한국통계학회논문집 제3권 1호
1996년 4월 pp. 155-168

Bayesian Estimation Procedure in Multiprocess Non-Linear Dynamic Normal Model[1)]

Joong Kweon Sohn[2)] and Sang Gil Kang[3)]

Abstract

In this paper we consider the multiprocess dynamic normal model with parameters having a time dependent non-linear structure. We develop and study the recursive estimation procedure for the proposed model with normality assumption. It turns out that the proposed model has nice properties such as insensitivity to outliers and quick reaction to abrupt changes of pattern.

1. Introduction

Dynamic systems have been used by communications and control engineers to the state of a system as it evolves through time since the works of Kalman(1960). Kalman(1960) developed an recursive estimation procedure for the state variables of a linear dynamic system. Ho and Lee(1964) studied the dynamic linear model with Bayesian framework. Duncan and Horn(1972) introduced the Kalman filter by relating the dynamic linear model to random β regression theory using the time varying random parameters as state variables. Harrison and Stevens(1976) summarized the foundations of Bayesian forecasting as the parametric or statespace model, the probabilistic information on model parameters, the sequential model definition which describes the dynamic behavior of model parameters and some uncertainty in choosing the underlying model from a number of discrete alternatives. West, Harrison, and Migon(1985) developed the dynamic generalized linear model for application in non-linear, non-normal time series and regression problems. Masrielez and Martin(1977) developed robust Bayesian estimates for a state space model where either the state noise is Gaussian and the observation noise is heavy-tailed, or vice versa. West(1981) developed an approximation to the sequential updating of the distribution of location parameters of a linear time series model. He

---

1) The Present Studies were Supported by the Basic Science Research Institute Program, Ministry of Education, 1994, Project No. BSRI-94-1403.
2) Professor, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea
3) Lecturer, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea

examined the behavior of the resulting non-linear filter algorithm. Kitagawa(1987) developed a non-normal state space model for non-stationary time series, where the observation and system noise distributions are non-normal.

The multiprocess dynamic linear model was developed by Harrison and Stevens (1971, 1976) for the time series that contain outliers and are subject to abrupt changes in pattern. Smith and West(1983) and Smith, Gordon, Knapp and Trimble(1983) described a related monitoring procedure for detecting various forms of kidney failure in renal transplant patients. West(1986) introduced a method of monitoring the predictive performance of a class of Bayesian models. West and Harrison(1986) studied the method of model monitoring and adapting to structural changes in the time series. Bolstad(1986) presented Harrison-Stevens forecasting algorithm and the multiprocess dynamic linear model. Bolstad(1988) developed the multiprocess dynamic generalized linear model. Bolstad(1995) developed the multiprocess dynamic poisson model for estimating and forecasting a poisson random variable with a time-varying parameter. Whittaker and Frühwirth-Schnatter(1994) used to a triangular multiprocess Kalman filter for detecting bacteriological growth in routine monitoring of feedstuff.

In this paper, we develop multiprocess dynamic normal models with non-linear structure by incorporating the perturbation index variable which determines the perturbation distribution. In Section 2, we develop the recursive estimation for the multiprocess dynamic normal model with parameter non-linearities. Here the model is assumed to follow normal distribution. In Section 3, we study the proposed recursive estimations for the generalized exponential growth model by using Monte Carlo simulation study.

2. Recursive Estimation of Multiprocess Non-linear Dynamic Normal Model

In this Section, we are concerned with the multiprocess non-linear dynamic normal models. Non-linear dynamic models can be written in the following twoforms:

$$\text{Observation equation : } \quad Y_t = F_t(\beta_t) + w_t,$$

and

$$\text{Evolution equation : } \quad \beta_t = g_t(\beta_{t-1}) + r_t,$$

where $F_t = (\cdot)$ is a known non-linear regression function, $g_t = (\cdot)$ is a known non-linear vector evolution function and w_t and r_t are error terms. In these models, we encounter difficulties with determining the posterior distribution of β_t given Y_t since $F_t(\cdot)$ and/or $g_t(\cdot)$ are non-linear function of β_t and β_{t-1}, respectively. Implementation of this analysis requires numerical integration to well approximate mathematically defined integrals, and so is in fact impossible to do it analytically. Thus whatever the particular structure of parameter non-linearities is in application, the base dynamic linear model analysis must be extended to cater for it. Therefore we use some approximation techniques. We use the first order Taylor series approximation technique.

The multiprocess dynamic model can be regarded as the dynamic model in that the parameter vector subject to perturbation. However, in the multiprocess dynamic model the distribution of the perturbation depends on the perturbation index random variable at that time. The sequences of perturbation index variables are independent of each other and each can be considered to be the outcome of a single multinomial trial with known prior probabilities. The prior probabilities do not have to remain constant over time. This allows prior knowledge by the forecaster into the model and hence the forecasting system is very flexible. Thus multiprocess non-linear dynamic model can be expressed as follows.

Let I_t be the perturbation index variable at time t.

$$P(I_t=j)=\pi_t^{(j)}, \quad \text{for } j=1,2,\cdots,k .$$

When $I_t=j$,

$$\beta_t=g_t(\beta_{t-1})+r_t ,$$

where $g_t(\cdot)$ is a known non-linear vector evolution function and r_t is the perturbation vector, which is normally distributed with mean vector 0 and known variance-covariance matrix $R_t^{(j)}$. The variance-covariance matrix depends on the perturbation index variable $I_t=j$ and can be changed over time. The observation equation is given by

$$Y_t=F_t(\beta_t)+w_t ,$$

where $F_t(\cdot)$ is a known non-linear regression function mapping the n-vector β_t to the real line and observation errors and w_t are independent and normally distributed with mean 0 and variance W_t .

Various linearization techniques have been developed for dynamic non-linear models, all being based on the use of linear approximations to non-linearities. The most straightforward and easily interpreted approach is the first order Taylor series approximations technique. This requires the assumptions that both non-linear regression function $F_t(\cdot)$ and non-linear evolution function $g_t(\cdot)$ be differentiable functions of their vector arguments. A Taylor series expansion of the evolution function and observation equation for multiprocess dynamic model are as follows.

At time $t-1$, the posterior distribution of β_{t-1} given I_{t-1} and Y_{t-1} is normally distributed with mean $m_{t-1}^{(i)}$ and variance-covariance matrix $V_{t-1}^{(i)}$, that is,

$$(\beta_{t-1}|I_{t-1}=i, Y_{t-1}) \sim N(m_{t-1}^{(i)}, V_{t-1}^{(i)}) .$$

A Taylor series expansion of the evolution functions about the estimate $m_{t-1}^{(i)}$ of β_{t-1}, gives

$$g(\beta_{t-1}) = g_t(m_{t-1}^{(i)}) + G_t(\beta_{t-1} - m_{t-1}^{(i)}) + R_1(\beta_{t-1} - m_{t-1}^{(i)}) ,$$

where $R_1(\beta_{t-1} - m_{t-1}^{(i)})$ is a remainder term which is a function of quadratic and higher order terms of $(\beta_{t-1} - m_{t-1}^{(i)})$ and G_t is the known $n \times n$ matrix derivative of the evolution matrix evaluated at the estimate $m_{t-1}^{(i)}$,

$$G_t = \left[\frac{\partial g_t(\beta_{t-1})}{\partial \beta_{t-1}'} \right]_{\beta_{t-1} = m_{t-1}^{(i)}} .$$

Assuming that terms other than the linear term are negligible, the linearized expression of the evolution equation becomes

$$\begin{aligned} \beta_t &\approx g_t(m_{t-1}^{(i)}) + G_t(\beta_{t-1} - m_{t-1}^{(i)}) + r_t \\ &= h_t + G_t\beta_{t-1} + r_t, \end{aligned} \tag{2.1}$$

where $h_t = g_t(m_{t-1}^{(i)}) - G_t m_{t-1}^{(i)}$ is also known.

Similarly the non-linear regression function is also linearized about the expected value $a_t = h_t + G_t m_{t-1}^{(i)}$ for β_t,

$$F_t(\beta_t) = F_t(a_t) + H_t'(\beta_t - a_t) + R_2(\beta_t - a_t),$$

where $R_2(\beta_t - a_t)$ is a remainder term which is a function of quadratic and higher order terms of $(\beta_t - a_t)$ and H_t is the known n-vector derivative of F_t evaluated at the prior mean a_t,

$$H_t = \left[\frac{\partial F_t(\beta_t)}{\partial \beta_t} \right]_{\beta_t = a_t} .$$

Assuming the linear term dominates the expansion, the non-linear regression function is linearized as

$$\begin{aligned} y_t &= F_t(\beta_t) + w_t \\ &\approx f_t + H_t'(\beta_t - a_t) + w_t, \end{aligned} \tag{2.2}$$

where $a_t = h_t + G_t m_{t-1}^{(i)}$ and $f_t = F_t(a_t)$. Thus the model with (2.1) and (2.2) as observation equation and evolution equation, respectively, is multiprocess dynamic linear model.

2.1 Recursive Estimation

Assume that at $t=0$ the initial prior is the usual normal form

$$(\beta_0 | Y_0) \sim N(m_0, V_0),$$

where mean vector m_0 and variance-covariance matrix V_0 are known at t_0. k posterior distributions of β_{t-1} given $I_{t-1} = i$ and Y_{t-1} are known at time $t-1$. Each posterior distribution has the usual conjugate normal form. Thus posterior distribution of β_{t-1} given $I_{t-1} = i$ and Y_{t-1} is normally distributed with mean vector $m_{t-1}^{(i)}$ and variance-covariance matrix $V_{t-1}^{(i)}$, that is,

$$(\beta_{t-1} | I_{t-1} = i, Y_{t-1}) \sim N(m_{t-1}^{(i)}, V_{t-1}^{(i)}). \tag{2.3}$$

The notation $Y_{t-1} = y_{t-1}, y_{t-2}, \cdots, y_1$ denotes all observations up to and including y_{t-1}. Also at time $t-1$ the posterior probability of perturbation index variable, $q_{t-1}^{(i)} = P(I_{t-1} = i \,|\, Y_{t-1})$, is known. Evolving to time t, β_t and Y_t depend on the perturbation index variable I_{t-1} and I_t.

(1) Evolution Step

In this step, each of these k distributions is calculated to time t conditional on $I_t = j$ for $j = 1, 2, \cdots, k$.

For each $I_{t-1}=i$ and $I_t=j$, we now derive the prior distribution, one-step forecast distribution and joint distribution at time t. The linear normal evolution equation (2.1) coupled with posterior distribution (2.3) leads directly to the prior distribution of β_t given $I_{t-1}=i, I_t=j$ and Y_{t-1}. Clearly the prior distribution β_t given $I_{t-1}=i, I_t=j$, and Y_{t-1} is normally distributed since it is a linear function of β_{t-1} and r_t which are independent and normally distributed. The mean vector and variance-covariance matrix are

$$\begin{aligned} n_t^{(i,j)} &= E[\ \beta_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= E[\ h_t+G_t\beta_{t-1}+r_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= h_t+G_t m_{t-1}^{(i)} \end{aligned}$$

and

$$\begin{aligned} C_t^{(i,j)} &= Var[\ \beta_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= Var[\ h_t+G_t\beta_{t-1}+r_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= G_t V_{t-1}^{(i)} G_t' + R_t^{(j)}, \end{aligned}$$

respectively. Therefore the prior distribution of β_t given $I_{t-1}=i, I_t=j$ and Y_{t-1} is normally distributed with mean vector $n_t^{(i,j)}=h_t+G_t m_{t-1}^{(i)}$ and variance- covariance matrix $C_t^{(i,j)}=G_t V_{t-1}^{(i)} G_t' + R_t^{(j)}$, that is,

$$(\beta_t | I_{t-1}=i, I_t=j, Y_{t-1}) \sim N(n_t^{(i,j)}, C_t^{(i,j)}). \tag{2.4}$$

By using prior distribution(2.4) together with the observation equation (2.2), the one-step forecast distribution of y_t given $I_{t-1}=i, I_t=j$ and Y_{t-1} follows a normal distribution with the mean and variance

$$\begin{aligned} E[\ y_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] &= E[\ F_t(\beta_t)+w_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= E[\ f_t+H_t'(\beta_t-a_t)+w_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= f_t+H_t'(n_t^{(i,j)}-a_t) \end{aligned}$$

and

$$\begin{aligned} Var[\ y_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] &= Var[\ F_t(\beta_t)+w_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= Var[\ f_t+H_t'(\beta_t-a_t)+w_t \,|I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= H_t' C_t^{(i,j)} H_t + W_t, \end{aligned}$$

respectively. Therefore the one-step forecast distribution of y_t given $I_{t-1}=i, I_t=j$ and Y_{t-1} is normally distributed with mean $f_t+H_t'(n_t^{(i,j)}-a_t)$ and variance $H_t'C_t^{(i,j)}H_t$, that is,

$$(y_t \mid I_{t-1}=i, I_t=j, Y_{t-1}) \sim N(f_t+H_t'(n_t^{(i,j)}-a_t), H_t'C_t^{(i,j)}H_t+W_t). \qquad (2.5)$$

By using prior distribution(2.4) and one-step forecast distribution(2.5), β_t and y_t follow a normal distribution with mean vector $(n_t^{(i,j)}, f_t+H_t'(n_t^{(i,j)}-a_t))'$ and variance - covariance matrix

$$\begin{pmatrix} C_t^{(i,j)} & C_t^{(i,j)}H_t \\ H_t'C_t^{(i,j)} & H_t'C_t^{(i,j)}H_t+W_t \end{pmatrix},$$

that is, the joint distribution of β_t and y_t given $I_{t-1}=i, I_t=j$ and Y_{t-1} is

$$\left(\begin{matrix} \beta_t \\ y_t \end{matrix} \mid I_{t-1}=i, I_t=j, Y_{t-1}\right) \sim N\left[\begin{pmatrix} n_t^{(i,j)} \\ f_t+H_t'(n_t^{(i,j)}-a_t) \end{pmatrix}, \begin{pmatrix} C_t^{(i,j)} & C_t^{(i,j)}H_t \\ H_t'C_t^{(i,j)} & H_t'C_t^{(i,j)}H_t+W_t \end{pmatrix}\right]. \qquad (2.6)$$

Note that the covariance of β_t and y_t can be obtained as

$$\begin{aligned} Cov[\beta_t, y_t \mid I_{t-1}=i, I_t=j, Y_{t-1}] &= Cov[\beta_t, f_t+H_t'(\beta_t-a_t) \mid I_{t-1}=i, I_t=j, Y_{t-1}] \\ &= C_t^{(i,j)}H_t. \end{aligned}$$

(2) Updating Step

When y_t is observed, the prior distribution of β_t in the evolution step is updated to its posterior distribution. We call this the updating step. Now it needs to compute the posterior mean vectors and variance-covariance matrices of $k\times k$ posterior distributions of β_t given the combination of $I_{t-1}=i$ and $I_t=j$.

By using standard normal theory for joint distribution of β_t and y_t in evolution step, the conditional distribution of β_t given y_t can be obtained as

$$(\beta_t \mid I_{t-1}=i, I_t=j, y_t, Y_{t-1}) = (\beta_t \mid I_{t-1}=i, I_t=j, Y_t).$$

Also the mean vector and variance-covariance matrix are

$$\begin{aligned} m_t^{(i,j)} &= E[\ \beta_t \mid I_{t-1}=i, I_t=j, Y_t] \\ &= n_t^{(i,j)} + C_t^{(i,j)} H_t (H_t' C_t^{(i,j)} H_t + W_t)^{-1} (y_t - (f_t + H_t'(n_t^{(i,j)} - a_t))) \end{aligned}$$

and

$$\begin{aligned} V_t^{(i,j)} &= Var[\ \beta_t \mid I_{t-1}=i, I_t=j, Y_t] \\ &= C_t^{(i,j)} - C_t^{(i,j)} H_t (H_t' C_t^{(i,j)} H_t + W_t)^{-1} H_t' C_t^{(i,j)}, \end{aligned}$$

respectively. Therefore the posterior distributions of β_t given $I_{t-1}=i$, $I_t=j$ and Y_t is normally distributed with mean vector

$$m_t^{(i,j)} = n_t^{(i,j)} + C_t^{(i,j)} H_t (H_t' C_t^{(i,j)} H_t + W_t)^{-1} (y_t - (f_t + H_t'(n_t^{(i,j)} - a_t)))$$

and variance-covariance matrix

$$V_t^{(i,j)} = C_t^{(i,j)} - C_t^{(i,j)} H_t (F_t' C_t^{(i,j)} H_t + W_t)^{-1} H_t' C_t^{(i,j)}.$$

That is,

$$(\beta_t \mid I_{t-1}=i, I_t=j, Y_t) \sim N(m_t^{(i,j)}, V_t^{(i,j)}). \tag{2.7}$$

To carry out the recursive estimation further, we need to derive the posterior probabilities of the perturbation indices given the present observation. This probability is called the posterior index probability. By using the Bayes theorem, we have

$$\begin{aligned} P_t^{(i,j)} &= P(I_{t-1}=i, I_t=j \mid Y_t) \\ &= \frac{P(y_t \mid I_{t-1}=i, I_t=j, Y_{t-1}) P(I_t=j \mid I_{t-1}=i, Y_{t-1}) P(I_{t-1}=i \mid Y_{t-1})}{P(y_t \mid Y_{t-1})} \\ &= \pi_t^{(j)} q_{t-1}^{(i)} \frac{P(y_t \mid I_{t-1}=i, I_t=j, Y_{t-1})}{P(y_t \mid Y_{t-1})} \end{aligned}$$

for $i=1,\cdots,k$ and $j=1,\cdots,k$. The quantity $P(y_t \mid Y_{t-1})$ is a normalizing constant. Hence the $P_t^{(i,j)}$ are all completely determined. If our interest is not the forecast itself but the change of pattern, these probabilities are useful in detecting the change of pattern .

(3) Collapsing Step

Estimation of β_t is based on the unconditional $k \times k$ component mixtures that average posterior distribution (2.7) with respect to the posterior index probabilities $P_t^{(i,j)}$. Thus

$$\begin{aligned} P(\beta_t \mid Y_t) &= \sum_{i=1}^{k} \sum_{j=1}^{k} P(\beta_t, I_{t-1}=i, I_{t=j} \mid Y_t) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} P(\beta_t \mid I_{t-1}=i, I_t=j, Y_t) P_t^{(i,j)}. \end{aligned}$$

This completes essentially the evolution and updating steps at time t. To proceed to time $t+1$, however, we need to remove the dependence of the joint posterior $P(\beta_t \mid Y_t)$ on $k\times k$ possible combination of $I_{t-1}=i$ and $I_t=j$ for $i=1,\cdots,k$ and $j=1,\cdots,k$. If we evaluate $P(\beta_t \mid Y_t)$ to time $t+1$ directly, the mixture will expand to k^3 components for β_{t+1}, dependently on all possible combinations of $I_{t+1}=l$, $I_t=j$ and $I_{t-1}=i$. However, the principle that the effect of different models at time $t-1$ are negligible for time $t+1$ is applied for approximating such mixtures. After collapsing the posterior distribution, mean vector and variance-covariance matrix of β_t are obtained as follows.

By using the posterior index probabilities at time t , the posterior distribution of $(\beta_t \mid I_t=j, Y_t)$ is represented as a k component mixture of $(\beta_t \mid I_{t-1}=i, I_t=j, Y_t)$. Thus the posterior distribution of β_t given $I_t=j$ and Y_t is

$$\begin{aligned} f(\beta_t \mid I_t=j, Y_t) &= \sum_{i=1}^{k} f(\beta_t \mid I_{t-1}=i, I_t=j, Y_t) \cdot \frac{P(I_{t-1}=i, I_t=j \mid Y_t)}{P(I_t=j \mid Y_t)} \\ &= \sum_{i=1}^{k} (q_t^{(j)})^{-1} P_t^{(i,j)} f(\beta_t \mid I_{t-1}=i, I_t=j, Y_t), \end{aligned}$$

where $q_t^{(j)} = \sum_{i=1}^{k} P_t^{(i,j)}$.

By using the method of approximation of mixture, the mean vector and variance-covariance matrix are

$$m_t^{(j)} = E(\beta_t \mid I_t=j, Y_t) = \sum_{i=1}^{k} (q_t^{(j)})^{-1} P_t^{(i,j)} m_t^{(i,j)}$$

and

$$V_t^{(j)} = Var(\beta_t \mid I_{t=j}, Y_t) = \sum_{i=1}^{k} (q_t^{(j)})^{-1} P_t^{(i,j)} [\ V_t^{(i,j)} + (m_t^{(i,j)} - m_t^{(j)})(m_t^{(i,j)} - m_t^{(j)})'\ ] \ ,$$

respectively. Therefore the posterior distributions of β_t given $I_t=j$ and Y_t is normally distributed with mean vector

$$m_t^{(j)} = \sum_{i=1}^{k} (q_t^{(j)})^{-1} P_t^{(i,j)} m_t^{(i,j)} \tag{2.8}$$

and variance-covariance matrix

$$V_t^{(j)} = \sum_{i=1}^{k} (q_t^{(j)})^{-1} P_t^{(i,j)} [\ V_t^{(i,j)} + (m_t^{(i,j)} - m_t^{(j)})(m_t^{(i,j)} - m_t^{(j)})'\ ] \ , \tag{2.9}$$

respectively. That is,

$$(\beta_t \mid I_t = j, Y_t) \sim N(m_t^{(j)}, V_t^{(j)}). \tag{2.10}$$

At this point, we complete the recursive estimation procedure up to time t and we are ready to repeat the process at time $t+1$ when the next observation becomes y_{t+1} available. Now we want to forecast the values of β_{t+i} and y_{t+i} if the further observation is not available, $i = 1, 2, \cdots$. Here we restrict ourselves to one-step forecast.

2.2 Forecast Distributions

To forecast the values of β_{t+1} and y_{t+1}, first we need to derive the forecast distribution of β_{t+1} and y_{t+1}. It can be easily seen.

The forecast distribution of β_{t+1} given $I_t = i$, $I_{t+1} = j$ and Y_t is normally distributed. Also the mean vector and variance-covariance matrix of β_{t+1} are

$$n_{t+1}^{(i,j)} = E[\ \beta_{t+1} \mid I_t = i, I_{t+1} = j, Y_t] = h_{t+1} + G_{t+1} m_t^{(i)}$$

and

$$C_{t+1}^{(i,j)} = Var[\ \beta_{t+1} \mid I_t = i, I_{t+1} = j, Y_t] = G_{t+1} V_t^{(i)} G_{t+1}' + R_{t+1}^{(j)},$$

where

$$h_{t+1} = g_{t+1}(m_t^{(i)}) - G_{t+1} m_t^{(i)}$$

and

$$G_{t+1} = \left[\frac{\partial g_{t+1}(\beta_t)}{\partial \beta_t'} \right]_{\beta_t = m_t^{(i)}},$$

respectively.

Therefore the forecast distribution of β_{t+1} given $I_t = i$, $I_{t+1} = j$ and Y_t is normally distributed with mean vector $n_{t+1}^{(i,j)} = h_{t+1} + G_{t+1} m_t^{(i)}$ and variance-covariance matrix $C_{t+1}^{(i,j)} = G_{t+1} V_t^{(i)} G_{t+1}' + R_{t+1}^{(j)}$, that is,

$$(\beta_{t+1} \mid I_t = i, I_{t+1} = j, Y_t) \sim N(n_{t+1}^{(i,j)}, C_{t+1}^{(i,j)}). \tag{2.11}$$

By using forecast distribution(2.11) and the observation equation(2.2), the distribution of y_{t+1} given $I_t = i$, $I_{t+1} = j$ and Y_t is normally distributed. Also its mean is

$$E[\, y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t] = f_{t+1} + H_{t+1}'(n_{t+1}^{(i,j)} - a_{t+1}),$$

where

$$a_{t+1} = h_{t+1} + G_{t+1} m_t^{(i)}$$

and

$$H_{t+1} = \left[\frac{\partial F_{t+1}(\beta_{t+1})}{\partial \beta_{t+1}} \right]_{\beta_{t+1} = a_{t+1}}.$$

The variance of y_{t+1} is

$$Var[\, y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t] = H_{t+1}' C_{t+1}^{(i,j)} H_{t+1} + W_{t+1}.$$

Therefore the distribution of y_{t+1} given $I_t = i$, $I_{t+1} = j$ and Y_t is normally distributed with mean $f_{t+1} + H_t'(n_{t+1}^{(i,j)} - a_{t+1})$ and variance $H_{t+1}' C_{t+1}^{(i,j)} H_{t+1} + W_{t+1}$. That is,

$$(y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t) \sim N\,(f_{t+1} + H_{t+1}'(n_{t+1}^{(i,j)} - a_{t+1}), H_{t+1}' C_{t+1}^{(i,j)} H_{t+1} + W_{t+1}).$$

The marginal predictive distribution for y_{t+1} is the mixture of the $k \times k$ components $P(y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t)$ with respect to posterior index probabilities. Thus the distributions of y_{t+1} given Y_t is given by

$$\begin{aligned} P(y_{t+1} \mid Y_t) &= \sum_{i=1}^{k} \sum_{j=1}^{k} P(y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t) P(I_t = i \mid Y_t) P(I_{t+1} = j \mid I_t = i, Y_t) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} P(y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t) q_t^{(i)} \pi_{t+1}^{(j)}. \end{aligned}$$

Therefore the unconditional forecast distribution of y_{t+1} given $I_t = i$, $I_{t+1} = j$ and Y_t is

$$P(y_{t+1} \mid Y_t) = \sum_{i=1}^{k} \sum_{j=1}^{k} q_t^{(i)} \pi_{t+1}^{(j)} P(y_{t+1} \mid I_t = i, I_{t+1} = j, Y_t).$$

3. Monte Carlo Simulation Study

In this section, we study the performances of the Bayesian estimation proposed in Section 2 via Monte Carlo simulation for the multiprocess non-linear dynamic normal model.

We consider a member of the generalized exponential growth models by Gamerman and Migon(1991). Let $y_t, t = 1, 2, \cdots, n$, be a time series of interest. The model is normally distribution with mean μ_t and variance $K(\mu_t)\sigma^2$, that is,

$$(y_t \mid \mu_t) \sim N(\mu_t, K(\mu_t)\sigma^2)$$

where $\beta_t = \mu_t^\lambda$ and non-linear equations are

$$\begin{aligned} \beta_t &= \beta_{t-1} + \gamma_{t-1} + w_{t1} \\ \gamma_t &= \phi_{t-1}\gamma_{t-1} + w_{t2} \\ \phi_t &= \phi_{t-1} + w_{t3}. \end{aligned}$$

β_t is the level, γ_t is the growth in the level and ϕ_t is the damping factor for this model. The non-linearity in the model is due to multiplicative effect of ϕ_t. The simulation study was carried out with the following example on an artificially generated time series. The time series consists of 80 normally distributed random variables and are the following change pattern. The time series data start with no change with an outlier of at the 12th observation. At the 21st observation, the growth change starts and continues up to the 30th observation. From the 31th observation to the 50th observation there is no change. From the 51th observation, the damping factor change starts and continues up to the 60th observation. At the 61th observation, level change starts and continues up to the 80th observation with an outlier at the 72th observation.

The forecast and the actual observations are shown in Figure 3.1 and the forecast errors are shown in Figure 3.2. From these figures, it can be summarized as follows:

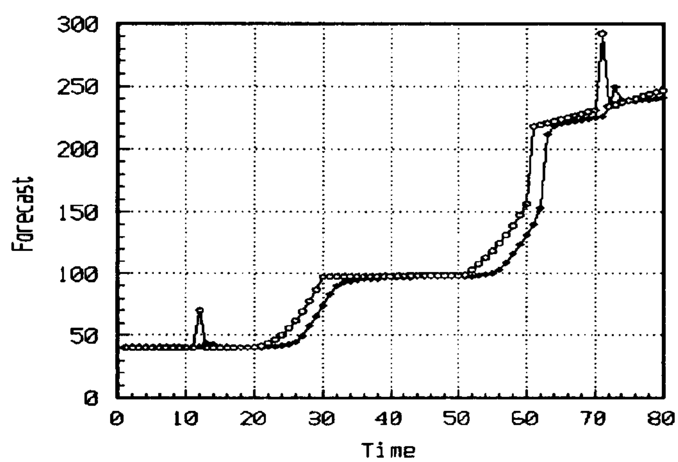

Figure 3.1 Observed(∘) and Forecast(•) Value

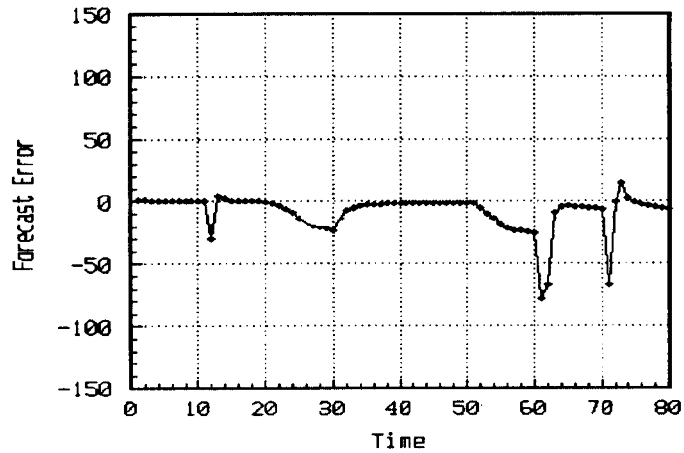

Figure 3.2 Forecast Error

(i) The developed model gives good estimates by using past data as well as present data when the time series is in a stable pattern.

(ii) The developed model is not sensitive to an outlier.

(iii) The developed model reacts quickly when a change occurs. But when a change occurs, the forecast error is slightly increasing.

References

[1] Bolstad, W. M. (1986), Harrison-Stevens Forecasting and the Multiprocess Dynamic Model, *The American Statistician*, 40, 129-135.

[2] Bolstad, W. M. (1988), Estimation in the Multiprocess Dynamic Generalized Linear Model, *Communications in Statistics: Theory and Method*, 17, 4179-4204.

[3] Bolstad, W. M. (1995), The Multiprocess Dynamic Poisson Model, *Journal of the American Statistical Association*, 90, 227-232.

[4] Duncan, B. and Horn, S. D. (1972), Linear Dynamic Recursive Estimation from the Viewpoint of Regression Analysis, *Journal of the American Statistical Association*, 67, 815-821.

[5] Gamerman, D. and Migon, H. S. (1991), Forecasting the number of AIDS cases in Brazil, *Statistician*, 40, 427-442.

[6] Harrison, P. J. and Stevens, C. F. (1971), A Bayesian Approach to Short-term Forecasting, *Operations Research Quarterly*, 22, 341-362.

[7] Harrison, P. J. and Stevens, C. F. (1976), Bayesian Forecasting, *Journal of the Royal Statistical Society* B, 38, 205-247.

[8] Ho, Y. C. and Lee, R. C. K. (1964), A Bayesian Approach to Problems in Stochastic Estimation and Control, *IEEE Transactions on Automatic Control*, 33-39.

[9] Kalman, R. E. (1960), A New Approach to Linear Filtering and Prediction Problems, *Journal of Basic Engineering*, 95-108.

[10] Kitagawa, W. (1987), Non-Gaussian State-Space Modeling of Nonstationary Time Series, *Journal of the American Statistical Association*, 82, 1041-1061.

[11] Masreliez, C. J. and Martin, R. D. (1977), Robust Bayesian Estimation for the Linear Model and Robustifying the Kalman Filter, *IEEE Transactions on Automatic Control*, 22, 361-371.

[12] Smith, A. F. M. and West, M. (1983), An Application of Multiprocess Kalman Filter to Renal Transplant Data, *Biometrics*, 39, 867-878.

[13] Smith, A. F. M., West, M., Gordon, K., Knapp, M. and Trimble, I. C. (1983), Monitoring Kidney Transplant Patients, *The Statistician*, 32, 46-54.

[14] West, M. (1981), Robust Sequential Approximate Bayesian Estimation, *Journal of the Royal Statistical Society* B, 43, 157-166.

[15] West, M. (1986), Bayesian Model Monitoring, *Journal of the Royal Statistical Society* B, 48, 70-78.

[16] West, M. and Harrison, P. J. (1986), Monitoring and Adaptation in Bayesian Forecasting Models, *Journal of the American Statistical Association*, 81, 741-750.

[17] West, M., Harrison, P. J. and Migon, H. S. (1985), Dynamic Generalized Linear Models and Bayesian Forecasting, *Journal of the American Statistical Association*, 80, 73-97.

[18] Whittaker, J. and Frühwirth-Schnatter, S. (1994), A Dynamic Changepoint Model for Detecting the Onset of Growth in Bacteriological Infections, *Applied Statistics*, 43, 625-640.