한국통계학회논문집 제3권 1호
1996년 4월 pp. 195-206

Effects of Identified Outliers for Parametric Estimators in a Pareto[1]

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Abstract

Several parametric estimators of the scale parameter and reliability in an assumed Pareto distribution with the presence of identified outliers are proposed, and their efficiencies are compared numerically each other.

1. Introduction

Many authors considered the problems of parametric estimation in the Pareto distribution because of its application in many socio-economic studies, physical and biological phenomena. Here we consider the estimation of the scale parameter and reliability in an assumed Pareto distribution with the presence of identified outliers when the shape parameter is known. There are many situations in which it is reasonable to assume that the items may not be homogeneous and hence the assumption of i.i.d. random variables may be unrealistic, and then the model may have to be modified suitably. Recently, Gather and Kale(1988), Dixit(1989, 1991 & 1994), Rohatgi and Selvavel(1993), and Woo(1994) considered the problems of parametric estimation in several distributions with the presence of outliers.

In this paper, we propose the several estimators for the scale parameter and reliability in an assumed Pareto distribution with the known shape parameter when identified outliers are present and exactly derive the probability density functions of order statistics with non-identically distributied random variables in the assumed Pareto distribution. Also, we obtain the biases and mean square errors for their estimators, and compare numerically the proposed several estimators of the scale parameter and reliability in the sence of mean square error.

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2. Parametric Estimation

The Pareto law in the shape-scale form is defined in terms of its density function(pdf) by

$$f(x;\alpha,\lambda) = \alpha\lambda^{\alpha}x^{-(\alpha-1)}, \qquad x>\lambda>0, \quad \alpha>0, \tag{2.1}$$

where α and λ are referred as the shape and scale parameters, respectively, denoted by $X\sim PAR(\alpha,\lambda)$ It is often used as a model for incomes, city population sizes and other similar phenomena.

Suppose that $X_1, X_2, \cdots, X_n$ contain exactly k(known)-number of the outliers but the outliers themselves are not known whose minimum income is large (or small) as compared to rest of them. Thus, we assume that $X_1, X_2, \cdots, X_n$ are independent random variables such that n-k of them are from $PAR(\alpha,\lambda)$ and the remaining k are from $PAR(\alpha, b\lambda)$ where b is a known positive constant. Before the start of the experiment, we have no prior knowledge as to which k of those n are the outliers. Let $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ be the corresponding order statistics of the sample.

From the permanent theory(Vaught and Venables(1972)), the pdf of $X_{(r)}$, $r=1,\cdots,n$ are given as follows : for $b\ge 1$,

$$f_r(x) = \begin{cases} \sum_{j=m_1}^{m_2}\sum_{p=0}^{j}\sum_{q=0}^{r-1-j} A_1 k\alpha b^{\alpha(k+p-j)}\lambda^{\alpha(n+p+q+1-r)}x^{-[\alpha(n+p+q+1-r)+1]} & \\ \quad + \sum_{j=m_3}^{m_4}\sum_{p=0}^{j}\sum_{q=0}^{r-1-j} A_2(n-k)\alpha b^{\alpha(k+p-j)}\lambda^{\alpha(n+p+q+1-r)}x^{-[\alpha(n+p+q+1-r)+1]}, & x>b\lambda \\ \sum_{p=0}^{r-1} A_3(n-k)\alpha\lambda^{\alpha(n+p+1-r-k)}x^{-[\alpha(n+p+1-r-k)+1]}, & \lambda<x\le b\lambda \end{cases}$$

and for $b<1$,

$$f_r(x)= \begin{cases} \sum_{j=m_1}^{m_2}\sum_{p=0}^{j}\sum_{q=0}^{r-1-j} A_1 k\alpha b^{\alpha(k+p-j)}\lambda^{\alpha(n+p+q+1-r)}x^{-[\alpha(n+p+q+1-r)+1]} & \\ \quad + \sum_{j=m_3}^{m_4}\sum_{p=0}^{j}\sum_{q=0}^{r-1-j} A_2(n-k)\alpha b^{\alpha(k+p-j)}\lambda^{\alpha(n+p+q+1-r)}x^{-[\alpha(n+p+q+1-r)+1]}, & x>\lambda \\ \sum_{p=0}^{r-1} A_4 k\alpha\lambda^{\alpha(k+p+1-r)}x^{-[\alpha(k+p+1-r)+1]}, & b\lambda<x\le\lambda \end{cases}$$

where,

$$A_1 = (-1)^{p+q} C(j, p) C(r-1-j, q) C(k-1, j) C(n-k, r-1-j),$$

$$A_2 = (-1)^{p+q} C(j, p) C(r-1-j, q) C(k, j) C(n-k-1, r-1-j),$$

$$A_3 = (-1)^p C(r-1, p) C(n-k-1, r-1), \quad A_4 = (-1)^p C(r-1, p) C(k-1, r-1),$$

$$C(n, r) = \frac{n!}{(n-r)!\, r!},$$

$$m_1 = \max(0, k+r-1-n), \quad m_2 = \min(k-1, r-1),$$

$$m_3 = \max(0, k+r-n) \text{ and } m_4 = \min(k, r-1).$$

Therefore, the l-th moment of $X_{(r)}$, $r = 1, 2, \cdots, n$, can be obtained as

$$E(X^l_{(r)}) = \begin{cases} \{ \sum_{j=m_1}^{m_2} \sum_{p=0}^{j} \sum_{q=0}^{r-1-j} \frac{A_1 ka}{a(n+p+q+1-r)-l} b^{-[a(n+j+q+1-r-k)-l]} \\ \quad + \sum_{j=m_3}^{m_4} \sum_{p=0}^{j} \sum_{q=0}^{r-1-j} \frac{A_2 (n-k) a}{a(n+p+q+1-r)-l} b^{-[a(n+j+q+1-r-k)-l]} \\ \quad + \sum_{p=0}^{r-1} \frac{A_3 (n-k) a}{a(n+p+1-k-r)-l} (1 - b^{-[a(n+p+1-k-r)-l]}) \} \lambda^l, & b \geq 1 \\ \{ \sum_{j=m_1}^{m_2} \sum_{p=0}^{j} \sum_{q=0}^{r-1-j} \frac{A_1 ka}{a(n+p+q+1-r)-l} b^{a(k+p-j)} \\ \quad + \sum_{j=m_3}^{m_4} \sum_{p=0}^{j} \sum_{q=0}^{r-1-j} \frac{A_2 (n-k) a}{a(n+p+q+1-r)-l} b^{a(k+p-j)} \\ \quad + \sum_{p=0}^{r-1} \frac{A_4 ka}{a(k+p+1-r)-l} (b^l - b^{-a(k+p+1-r)}) \} \lambda^l, & b < 1. \end{cases} \tag{2.2}$$

Here we consider the estimation of the scale parameter in an assumed Pareto distribution with the presence of identified outliers when the shape parameter is known. In an assumed Pareto distribution, the MLE of λ is $\hat{\lambda}_1 = X_{(1)}$ From the result (2.2), we can obtian the l-th moment of $X_{(1)}$ as following ;

$$E(X^l_{(1)}) = \begin{cases} \{ \frac{(n-k)a}{(n-k)a-l} - \frac{kla}{(na-l)[(n-k)a-l]} b^{-[(n-k)a-l]} \} \lambda^l, & b \geq 1 \\ \{ \frac{ka}{ka-l} b^l - \frac{(n-k) la}{(na-l)(ka-l)} b^{ka} \} \lambda^l, & b < 1, \end{cases} \tag{2.3}$$

$$= C_l \cdot \lambda^l, \quad l = 1, 2, \cdots.$$

Therefore, the bias and MSE of $\hat{\lambda}_1$ are given by

$$BIAS(\hat{\lambda}_1)=\begin{cases}\left(\frac{1}{(n-k)a-1}-\frac{ka}{(na-1)[(n-k)a-1]}\,b^{-[(n-k)a-1]}\right)\lambda, & b\geq 1\\ \left(\frac{ka(b-1)+1}{ka-1}-\frac{(n-k)a}{(na-1)(ka-1)}\,b^{ka}\right)\lambda, & b<1.\end{cases} \quad (2.4)$$

and

$$MSE(\hat{\lambda}_1)=\begin{cases}\{\frac{2}{[(n-k)a-1][(n-k)a-2]}+2kab^{-[(n-k)a-1]} \\ \quad\cdot\left(\frac{1}{(na-1)[(n-k)a-1]}-\frac{b}{(na-2)[(n-k)a-2]}\right)\}\lambda^2, & b\geq 1\\ \{\frac{bka[ka(b-1)-(b-2)]}{(ka-1)(ka-2)}-\frac{2(n-k)ab^{ka}[(n-k)a-3)]}{(na-2)(ka-2)}+1\}\lambda^2, & b<1\end{cases}$$

(2.5)

From the result (2.3), an unbiased estimator for λ is given by

$$\hat{\lambda}_2=C_1^{-1}\cdot X_{(1)}$$

From the result (2.3), the variances of $\hat{\lambda}_2$ is given by

$$VAR(\hat{\lambda}_2)=\begin{cases}\{\left(\frac{(n-k)a}{(n-1)a-2}-\frac{2ka}{(na-2)[(n-k)a-2]}\,b^{-[(n-k)a-2]}\right) \\ \quad\left(\frac{(n-k)a}{(n-k)a-1}-\frac{ka}{(na-1)[(n-k)a-1]}\,b^{-[(n-k)a-1]}\right)^{-2}-1\}\,\lambda^2, & b\geq 1\\ \{\left(\frac{ka}{ka-2}\,b^2-\frac{2(n-k)a}{(na-1)(ka-1)}\,b^{ka}\right) \\ \quad\left(\frac{ka}{(ka-1)}\,b-\frac{(n-k)a}{(na-1)(ka-1)}\,b^{ka}\right)^{-2}-1\}\,\lambda^2, & b<1\end{cases}$$

(2.6)

Next, we shall consider a class of estimators of scale parameter λ by

$$C=\{\, c\cdot X_{(1)} \mid 0<c \,\}$$

Then, we can propose an estimator which minimizes mean square errors among the class :

$$\hat{\lambda}_3=\frac{C_2}{C_1^2}X_{(1)}.$$

From result (2.2), we can obtain the bias and mean square error for $\hat{\lambda}_3$ as following;

$$BIAS(\widehat{\lambda_3}) = (\frac{C_2}{C_1} - 1)\lambda, \qquad MSE(\widehat{\lambda_3}) = (\frac{C_2^3}{C_1^4} - \frac{2C_2}{C_1} + 1)\lambda^2. \tag{2.7}$$

Now we consider the estimators of the reliability in an assumed Pareto distribution with the known shape parameter when identified outliers are present. In a Pareto distribution, the reliability is given by

$$R(t \mid \lambda) = \begin{cases} 1, & t \le \lambda \\ \lambda^{\alpha} \cdot t^{-\alpha}, & t > \lambda . \end{cases}$$

Therefore, the MLE $\widehat{R_1}(t)$ for the reliability is given as follow ;

$$\widehat{R_1}(t) = \begin{cases} 1, & t \le X_{(1)} \\ X_{(1)}^{\alpha} \cdot t^{-\alpha}, & t > X_{(1)} . \end{cases}$$

Since the pdf of $X_{(1)}$ is given as following ;

for $b \ge 1$,

$$f_1(x) = \begin{cases} nab^{k\alpha}\lambda^{n\alpha}x^{-(n\alpha+1)}, & x > b\lambda \\ (n-k)a\lambda^{(n-k)\alpha}x^{-[(n-k)\alpha+1]}, & \lambda < x \le b\lambda, \end{cases} \tag{2.8}$$

and for $b < 1$,

$$f_1(x) = \begin{cases} nab^{k\alpha}\lambda^{n\alpha}x^{-(n\alpha+1)}, & x > \lambda \\ kab^{k\alpha}\lambda^{k\alpha}x^{-(k\alpha+1)}, & b\lambda < x \le \lambda, \end{cases}$$

the l-th moment of $\widehat{R_1}(t)$ is given as ;

for $b \ge 1$,

$$E[\widehat{R_1^l}(t)] = [1 - b^{k\alpha}(\frac{\lambda}{t})^{n\alpha}](\frac{\lambda}{t})^{l\alpha}\{\frac{n}{n-l}b^{-(n-k-l)\alpha}[1-(\frac{b\lambda}{t})^{(n-l)\alpha}] + \frac{n-k}{n-k-l}[1-b^{-(n-k-l)\alpha}]\} + b^{2k\alpha}(\frac{\lambda}{t})^{2n\alpha}$$

and for $b < 1$,

$$E[\widehat{R_1^l}(t)] = \begin{cases} [1-b^{k\alpha}(\frac{\lambda}{t})^{n\alpha}]\, b^{k\alpha}(\frac{\lambda}{t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{\lambda}{t})^{(n-l)\alpha}] + \frac{k}{k-l}[b^{-(k-l)\alpha}-1]\} + b^{2k\alpha}(\frac{\lambda}{t})^{2n\alpha}, & k-l \ne 0 \\ [1-b^{k\alpha}(\frac{\lambda}{t})^{n\alpha}]\, b^{k\alpha}(\frac{\lambda}{t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{\lambda}{t})^{(n-l)\alpha}] + -k(\log b)a\lambda^{(k-l)\alpha}\} + b^{2k\alpha}(\frac{\lambda}{t})^{2n\alpha}, & k-l = 0. \end{cases} \tag{2.9}$$

Next, using the unbiased estimator for λ, we propose the following estimator $\widehat{R_2}(t)$ for the reliability in an assumed Pareto distribution with the presence of identified outliers when the shape parameter is known;

$$\widehat{R_2}(t)=\begin{cases}1, & t\le C_1^{-1}X_{(1)}\\ C_1^{-\alpha}\cdot X_{(1)}^{\alpha}\cdot t^{-\alpha}, & t> C_1^{-1}X_{(1)}.\end{cases}$$

From the density function (2.8), the l-th moment of $\widehat{R_2}(t)$ is given as following ;

for $b\ge 1$,

$$E[\widehat{R_2^l}(t)] = [1-b^{k\alpha}(\frac{\lambda}{C_1 t})^{n\alpha}](\frac{\lambda}{C_1 t})^{l\alpha}\{\frac{n}{n-l}b^{-(n-k-l)\alpha}[1-(\frac{b\lambda}{C_1 t})^{(n-l)\alpha}] + \frac{n-k}{n-k-l}[1-b^{-(n-k-l)\alpha}]\} + b^{2k\alpha}(\frac{\lambda}{C_1 t})^{2n\alpha},$$

and for $b<1$,

$$E[\widehat{R_2^l}(t)] = \begin{cases}[1-b^{k\alpha}(\frac{\lambda}{C_1 t})^{n\alpha}]\, b^{k\alpha}(\frac{\lambda}{C_1 t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{\lambda}{C_1 t})^{(n-l)\alpha}] \\ \quad +\frac{k}{k-l}[b^{-(k-l)\alpha}-1]\} + b^{2k\alpha}(\frac{\lambda}{C_1 t})^{2n\alpha}, & k-l\neq 0\\ [1-b^{k\alpha}(\frac{\lambda}{C_1 t})^{n\alpha}]\, b^{k\alpha}(\frac{\lambda}{C_1 t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{\lambda}{C_1 t})^{(n-l)\alpha}] \\ \quad -k(\log b)\alpha\lambda^{(k-l)\alpha}\} + b^{2k\alpha}(\frac{\lambda}{C_1 t})^{2n\alpha}, & k-l=0.\end{cases} \tag{2.10}$$

Finally, we propose an estimator $\widehat{R_3}(t)$ for reliability which the scale parameter in the reliability is replaced by the $\widehat{\lambda_3}$

$$\widehat{R_3}(t)=\begin{cases}1, & t\le \frac{C_2}{C_1^2}X_{(1)}\\ (\frac{C_2}{C_1^2})^{\alpha}X_{(1)}^{\alpha}\cdot t^{-\alpha}, & t> \frac{C_2}{C_1^2}X_{(1)}.\end{cases}$$

From the result (2.8), the l-th moment of $\widehat{R_3}(t)$ is given as ;

for $b\ge 1$,

$$E[\widehat{R_3^l}(t)] = [1-b^{k\alpha}(\frac{C_2\lambda}{C_1^2 t})^{n\alpha}](\frac{C_2\lambda}{C_1^2 t})^{l\alpha}\{\frac{n}{n-l}b^{-(n-k-l)\alpha}[1-(\frac{bC_2\lambda}{C_1^2 t})^{(n-l)\alpha}] + \frac{n-k}{n-k-l}[1-b^{-(n-k-l)\alpha}]\} + b^{2k\alpha}(\frac{C_2\lambda}{C_1^2 t})^{2n\alpha},$$

and for $b<1$,

$$E[\widehat{R}_3^2(t)] = \begin{cases} [1-b^{k\alpha}(\frac{C_2\lambda}{C_1^2t})^{n\alpha}]\, b^{k\alpha}(\frac{C_2\lambda}{C_1^2t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{C_2\lambda}{C_1^2t})^{(n-l)\alpha}] \\ \quad +\frac{k}{k-l}[b^{-(k-l)\alpha}-1]\}+b^{2k\alpha}(\frac{C_2\lambda}{C_1^2t})^{2n\alpha}, & k-l\neq 0 \\ [1-b^{k\alpha}(\frac{C_2\lambda}{C_1^2t})^{n\alpha}]\, b^{k\alpha}(\frac{C_2\lambda}{C_1^2t})^{l\alpha}\{\frac{n}{n-l}[1-(\frac{C_2\lambda}{C_1^2t})^{(n-l)\alpha}] \\ \quad -k(\log b)\alpha\lambda^{(k-l)\alpha}\}+b^{2k\alpha}(\frac{C_2\lambda}{C_1^2t})^{2n\alpha}, & k-l=0. \end{cases} \tag{2.11}$$

3. Numerical Comparisons

From the results (2.4) through (2.11), we evaluate the exact numerical values of mean square errors for the proposed several estimators of the scale parameter and reliability in an assumed Pareto distribution with presence of identified outliers. The numerical values of mean square errors for the several estimators are given in Tables 1 through 4 for the sample size n=25, k=0(1)5, $b=1\pm 1/2^p$, $p=1,2,3$, and $R(t)=0.01,\ 0.05,\ 0.1$.

In the sense of mean square error, $\widehat{\lambda_3}$ is more efficient than the MLE and an unbiased estimator of the scale parameter, and $\widehat{R_3}(t)$ is more efficient than other two estimators for the reliability, even though the identified outliers exist.

Table 1. MSE of estimators for the scale parameter in an assumed Pareto distribution with presence of identified outliers.

($\alpha = 3$, $\lambda = 1$)

n	k	b	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
25	0	1	.0003702	.0001826	.0001826
	1	1.5	.0004024	.0001984	.0001984
		1.25	.0004024	.0001984	.0001984
		1.125	.0004023	.0001983	.0001984
		0.875	.0022012	.0020767	.0020723
		0.75	.0157811	.0110445	.0109238
		0.5	.1250462	.0598821	.0564988
	2	1.5	.0004389	.0002163	.0002163
		1.25	.0004389	.0002163	.0002163
		1.125	.0004388	.0002162	.0002162
		0.875	.0037241	.0027725	.0027648
		0.75	.0260180	.0110177	.0108977
		0.5	.1734432	.3114542	.0302047
	3	1.5	.0004808	.0002367	.0002366
		1.25	.0004807	.0002367	.0002366
		1.125	.0004805	.0002364	.0002364
		0.875	.0049985	.0028573	.0028492
		0.75	.0330604	.0085410	.0084687
		0.5	.1963595	.0150641	.0148406
	4	1.5	.0005288	.0002602	.0002601
		1.25	.0005288	.0002602	.0002601
		1.125	.0005282	.0002597	.0002597
		0.875	.0060715	.0026601	.0026530
		0.75	.0380722	.0061868	.0061488
		0.5	.2090865	.0082615	.0081939
	5	1.5	.0005845	.0002874	.0002872
		1.25	.0005844	.0002874	.0002872
		1.125	.0005834	.0002864	.0002863
		0.875	.0069806	.0023562	.0023506
		0.75	.0417537	.0044452	.0044255
		0.5	.2170326	.0051213	.0050953

Table 2. MSE of estimators for the reliability in an assumed Pareto distribution with presence of identified outliers.

($\alpha = 3$, $\lambda = 1$, $R(t) = 0.01$)

n	k	b	$\widehat{R_1}(t)$	$\widehat{R_2}(t)$	$\widehat{R_3}(t)$
25	0	1	.0000004	.0000001	.0000001
	1	1.5	.0000004	.0000002	.0000002
		1.25	.0000004	.0000002	.0000002
		1.125	.0000004	.0000002	.0000002
		0.875	.0000017	.0000017	.0000016
		0.75	.0000095	.0000094	.0000087
		0.5	.0000465	.0000405	.0000398
	2	1.5	.0000004	.0000002	.0000002
		1.25	.0000004	.0000002	.0000002
		1.125	.0000004	.0000002	.0000002
		0.875	.0000028	.0000024	.0000023
		0.75	.0000154	.0000113	.0000104
		0.5	.0000612	.0000601	.0000496
	3	1.5	.0000005	.0000002	.0000002
		1.25	.0000005	.0000002	.0000002
		1.125	.0000005	.0000002	.0000002
		0.875	.0000037	.0000026	.0000025
		0.75	.0000193	.0000095	.0000090
		0.5	.0000669	.0000266	.0000241
	4	1.5	.0000005	.0000003	.0000003
		1.25	.0000005	.0000003	.0000003
		1.125	.0000005	.0000003	.0000003
		0.875	.0000045	.0000025	.0000024
		0.75	.0000221	.0000071	.0000068
		0.5	.0000698	.0000125	.0000118
	5	1.5	.0000006	.0000003	.0000003
		1.25	.0000006	.0000003	.0000003
		1.125	.0000006	.0000003	.0000003
		0.875	.0000051	.0000023	.0000022
		0.75	.0000240	.0000051	.0000049
		0.5	.0000714	.0000068	.0000066

Table 3. MSE of estimators for the reliability in an assumed Pareto distribution with presence of identified outliers.

($\alpha = 3$, $\lambda = 1$, $R(t) = 0.05$)

n	k	b	$\widehat{R_1}(t)$	$\widehat{R_2}(t)$	$\widehat{R_3}(t)$
25	0	1	.0000091	.0000043	.0000043
	1	1.5	.0000099	.0000047	.0000047
		1.25	.0000099	.0000047	.0000047
		1.125	.0000099	.0000047	.0000047
		0.875	.0000422	.0000421	.0000421
		0.75	.0002383	.0002371	.0002194
		0.5	.0011624	.0022624	.0015311
	2	1.5	.0000108	.0000052	.0000052
		1.25	.0000108	.0000052	.0000052
		1.125	.0000108	.0000052	.0000052
		0.875	.0000697	.0000608	.0000596
		0.75	.0003857	.0002826	.0002620
		0.5	.0015298	.0015018	.0012269
	3	1.5	.0000119	.0000057	.0000057
		1.25	.0000119	.0000057	.0000057
		1.125	.0000119	.0000057	.0000057
		0.875	.0000926	.0000657	.0000645
		0.75	.0004837	.0002387	.0002251
		0.5	.0016748	.0006641	.0006015
	4	1.5	.0000131	.0000063	.0000063
		1.25	.0000131	.0000063	.0000063
		1.125	.0000131	.0000063	.0000063
		0.875	.0001118	.0000634	.0000622
		0.75	.0005513	.0001785	.0001712
		0.5	.0017445	.0003125	.0002958
	5	1.5	.0000146	.0000069	.0000069
		1.25	.0000146	.0000069	.0000069
		1.125	.0000146	.0000069	.0000069
		0.875	.0001279	.0000575	.0000565
		0.75	.0005994	.0001286	.0001247
		0.5	.0017838	.0001711	.0001653

Table 4. MSE of estimators for the reliability in an assumed Pareto distribution with presence of identified outliers.

($\alpha = 3$, $\lambda = 1$, $R(t) = 0.1$)

n	k	b	$\widehat{R_1}(t)$	$\widehat{R_2}(t)$	$\widehat{R_3}(t)$
25	0	1	.0000362	.0000174	.0000174
	1	1.5	.0000395	.0000189	.0000189
		1.25	.0000395	.0000189	.0000189
		1.125	.0000395	.0000189	.0000189
		0.875	.0001689	.0001712	.0001687
		0.75	.0009535	.0009482	.0008777
		0.5	.0046497	.0090493	.0061246
	2	1.5	.0000433	.0000207	.0000207
		1.25	.0000433	.0000207	.0000207
		1.125	.0000433	.0000207	.0000206
		0.875	.0002789	.0002432	.0002385
		0.75	.0015423	.0011307	.0010481
		0.5	.0061191	.0060071	.0049077
	3	1.5	.0000476	.0000227	.0000227
		1.25	.0000476	.0000227	.0000227
		1.125	.0000475	.0000227	.0000226
		0.875	.0003705	.0002630	.0002578
		0.75	.0019349	.0009548	.0009007
		0.5	.0066993	.0026564	.0024061
	4	1.5	.0000526	.0000251	.0000250
		1.25	.0000526	.0000251	.0000250
		1.125	.0000525	.0000251	.0000249
		0.875	.0004472	.0002534	.0002488
		0.75	.0022051	.0007143	.0006848
		0.5	.0069783	.0012502	.0011832
	5	1.5	.0000585	.0000277	.0000277
		1.25	.0000584	.0000278	.0000277
		1.125	.0000583	.0000278	.0000276
		0.875	.0005119	.0002300	.0002262
		0.75	.0023976	.0005144	.0004991
		0.5	.0071354	.0006845	.0006613

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