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Moments of Probability Distributions Derived Using Differential Operators [1)]

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Abstract

In 1992, Boullion obtained the method of the calculus of the moments of discrete probability distributions using differential operator, and he published the method of calculus of the moments.

The purpose of this paper is to introduce an idea that this method can be applied to calculate the moments of continuous probability distributions.

1. Introduction

Up to now, we have generally used the moment generating function and the characteristic function for the calculation of the moments. Unfortunately this method does not work in some calculations of moments.

In 1981, Link demonstrated a new calculus of moments using the finite difference operator and showed some instances of calculating method for the moments of discrete probability distributions. Afterward, this method has been evolved by subsequent scholars: Chan(1982), Rao and Janardan(1982), Janardan(1984), Charalambides(1984, 1986). And the problems could be partially solved which could not be calculated by the moment generating function and the characteristic function.

In 1992, Boullion succeeded in developing another new method of calculation using the differential operator and introduced the calculus of moment of binomial distribution, Poisson distribution, negative binomial distribution, and Neyman type A distribution.

However, these two methods only involved the calculus of moments of discrete probability distributions.

In this paper, we intend to introduce a method which can calculate moments of the continuous probability distributions. In chapter 2, 3, we define the differential operator, and introduce main results and some examples.

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2. The Calculation of the Moments

We will use the following differential operator defined by Boullion(1992):

$$\delta_x = x\frac{d}{dx}, \; u\frac{d}{du}u^x g(x) = \delta_u * u^x g(x).$$

Let D_u^r be r times differentiation with respect to u and denoted by $D_u^r = \frac{d^r}{du^r}u^x$ where x is a random variable $X=x$ and let I_u^r be r-times integration with respect to u and denoted by $I_u^r = \int \ldots \int u^x du \ldots du$.

Definition 2.1. For the random variable X, $E(X^{(r)})$ is called r-th decending factorial moment if $X^{(r)} = X(X-1)\ldots(X-r+1)$.

Definition 2.2. For the random variable X, $E(X^{[r]})$ is called r-th ascending factorial moment if $X^{[r]} = X(X+1)\ldots(X+r-1)$.

Theorem 2.3. Let X be a continuous random variable with density $cu^x g(x)$, where c is a constant and u is not a function of x. Then, there exists $h(u)$ such that the k-th moment of X about the point b, μ_b^k is $c(\delta_u - b)^k * h(u)$.

Proof. Let X be a continuous random variable with density $cu^x g(x)$. Then

$$\begin{aligned}
\mu_b^k &= \int_{-\infty}^{\infty} (x-b)^k cu^x g(x)dx \\
&= c\int_{-\infty}^{\infty} \Sigma_i \, {}_kC_i(-b)^{k-i} x^i u^x g(x)dx \\
&= c\int_{-\infty}^{\infty} \Sigma_i \, {}_nC_i(-b)^{k-i} \delta_u^i * u^x g(x)dx \\
&= c(\delta_u - b)^k * \int_{-\infty}^{\infty} u^x g(x)dx \\
&= c(\delta_u - b)^k * h(u),
\end{aligned}$$

where $\int_{-\infty}^{\infty} u^x g(x)dx$ exists, and $h(u) = \int_{-\infty}^{\infty} u^x g(x)dx$.

Example 2.4. Let X be an exponential random variable with density ae^{-ax}. Then since $ae^{-ax}=a(e^{-a})^x, c=a>0, u=e^{-a}, g(x)=1$ and $h(u)=\int_0^{\infty}u^x dx=[\frac{u^x}{\log u}]_0^{\infty}=-\frac{1}{\log u}$ thus we have $\mu_b^k=a(\delta_u-b)^k*(-\frac{1}{\log u})$.

Example 2.5. Let X be a gamma random variable with density $\frac{a}{\Gamma(r)}(ax)^{r-1}e^{-ax}$. Then since $f(x)=\frac{a^r}{\Gamma(r)}(e^{-a})^x x^{r-1}, c=\frac{a^r}{\Gamma(r)}$ and $h(u)=\int_0^{\infty}u^x x^{r-1}dx=(-1)^{r-1}(r-1)!\{-\frac{1}{(\log u)^r}\}$ thus we have $\mu_b^k=a^r(\delta_u-b)^k*(-1)^{r-1}\{-\frac{1}{(\log u)^r}\}$.

Example 2.6. Let X be a Laplace random variable with density $\frac{1}{2\phi}e^{-\frac{x-\theta}{\phi}}$, $\phi>0$. Then since $f(x)=\frac{1}{2\phi}(e^{-\frac{1}{\phi}})^x$, $x-\theta>0$, $\phi>0$, $c=\frac{1}{2\phi}$, $g(x)=1$ and $h(u)=-\frac{1}{\log u}$ thus we have $\mu_b^k=\frac{1}{2\phi}(\delta_u-b)^k*(-\frac{1}{\log u})$.

3. The Calculation of Factorial Moments

Theorem 3.1. For the countinuous random variable X with probability density function $f(x)$, the r-th decending factorial moment of X, $E(X^{(r)})$ is $cu^r D_u^r\int_{-\infty}^{\infty}u^x g(x)dx$, where $f(x)=cu^x g(x)$.

Proof. Since $f(x)$ is $cu^x g(x)$,

$$E(X^{(r)})=\int_{-\infty}^{\infty}x(x-1)\dots(x-r+1)f(x)dx$$

$$=c\int_{-\infty}^{\infty}x^{(r)}u^x g(x)dx$$

$$=cu^r\int_{-\infty}^{\infty}x^{(r)}u^{x-r}g(x)dx$$

$$=cu^r\int_{-\infty}^{\infty}D_u^r u^x g(x)dx$$

$$=cu^r D_u^r\int_{-\infty}^{\infty}u^x g(x)dx.$$

If X is the discrete random variable, $E(X^{(r)})$ is obtained by similarity to that of the continuous random variable case.

Theorem 3.2. For the continuous random variable X with probability function $f(x)$, r-th ascending factorial moment of X, $E(X^{[r]})$ is $cuD_u^r\int_{-\infty}^{\infty}u^{x+r-1}g(x)dx$, where $f(x) = cu^x g(x)$.

Proof. Since $f(x)$ is $cu^x g(x)$,

$$E(X^{[r]}) = \int_{-\infty}^{\infty}x(x+1)\dots(x+r-1)f(x)dx$$

$$= c\int_{-\infty}^{\infty}x^{[r]}u^x g(x)dx$$

$$= cu\int_{-\infty}^{\infty}x^{[r]}\{D_u^r(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}u^{x-1}du\dots du)\}g(x)dx$$

$$= cu\int_{-\infty}^{\infty}x^{[r]}\{D_u^r(I_u^r u^{x-1})\}g(x)dx$$

$$= cuD_u^r\int_{-\infty}^{\infty}u^{x+r-1}g(x)dx.$$

If X is the discrete random variable, $E(X^{[r]})$ is obtained by similarity to that of the continuous random variable case.

Example 3.3. Let X be an exponential random variable with density ae^{-ax}. Then since $f(x) = a(e^{-a})^x$, where $c = a, u = e^{-a}$ and $g(x) = 1$,

$$E(X^{(r)}) = au^r D_u^r\int_0^{\infty}u^x dx$$

$$= au^r D_u^r(-\frac{1}{\log u}).$$

and

$$E(X^{[r]}) = auD_u^r u^{r-1}\int_0^{\infty}u^x dx$$

$$= auD_u^r u^{r-1}(-\frac{1}{\log u}).$$

Example 3.4. Let X be a gamma random variable with density $\frac{a}{\Gamma(s)}(ax)^{s-1}e^{-ax}$. Then since $f(x)=\frac{a^s}{\Gamma(s)}x^{s-1}u^x$, where $c=\frac{a^s}{\Gamma(s)}$, $u=e^{-a}$, and $g(x)=x^{s-1}$,

$$E(X^{(r)})=\frac{a^s}{\Gamma(s)}u^r D_u^r \int_0^\infty x^{s-1}u^x dx$$

$$=\frac{a^s}{\Gamma(s)}u^r D_u^r (-1)^s(s-1)!\{\frac{1}{(\log u)^s}\},$$

and

$$E(X^{[r]})=\frac{a^s}{\Gamma(s)}uD_u^r u^{r-1}\int_0^\infty x^{s-1}u^x dx$$

$$=\frac{a^s}{\Gamma(s)}uD_u^r(-1)^s(s-1)!\,\frac{u^{r-1}}{(\log u)^s}.$$

Remark. By this method, the moments are obtained in case that the probability function is $cu^x g(x)$. But in case of the general forms of the probability function, if we want to use this method, we need to change the form to $cu^x g(x)$.

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