estimate it by δ_i, the solution of the equation

$$y_i = y_{i-1}\exp\{\delta\Delta_i\} \tag{11}$$

If $y_{i-1} > 0$, then (11) gives

$$\delta_i = (\log y_i - \log y_{i-1})/\Delta_i\ .$$

Let $\bar{\delta}$ be the median of δ_i's.

Replacing δ in (10) by $\bar{\delta}$ gives

$$y_{i-1} \exp\{\bar{\delta}(t-t_{i-1})\}, \quad t_{i-1} \le t \le t_i$$

so that A_i may be approximated by

$$\int_{t_{i-1}}^{t_i} y_{i-1} \exp\{\bar{\delta}_i (t-t_{i-1})\} dt = y_{i-1} \exp(\bar{\delta}\Delta_i - 1)/\bar{\delta} \tag{12}$$

If $y_{i-1} > 0$ and $y_i = 0$, then there is no solution for δ in (11). In this case we approximate A_i in (9) by its trapezoidal approximation, namely,

$$\Delta_i y_{i-1}/2 \tag{13}$$

Thus from (12) and (13), we have the corresponding estimator $\hat{\delta}_E$ of δ as in (5).

It should be noted that all denominators of $\hat{\delta}_L$, $\hat{\delta}_E$ and $\hat{\delta}_W$ approximate s.

Now we estimate ϕ. Two cases will be considered. In the case in which $\hat{\delta}_E \neq 0$, we use the expression for the variance of Y_i given $Y_{i-1} = y_{i-1}$ valid for $\delta \neq 0$. In the case in which $\hat{\delta}_E = 0$, we use the corresponding variance expression valid for $\delta = 0$.

It is well known that for $\delta \neq 0$, the variance of Y_i given $Y_{i-1} = y_{i-1} > 0$ is given by

$$Var(Y_i | Y_{i-1} = y_{i-1}) = y_{i-1} \frac{\phi}{\delta} \exp(\delta\Delta_i)\{\exp(\delta\Delta_i) - 1\} \tag{14}$$

for $i = 1, \cdots, n$ (see, e.g., Chiang (1980, p. 275)). Therefore when $\hat{\delta}_E \neq 0$, we replace the left-hand-side of (14) by

$$\{y_i - E_{\hat{\delta}_E}(Y_i | Y_{i-1} = y_{i-1})\}^2 = \{y_i - y_{i-1} \exp(\hat{\delta}_E \Delta_i)\}^2$$

and replace δ in the right-hand-side of (14) by $\hat{\delta}_E$ to get

$$\begin{aligned} &\{y_i - y_{i-1} \exp(\hat{\delta}_E \Delta_i)\}^2 \\ &= y_{i-1} \frac{\phi}{\hat{\delta}_E} \exp(\hat{\delta}_E \Delta_i)\{\exp(\hat{\delta}_E \Delta_i) - 1\} \end{aligned} \tag{15}$$

The solution of (15) for ϕ is given by

$$\frac{\hat{\delta}_E \{y_i - y_{i-1} \exp(\hat{\delta}_E \Delta_i)\}^2}{y_{i-1} \exp(\hat{\delta}_E \Delta_i)\{\exp(\hat{\delta}_E \Delta_i) - 1\}} \, . \tag{16}$$

Then an estimate of ϕ is given by

$$\text{median of } \frac{\hat{\delta}_E\{y_i - y_{i-1}\exp(\hat{\delta}_E \Delta_i)\}^2}{y_{i-1}\exp(\hat{\delta}_E \Delta_i)\{\exp(\hat{\delta}_E \Delta_i) - 1\}} \tag{17}$$

Next we consider the case $\hat{\delta}_E = 0$. When $\delta = 0$, the expectation of Y_i given $Y_{i-1} = y_{i-1}$ is

$$E(Y_i | Y_{i-1} = y_{i-1}) = y_{i-1}$$

and the variance of Y_i given $Y_{i-1} = y_{i-1}$ is given by

$$Var(Y_i | Y_{i-1} = y_{i-1}) = \phi y_{i-1} \Delta_i \tag{18}$$

for $i = 1, \cdots, n$ (see, e.g., Chiang (1980, p. 276)). We replace the left-hand-side of (18) by

$$\{y_i - E_{\delta=0}(Y_i | Y_{i-1} = y_{i-1})\}^2 = (y_i - y_{i-1})^2$$

to get the equation

$$(y_i - y_{i-1})^2 = \phi y_{i-1} \Delta_i \tag{19}$$

The solution of (19) for ϕ is given by

$$\frac{(y_i - y_{i-1})^2}{y_{i-1} \Delta_i} . \tag{20}$$

Then an estimate of ϕ is given by

$$\text{median of } \frac{(y_i - y_{i-1})^2}{y_{i-1} \Delta_i} .$$

4. Monte Carlo experiments

In order to compare the performance of various estimators for δ and ϕ, simulation studies with 2000 replications have been conducted for δ = 0.05, ϕ = 0.06, t^* = 100, a = 5 and n = 10, 20, 25, 50, and 100 under the equidistant sampling scheme. Averages and square roots of mean square errors of various estimators for δ and ϕ are obtained. The results are summarized in Table 1. For each n, the first row is for empirical means of estimators and the second row is for square roots of empirical mean square errors of estimators.

Under the nonequidistant sampling scheme in which t_i is determined by

$$t_i = \frac{3}{\delta} \log\left[1 + \frac{i}{n}\{\exp(\delta t^*/3) - 1\}\right] \tag{21}$$

for $i = 1, \cdots, n$. Simulation studies with 2000 replications also have been conducted for δ = 0.05, ϕ = 0.06, t^* = 100, a = 5 and n = 10, 20, 25, 50, and 100. The result is summarized in Table 2.

The sampling scheme (21) is an approximate optimal sampling scheme for the linear growth pure birth process given by Becker and Kersting (1983). Since the linear growth birth and death process Y(t) with $\delta > 0$ behaves like a linear growth pure birth process with birth rate δ, at least in mean, the sampling scheme (21) intuitively might be applicable to the linear growth birth and death process.

Table 1. Empirical means and square roots of empirical mean square errors, in the first and second row for each n, respectively, of various estimators of 2000 replications for δ = 0.05, ϕ =0.06, t^* = 100, and a = 5 under the equidistant sampling scheme.

n	$\hat{\delta}_C$	$\hat{\delta}_E$	$\hat{\delta}_W$	$\hat{\delta}_D$	$\hat{\phi}_C$	$\hat{\phi}_E$	$\hat{\phi}_W^*$
10	0.0495	0.0496	0.0388	0.0496	0.0597	0.0593	0.0254
	0.0025	0.0056	0.0112	0.0057	0.0024	0.0314	0.0368
20	0.0496	0.0495	0.0440	0.0494	0.0597	0.0594	0.0373
	0.0026	0.0042	0.0064	0.0043	0.0025	0.0219	0.0257
25	0.0496	0.0496	0.0450	0.0495	0.0597	0.0595	0.0410
	0.0025	0.0034	0.0056	0.0036	0.0024	0.0192	0.0220
50	0.0496	0.0495	0.0473	0.0495	0.0597	0.0596	0.0468
	0.0025	0.0028	0.0035	0.0029	0.0024	0.0138	0.0164
100	0.0496	0.0496	0.0483	0.0496	0.0598	0.0597	0.0528
	0.0024	0.0025	0.0028	0.0025	0.0024	0.0102	0.0122

Table 2. Empirical means and square root of empirical mean square errors, in the first and second row for each n, respectively, of various estimators of 2000 replications for δ = 0.05, ϕ = 0.06, t^* = 100, and a = 5 under the nonequidistant sampling scheme (21).

n	$\hat{\delta}_C$	$\hat{\delta}_E$	$\hat{\phi}_C$	$\hat{\phi}_E$
10	0.0496	0.0495	0.0597	0.0616
	0.0025	0.0055	0.0024	0.0316
20	0.0497	0.0496	0.0598	0.0604
	0.0025	0.0039	0.0025	0.0211
25	0.0497	0.0496	0.0597	0.0595
	0.0025	0.0034	0.0023	0.0204
50	0.0497	0.0497	0.0598	0.0596
	0.0024	0.0028	0.0023	0.0132
100	0.0497	0.0498	0.0598	0.0597
	0.0025	0.0026	0.0024	0.0104

In both Table 1 and Table 2, $\hat{\delta}_E$ compares favorably with $\hat{\delta}_C$. For all cases empirical means of $\hat{\phi}_W$ are smaller than those of $\hat{\delta}_C$ and increase as n increases. The empirical means of $\hat{\phi}_E$ are close to the true value ϕ = 0.6. As expected, $\hat{\phi}_W$ is not a good competitor of $\hat{\phi}_E$.

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