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# A Comparison of Confidence Intervals for the Reliability of the Stress-Strength Models with Explanatory Variables[1]

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## Abstract

In this paper, we consider the distribution-free confidence intervals for the reliability of the stress-strength model when the stress $X$ and strength $Y$ depend linearly on some explanatory variables $\boldsymbol{z}$ and $\boldsymbol{w}$, respectively. We apply these confidence intervals to the Rocket-Motor data and compare the results to those of Guttman et al. (1988). Some simulation results show that the distribution-free confidence intervals have better performance for nonnormal errors compared to those of Guttman et al. (1988), which are designed for normal random variables in respect that the former yield the coverage levels closer to the nominal coverage level than the latter.


## 1. Introduction

Suppose $Y$ is the strength of a unit subjected to a stress $X$. In the stress-strength model, the reliability of the stress-strength model is defined as $P(X< Y)$, which is the probability that the strength of the unit exceeds the applied stress. The above model was first considered by Birnbaum (1956) and has been found an increasing number of applications in many different areas, especially in the structural and aircraft industries.

In this paper, we consider the reliability of the stress and strength when they are linearly related to explanatory variables. Suppose that $X$ is related to $p$ explanatory variables $\boldsymbol{z}$ and $Y$ is related to $q$ explanatory variables $\boldsymbol{w}$ according to the linear relations,

$$X=\mu+ \boldsymbol{\beta}'(\boldsymbol{z}-\overline{\boldsymbol{z}})+\delta \text{ and } Y=\nu+\boldsymbol{\gamma}'(\boldsymbol{w}-\overline{\boldsymbol{w}})+\varepsilon, \tag{1.1}$$

where $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_p)'$ and $\boldsymbol{\gamma}=(\gamma_1,\cdots,\gamma_q)'$ are regression coefficients and the errors $\delta$ and $\varepsilon$ are independent random variables with distributions $F$ and $G$, respectively, such that

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$E\delta = E\varepsilon = 0$, $Var\delta = \sigma^2 < \infty$ and $Var\varepsilon = \tau^2 < \infty$.

Guttman et al. (1988) derived confidence limits for $P(X < Y \mid \boldsymbol{z}, \boldsymbol{w})$ with explanatory variables $\boldsymbol{z}$ and $\boldsymbol{w}$ under the presumption that $\delta$ and $\varepsilon$ are mutually independent and normally distributed. However, in this paper we consider a Mann-Whitney type statistic to estimate the reliability.

Park (1995) has considered the problem of estimating $P(X < Y \mid \boldsymbol{z}, \boldsymbol{w})$ using the Mann-Whitney type statistic. Here, the errors $\delta$ and $\varepsilon$ are not necessarily normal. Suppose that $(X_i, \boldsymbol{z}_i)$ and $(Y_j, \boldsymbol{w}_j)$, $i = 1, \cdots, m$, $j = 1, \cdots, n$ are from the models in (1.1), and let $\overline{\boldsymbol{z}} = m^{-1} \sum_{i=1}^{m} \boldsymbol{z}_i$ and $\overline{\boldsymbol{w}} = n^{-1} \sum_{j=1}^{n} \boldsymbol{w}_j$. Let $P(\boldsymbol{\theta}) = P(X < Y \mid \boldsymbol{z}, \boldsymbol{w})$, where $\boldsymbol{\theta} = (\mu, \nu, \boldsymbol{\beta}', \boldsymbol{\gamma}')'$.

In view of (1.1), we can write

$$P(\boldsymbol{\theta}) = P(\delta - \varepsilon < -\mu + \nu - \boldsymbol{\beta}'(\boldsymbol{z} - \overline{\boldsymbol{z}}) + \boldsymbol{\gamma}'(\boldsymbol{w} - \overline{\boldsymbol{w}})) = P(\delta - \varepsilon < x),$$

where $x = -\mu + \nu - \boldsymbol{\beta}'(\boldsymbol{z} - \overline{\boldsymbol{z}}) + \boldsymbol{\gamma}'(\boldsymbol{w} - \overline{\boldsymbol{w}})$.

Providing that $\mu, \nu, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are known, $P(\boldsymbol{\theta})$ can be estimated by

$$U(\boldsymbol{\theta}) = (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} I(\delta_i - \varepsilon_j < x) = (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij},$$

where $I$ denotes the indicator function and $U_{ij} = I(\delta_i - \varepsilon_j < x)$.

However, since we do not know the true errors $\delta_i$ and $\varepsilon_j$ as well as the parameters $\mu, \nu, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, we have to estimate those. Let $\widehat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\nu}, \widehat{\boldsymbol{\beta}}', \widehat{\boldsymbol{\gamma}}')'$ be the least squares estimators of $\boldsymbol{\theta}$, and $\hat{\delta}_i$ and $\hat{\varepsilon}_j$ be the residuals computed by $\hat{\delta}_i = X_i - \hat{\mu} - \widehat{\boldsymbol{\beta}}'(\boldsymbol{z}_i - \overline{\boldsymbol{z}})$ and $\hat{\varepsilon}_j = Y_j - \hat{\nu} - \widehat{\boldsymbol{\gamma}}'(\boldsymbol{w}_j - \overline{\boldsymbol{w}})$. Then an estimator of $P(\boldsymbol{\theta})$ is given as the following :

$$\hat{U}(\widehat{\boldsymbol{\theta}}) = (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} I(\hat{\delta}_i - \hat{\varepsilon}_j < \hat{x}) = (mn)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{U}_{ij},$$

where $\hat{x} = -\hat{\mu} + \hat{\nu} - \widehat{\boldsymbol{\beta}}'(\boldsymbol{z} - \overline{\boldsymbol{z}}) + \widehat{\boldsymbol{\gamma}}'(\boldsymbol{w} - \overline{\boldsymbol{w}})$ and $\hat{U}_{ij} = I(\hat{\delta}_i - \hat{\varepsilon}_j < \hat{x})$.

She has also shown that $\sqrt{N}[\hat{U}(\widehat{\boldsymbol{\theta}}) - P(\boldsymbol{\theta})]$ is asymptotically normal when $N = m + n$ and proposed some estimators of the asymptotic variance of $\hat{U}(\widehat{\boldsymbol{\theta}})$ following the ideas of Mee (1990) and Sen (1967). Using the consistency of such estimators, we obtain distribution-free confidence intervals for $P(\boldsymbol{\theta})$.

In Section 2, we review some regularity conditions for the asymptotic normality of

$\sqrt{N}[\ \hat{U}(\hat{\theta})-P(\theta)]$ and some estimators of the asymptotic variance of $\hat{U}(\hat{\theta})$, and construct distribution-free confidence intervals for $P(\theta)$. In Section 3, we apply our confidence bounds to the Rocket-Motor data and compare the results to those of Guttman et al. (1988), and some Monte Carlo simulations are conducted to evaluate the performance of confidence intervals for various sample sizes, $P(\theta)$ and the distributions of errors.

## 2. Estimation for $P(X<Y|\mathbf{z},\mathbf{w})$

Throughout the sequel, we will make use of the following notations.

$$\mathbf{Z}=(\mathbf{z}_1,\ \mathbf{z}_2,\cdots,\ \mathbf{z}_m)',\ \mathbf{W}=(\mathbf{w}_1,\ \mathbf{w}_2,\cdots,\ \mathbf{w}_n)',$$

$$\overline{\mathbf{Z}}=(\overline{\mathbf{z}},\ \overline{\mathbf{z}},\cdots,\ \overline{\mathbf{z}})' \text{ and } \overline{\mathbf{W}}=(\overline{\mathbf{w}},\ \overline{\mathbf{w}},\cdots,\ \overline{\mathbf{w}})',$$

where $\mathbf{Z}$ and $\overline{\mathbf{Z}}$ are $m\times p$ matrices and $\mathbf{W}$ and $\overline{\mathbf{W}}$ are $n\times q$ matrices.

To derive asymptotic results for $\hat{U}(\hat{\theta})$, Park (1995) imposed the following conditions.

**Condition A** : For a real number $\rho$ in the interval (0,1), $m/N\rightarrow\rho$ and $n/N\rightarrow 1-\rho$, as $N\rightarrow\infty$.

**Condition B** : $\sup_x\{|F'(x)|+|F''(x)|+|G'(x)|+|G''(x)|\}<\infty$.

**Condition C** : $\overline{\mathbf{z}}\rightarrow\xi_1,\ m^{-1}\sum_{i=1}^{m}(\mathbf{z}_i-\overline{\mathbf{z}})(\mathbf{z}_i-\overline{\mathbf{z}})'\rightarrow\Gamma_z$ as $m\rightarrow\infty$

and

$$\overline{\mathbf{w}}\rightarrow\xi_2,\ n^{-1}\sum_{j=1}^{n}(\mathbf{w}_j-\overline{\mathbf{w}})(\mathbf{w}_j-\overline{\mathbf{w}})'\rightarrow\Gamma_w \text{ as } n\rightarrow\infty,$$

where $\xi_1\in R^p, \xi_2\in R^q$, and $\Gamma_z$ and $\Gamma_w$ are $p\times p$ and $q\times q$ positive definite matrices, respectively. Moreover, assume that, as $m,n\rightarrow\infty$,

$$\max_{1\le i\le m}\|m^{-1/2}(\mathbf{z}_i-\overline{\mathbf{z}})\|\rightarrow 0 \quad\text{and}\quad \max_{1\le j\le n}\|n^{-1/2}(\mathbf{w}_j-\overline{\mathbf{w}})\|\rightarrow 0,$$

where $\|\cdot\|$ denotes an Euclidean norm.

Under the above conditions Park (1995) established the asymptotic normality of $\sqrt{N}[\ \hat{U}(\hat{\theta})-P(\theta)]$ as follows :

$$\sqrt{N}[\hat{U}(\hat{\theta})-P(\theta)] \xrightarrow{d} N(0,\Sigma), \tag{2.1}$$

where

$$\Sigma = \rho^{-1}\Big[\int(1-G(t-x))^2 dF(t)-P^2(\theta)+(z-\xi_1)'\Gamma_z^{-1}(z-\xi_1)(H'(x))^2\sigma^2\Big]$$
$$+(1-\rho)^{-1}\Big[\int F^2(t+x)dG(t)-P^2(\theta)+(w-\xi_2)'\Gamma_w^{-1}(w-\xi_2)(H'(x))^2\tau^2\Big]$$

and $H(x)=P(\delta-\varepsilon<x)$.

The above fact implies that to obtain the estimator producing small asymptotic variance, one should choose the design vectors whose arithmetic means are close to given points $z$ and $w$.

To carry out statistical inference, one should estimate the asymptotic variance $\Sigma$. For the sake of convenience, let $p_1=P(U_{ij}U_{kj}=1)$ and $p_2=P(U_{ij}U_{ik}=1)$. Note that $p_1=\int F^2(t+x)dG(t)$ and $p_2=\int(1-G(t-x))^2dF(t)$. It is known in Park (1995) that under Conditions A, B and C,

$$\hat{p}_1=\frac{\sum_{i=1}^m\sum_{j=1}^n\sum_{k\neq i}^m \hat{U}_{ij}\hat{U}_{kj}}{mn(m-1)} \xrightarrow{p} p_1,$$

$$\hat{p}_2=\frac{\sum_{i=1}^m\sum_{j=1}^n\sum_{k\neq j}^n \hat{U}_{ij}\hat{U}_{ik}}{mn(n-1)} \xrightarrow{p} p_2 \tag{2.2}$$

and

$$\hat{p}_3=\frac{\hat{H}(\hat{x}+h)-\hat{H}(\hat{x}-h)}{2h} \xrightarrow{p} H'(x),$$

where $\hat{H}(x)=\sum_{i=1}^m\sum_{j=1}^n I(\hat{\delta}_i-\hat{\varepsilon}_j<x)/mn$ and $h$ is a bandwidth of order $N^b, -1/2<b<0$.

In addition,

$$\hat{\sigma}^2=\frac{\sum_{i=1}^m\left(X_i-\hat{\mu}-\hat{\beta}'(z_i-\bar{z})\right)^2}{(m-p-1)} \xrightarrow{p} \sigma^2$$

and (2.3)

$$\hat{\tau}^2=\frac{\sum_{j=1}^n\left(Y_j-\hat{\nu}-\hat{\gamma}'(w_j-\bar{w})\right)^2}{(n-q-1)} \xrightarrow{p} \tau^2$$

are well-known.

From (2.1) - (2.3), Park (1995) obtained the $\sqrt{N}$-consistent asymptotic variance estimator for $\hat{U}(\hat{\theta})$ in Theorem 3.3.3 denoted as $\hat{\Sigma}_b$. The asymptotic variance estimator of $\hat{U}(\hat{\theta})$ consists of two parts. One is from $U(\theta)$ and the other is from the regression coefficient estimators. In $\hat{\Sigma}_b$, the variance estimator of $\sqrt{N}U(\theta)$ is given by

$$\hat{V}_b = \frac{N}{m}[\hat{p}_2 - \hat{U}(\hat{\theta})^2] + \frac{N}{n}[\hat{p}_1 - \hat{U}(\hat{\theta})^2].$$

Mee (1990) pointed out that providing $m, n, \beta$ and $\gamma$ are known, $\hat{V}_b$ is a negatively biased estimator of variance for $\sqrt{N}U(\theta)$. As a result, $\hat{\Sigma}_b$ may have negative values at small $m$ and $n$ when $\hat{U}(\hat{\theta})$ is near to 0 or 1. In order to overcome this defectness, Park (1995) suggested to use unbiased or positively biased estimator of variance for $\sqrt{N}U(\theta)$. Hence following the ideas of Mee (1990) and Sen (1967), one might consider the following remark, which propose estimators of variance for $\sqrt{N}[\hat{U}(\hat{\theta}) - P(\theta)]$.

**Remark** Under Conditions A, B and C,

$$\begin{aligned}\hat{\Sigma}_u = \frac{N}{(m-1)(n-1)}[&(m-1)(\hat{p}_1 - \hat{U}(\hat{\theta})^2) + (n-1)(\hat{p}_2 - \hat{U}(\hat{\theta})^2) \\ &+ \hat{U}(\hat{\theta}) - \hat{U}(\hat{\theta})^2] \\ &+ N(z-\bar{z})'[(Z-\bar{Z})'(Z-\bar{Z})]^{-1}(z-\bar{z})\,\hat{p}_3^2\,\hat{\sigma}^2 \\ &+ N(w-\bar{w})'[(W-\bar{W})'(W-\bar{W})]^{-1}(w-\bar{w})\,\hat{p}_3^2\,\hat{\tau}^2\end{aligned}$$

and

$$\begin{aligned}\hat{\Sigma}_s = &\frac{N}{m(n-1)}[(m-1)(\hat{p}_1 - \hat{U}(\hat{\theta})^2) + \hat{U}(\hat{\theta}) - \hat{U}(\hat{\theta})^2] \\ &+ \frac{N}{n(m-1)}[(n-1)(\hat{p}_2 - \hat{U}(\hat{\theta})^2) + \hat{U}(\hat{\theta}) - \hat{U}(\hat{\theta})^2] \\ &+ N(z-\bar{z})'[(Z-\bar{Z})'(Z-\bar{Z})]^{-1}(z-\bar{z})\,\hat{p}_3^2\,\hat{\sigma}^2 \\ &+ N(w-\bar{w})'[(W-\bar{W})'(W-\bar{W})]^{-1}(w-\bar{w})\,\hat{p}_3^2\,\hat{\tau}^2\end{aligned}$$

are consistent estimators of the asymptotic variance for $\sqrt{N}[\hat{U}(\hat{\theta}) - P(\theta)]$.

Now, we construct distribution-free confidence intervals using the estimators of variance for $\hat{U}(\hat{\theta})$, which is given in Remark. Suppose that Conditions A, B and C hold. Let $\hat{\Sigma}$ be one of $\hat{\Sigma}_b$, $\hat{\Sigma}_u$ and $\hat{\Sigma}_s$. Then an approximate $100(1-2\alpha)\%$ distribution-free confidence interval for $P(\theta)$ is defined by

$$\sqrt{N}[\ \hat{U}(\widehat{\theta})-P(\theta)] \le Z_\alpha \hat{\Sigma}^{1/2},$$

where $Z_\alpha$ is the upper $100\alpha\%$ quantile of the standard normal distribution.

## 3. Real Examples and Simulation Results

In this section, our estimate and confidence intervals are compared to those which are obtained by Guttman et al. (1988). First we are going to apply our results to real data sets by Rocket-Motor experiment. These are very well-known data. $Y$ is the chamber burst strength of a Rocket-Motor case and $X$ is the operating pressure, which is the stress the motor must withstand. From a designed experiment, temperature $z$ is the explanatory variable which affect operating pressure $X$. The data set (with sample sizes $m=51, n=17$) is given in Guttman et al. (1988).

we compute our estimate and lower confidence bounds and those under the normal assumption. From Table 3.1, we obtain the results applied to the real data. In Table 3.1, $R(\widehat{\theta})$ is the estimate of $P(\theta)$ by Guttman et al. (1988) under the normal assumption when $\tau^2/\sigma^2$ is unknown. Further, $LCB_1$ and $LCB_2$ are the lower confidence bounds corresponding to the asymptotic variance estimates for $P(\theta)$ being $\hat{\Sigma}_u$ and $\hat{\Sigma}_s$, respectively. $LCB_3$ is the approximate lower confidence bound for $P(\theta)$ by Guttman et al. (1988) under the normal assumption when $\tau^2/\sigma^2$ is unknown.

Calculating from the data in Guttman et al. (1988), we find

$$m=51, n=17, p=1, q=0,$$

$$\hat{\mu}=6.937,\ \widehat{\beta}=0.018,\ \hat{\nu}=16.485,\ \hat{\sigma}^2=0.056,\ \hat{\tau}^2=0.341,$$

$$\bar{z}=11.824,\ \sum_{i=1}^{m} z_i^{\ 2}=120291.$$

Our estimate $\hat{U}(\widehat{\theta})$ and $R(\widehat{\theta})$ have similar values. Except for the cases $z=530, 540$, whose lower confidence bounds are near to zero, our lower confidence bounds $LCB_1$ and $LCB_2$ have larger values than $LCB_3$ under the normal assumption. Throughout, $LCB_1$ and $LCB_2$ have almost same values with $LCB_2$ being slightly smaller than $LCB_1$ due to slightly larger estimate of variance for $\hat{U}(\widehat{\theta})$.

We can also observe that all the lower bounds are largest in the neighborhood where $z$ is given near to $\bar{z}$ and gradually decrease as $z$ is far from $\bar{z}$. This reveals the fact that as

$\boldsymbol{z}$ and $\boldsymbol{w}$ are far from $\bar{\boldsymbol{z}}$ and $\bar{\boldsymbol{w}}$, they contribute largely to increase estimates of variance for $P(\boldsymbol{\theta})$, as one can verify from $\hat{\Sigma}_u$ and $\hat{\Sigma}_s$ in Remark. As a result, our confidence bounds decrease more rapidly than that under the normal assumption as they are close to 0.

**Table 3.1** Comparison of estimates and 95% lower confidence bounds for $P(X < Y \mid \boldsymbol{z}, \boldsymbol{w})$ with those under the normal assumption for Rocket-Motor experiment.

| | estimates | | lower confidence bounds | | |
|---|---|---|---|---|---|
| $\boldsymbol{z}$ | $\hat{U}(\widehat{\boldsymbol{\theta}})$ | $R(\widehat{\boldsymbol{\theta}})$ | $LCB_1$ | $LCB_2$ | $LCB_3$ |
| 410 | 1.000 | 1.000 | 1.000 | 1.000 | 0.993 |
| 420 | 1.000 | 1.000 | 0.999 | 0.999 | 0.987 |
| 430 | 1.000 | 0.999 | 0.997 | 0.997 | 0.976 |
| 440 | 0.999 | 0.998 | 0.992 | 0.992 | 0.958 |
| 450 | 0.997 | 0.995 | 0.980 | 0.980 | 0.929 |
| 460 | 0.992 | 0.989 | 0.957 | 0.956 | 0.887 |
| 470 | 0.982 | 0.977 | 0.925 | 0.925 | 0.828 |
| 480 | 0.955 | 0.956 | 0.855 | 0.855 | 0.750 |
| 490 | 0.932 | 0.923 | 0.784 | 0.784 | 0.655 |
| 500 | 0.905 | 0.872 | 0.678 | 0.678 | 0.548 |
| 510 | 0.840 | 0.802 | 0.543 | 0.542 | 0.434 |
| 520 | 0.708 | 0.713 | 0.339 | 0.339 | 0.324 |
| 530 | 0.571 | 0.608 | 0.155 | 0.155 | 0.225 |
| 540 | 0.452 | 0.495 | 0.035 | 0.035 | 0.145 |

In this section, we also evaluate the confidence intervals presented in Section 2 through simulation. The simulation studies were performed by using IBM/PC 486DX2-66-S, and IMSL was utilized to generate random numbers and to compute the regression estimates.

Monte Carlo studies are performed for investigating the adequacy of two confidence interval methods given in Section 2. The distributions of errors under the consideration are as follows:

Case 1 : $\delta, \varepsilon \sim N(0,1)$

Case 2 : $\delta \sim 0.95N(0,1)+0.05N(0,3^2)$ and $\varepsilon \sim 0.95N(0,1)+0.05N(0,10^2)$

Case 3 : $\delta, \varepsilon \sim 0.9N(0,1)+0.1N(0,10^2)$

As one can see, we consider the standard normal distributions in Case 1 and the variance-contaminated normal distributions in Cases 2 and 3. For all such cases, both $\delta$ and $\varepsilon$ have symmetric and unimodal distributions. Thus, every $\delta - \varepsilon$ have symmetric distributions which are unimodal.

Regression parameters $m, n, \boldsymbol{\beta}$ and $\gamma$ are chosen so that $P(X < Y \mid \boldsymbol{z}, \boldsymbol{w})$ takes the values of $0.1, 0.2, \cdots, 0.9$. For the sake of convenience, we only consider the simple linear regression models for $X$ and $Y$. We set $\nu=2P(\theta)\mu$, $\beta=2\mu$ and $\gamma=2\nu$, and take both $z_i$ and $w_i$ as $\pm(i-1)/m, i=1,2,\ldots,m/2$, symmetrically around the point zero. For each case, we try simulation when $z=\bar{z}=0$ and $w=\bar{w}=0$.

The equally chosen sample sizes $m$ and $n$ range from 10 to 80, and bandwidth $h$ is selected as $(m+n)^{-0.2}$. The number of pairs of samples generated for each combination of $P(\theta)$ and $m(=n)$ is 1000.

Throughout the Tables 3.2 - 3.4, the results are reported both without continuity correction and with continuity correction. The employed methods $1(1^c)$ and $2(2^c)$ represent those from the confidence bounds based on the estimates of the variance $\hat{\Sigma}_u$ and $\hat{\Sigma}_s$, which are given in Remark, without(with) continuity correction. In addition, the results denoted as 'normal' represent those based on the confidence intervals by Guttman et al. (1988).

For each combination of sample size and $P(\theta)$, the proportion of 95% lower confidence bounds that fall below $P(\theta)$ is calculated. The coverage is reported only for one-sided lower confidence bounds because an upper bound at $P(\theta)$ will perform identically with a lower bound at $1-P(\theta)$.

Tables 3.2 - 3.4 by methods named $1, 1^c, 2$ and $2^c$ display the simulated coverage levels without any distributional assumption. From those, we can make the comments as follows :

[1] The methods with continuity correction yield slightly higher coverage levels than those without continuity correction at the sample sizes 10 and 20. However, for the sample sizes 40 and 80, the effect of continuity correction is not remarkable.

[2] Every method ultimately yields the coverage levels close to the nominal level as sample size increases. Even in the case of the sample size being equal to 10, the coverage level is moderately close to the nominal 0.95 level unless $P(\theta)$ is near to 1.

[3] In general, as sample sizes come to small and $P(\theta)$ is near to 1, the coverage discrepancy increases. To reduce this discrepancy, using the lower bounds by method $2^c$ is preferable since this method has a effect of increasing confidence levels due to the continuity correction and slightly larger estimate of the variance.

[4] All the lower confidence bounds have desirable performance for all the applied distributions. As a result, even though errors are not from normal distributions, we have adequate coverages for the distributions with moderately or extremely heavy tails, unless sample size is small and $P(\theta)$ is near to 1.

The figures in Tables 3.2 - 3.4 obtained by the method 'normal' denote the lower confidence bounds yielded by the method of Guttman et al. (1988). From these simulation results, we make the following reports :

[1] On the whole, at $P(\theta)$ near to 0.5, the simulated coverage level is closest to the nominal coverage level. As $P(\theta)$ is far from 0.5, the simulated coverage level also tends to deviate for each case of the variance-contaminated normal errors.

[2] As sample size increases, the coverage discrepancy from 0.95 nominal coverage level increases in almost all cases of the variance-contaminated normal errors.

From the above, we conclude that the parametric method of Guttman et al. (1988) performs better than our nonparametric method when the errors are from normal distributions.

On the other hand, as one can see from Tables 3.3 - 3.4, when the true errors are from heavy tailed distributions, the method of Guttman et al. (1988) has severe defectness since the coverage level is rapidly apart from the nominal coverage level as sample size increases.

However, our estimation procedures hold successfully when the errors have distributions with moderately or extremely heavier tails than those of normal distributions, in respect that our confidence intervals yield the coverage levels closer to the nominal coverage level than those produced by Guttman et al. (1988). Further, our confidence intervals gradually tend to have nominal coverage level as sample size increases. So, our results are useful for handling heavy tailed nonnormal distributions.

**Table 3.2** Simulated coverage level of 95% lower confidence bounds for $P(X < Y \mid z = \bar{z}, w = \bar{w})$ when $\delta, \varepsilon \sim N(0,1)$

| $m=n$ | method | $P(X < Y \mid z = \bar{z}, w = \bar{w})$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 10 | 1 | .994 | .980 | .964 | .953 | .943 | .902 | .872 | .811 | .700 |
| 10 | 2 | .994 | .980 | .966 | .956 | .945 | .906 | .878 | .817 | .733 |
| 10 | $1^c$ | .994 | .980 | .966 | .954 | .944 | .907 | .876 | .817 | .700 |
| 10 | $2^c$ | .996 | .981 | .969 | .958 | .948 | .911 | .884 | .824 | .733 |
| 10 | normal | .952 | .952 | .950 | .956 | .940 | .951 | .942 | .949 | .958 |
| 20 | 1 | .986 | .972 | .968 | .951 | .940 | .928 | .914 | .883 | .817 |
| 20 | 2 | .986 | .974 | .969 | .952 | .941 | .931 | .919 | .890 | .822 |
| 20 | $1^c$ | .986 | .974 | .969 | .951 | .941 | .931 | .917 | .888 | .821 |
| 20 | $2^c$ | .986 | .976 | .969 | .953 | .942 | .933 | .920 | .893 | .829 |
| 20 | normal | .957 | .951 | .955 | .941 | .935 | .938 | .949 | .941 | .948 |
| 40 | 1 | .974 | .976 | .957 | .963 | .939 | .941 | .929 | .895 | .871 |
| 40 | 2 | .976 | .977 | .959 | .964 | .941 | .942 | .929 | .895 | .873 |
| 40 | $1^c$ | .976 | .977 | .957 | .963 | .940 | .941 | .929 | .895 | .873 |
| 40 | $2^c$ | .977 | .977 | .961 | .965 | .941 | .942 | .930 | .896 | .875 |
| 40 | normal | .961 | .944 | .944 | .951 | .953 | .953 | .942 | .941 | .944 |
| 80 | 1 | .979 | .967 | .964 | .943 | .945 | .942 | .927 | .923 | .898 |
| 80 | 2 | .979 | .967 | .965 | .943 | .945 | .943 | .927 | .925 | .899 |
| 80 | $1^c$ | .979 | .967 | .965 | .943 | .945 | .942 | .927 | .924 | .899 |
| 80 | $2^c$ | .979 | .967 | .965 | .944 | .945 | .943 | .927 | .925 | .900 |
| 80 | normal | .940 | .962 | .951 | .953 | .958 | .949 | .951 | .953 | .953 |

**Table 3.3** Simulated coverage level of 95% lower confidence bounds for $P(X < Y \mid z=\bar{z}, w=\bar{w})$ when $\delta \sim 0.95N(0,1)+0.05N(0,3^2)$ and $\varepsilon \sim 0.95N(0,1)+0.05N(0,10^2)$

| $m=n$ | method | $P(X < Y \mid z=\bar{z}, w=\bar{w})$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 10 | 1 | .959 | .965 | .934 | .958 | .916 | .900 | .866 | .827 | .684 |
| 10 | 2 | .961 | .966 | .942 | .961 | .922 | .903 | .870 | .833 | .726 |
| 10 | $1^c$ | .962 | .966 | .938 | .960 | .921 | .903 | .870 | .834 | .684 |
| 10 | $2^c$ | .964 | .967 | .945 | .961 | .924 | .906 | .881 | .842 | .726 |
| 10 | normal | .908 | .921 | .948 | .968 | .949 | .909 | .766 | .616 | .549 |
| 20 | 1 | .965 | .951 | .944 | .957 | .938 | .935 | .928 | .898 | .810 |
| 20 | 2 | .966 | .955 | .946 | .959 | .938 | .937 | .930 | .902 | .818 |
| 20 | $1^c$ | .966 | .955 | .945 | .959 | .938 | .937 | .929 | .899 | .811 |
| 20 | $2^c$ | .968 | .957 | .950 | .961 | .939 | .941 | .930 | .904 | .826 |
| 20 | normal | .867 | .910 | .941 | .963 | .956 | .871 | .728 | .605 | .485 |
| 40 | 1 | .965 | .958 | .945 | .949 | .956 | .939 | .937 | .929 | .888 |
| 40 | 2 | .967 | .962 | .946 | .950 | .956 | .941 | .937 | .932 | .889 |
| 40 | $1^c$ | .967 | .959 | .946 | .950 | .956 | .940 | .937 | .931 | .889 |
| 40 | $2^c$ | .967 | .963 | .946 | .950 | .956 | .942 | .938 | .932 | .891 |
| 40 | normal | .842 | .890 | .931 | .948 | .953 | .846 | .721 | .637 | .556 |
| 80 | 1 | .966 | .955 | .949 | .959 | .944 | .961 | .943 | .926 | .904 |
| 80 | 2 | .966 | .956 | .951 | .959 | .946 | .962 | .944 | .928 | .905 |
| 80 | $1^c$ | .966 | .956 | .949 | .959 | .945 | .961 | .944 | .926 | .905 |
| 80 | $2^c$ | .966 | .956 | .952 | .959 | .947 | .962 | .945 | .928 | .905 |
| 80 | normal | .798 | .846 | .884 | .939 | .950 | .873 | .774 | .683 | .631 |

**Table 3.4** Simulated coverage level of 95% lower confidence bounds for $P(X < Y \mid z = \bar{z}, w = \bar{w})$ when $\delta, \varepsilon \sim 0.9N(0,1) + 0.1N(0,10^2)$

| $m=n$ | method | $P(X < Y \mid z = \bar{z}, w = \bar{w})$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 10 | 1 | .937 | .909 | .940 | .946 | .951 | .928 | .917 | .842 | .707 |
| 10 | 2 | .938 | .912 | .942 | .952 | .957 | .934 | .926 | .854 | .721 |
| 10 | $1^c$ | .941 | .912 | .943 | .949 | .957 | .934 | .927 | .857 | .707 |
| 10 | $2^c$ | .942 | .918 | .945 | .954 | .960 | .938 | .933 | .865 | .721 |
| 10 | normal | .998 | .998 | .996 | .997 | .948 | .739 | .516 | .407 | .329 |
| 20 | 1 | .925 | .917 | .942 | .946 | .952 | .946 | .941 | .905 | .842 |
| 20 | 2 | .926 | .920 | .945 | .948 | .956 | .947 | .941 | .908 | .842 |
| 20 | $1^c$ | .926 | .919 | .944 | .948 | .955 | .946 | .941 | .907 | .842 |
| 20 | $2^c$ | .927 | .920 | .947 | .951 | .957 | .948 | .944 | .910 | .845 |
| 20 | normal | 1.000 | 1.000 | 1.000 | .994 | .954 | .694 | .378 | .255 | .179 |
| 40 | 1 | .937 | .914 | .940 | .941 | .953 | .948 | .948 | .939 | .918 |
| 40 | 2 | .938 | .916 | .940 | .941 | .953 | .948 | .950 | .939 | .918 |
| 40 | $1^c$ | .938 | .915 | .940 | .941 | .953 | .948 | .949 | .939 | .918 |
| 40 | $2^c$ | .938 | .916 | .940 | .941 | .955 | .948 | .954 | .940 | .920 |
| 40 | normal | 1.000 | .999 | .998 | .999 | .946 | .595 | .248 | .103 | .052 |
| 80 | 1 | .936 | .933 | .940 | .941 | .947 | .944 | .954 | .945 | .892 |
| 80 | 2 | .936 | .935 | .941 | .941 | .948 | .944 | .954 | .945 | .895 |
| 80 | $1^c$ | .936 | .934 | .941 | .941 | .947 | .944 | .954 | .945 | .894 |
| 80 | $2^c$ | .937 | .935 | .942 | .942 | .950 | .944 | .955 | .946 | .895 |
| 80 | normal | 1.000 | 1.000 | 1.000 | .999 | .942 | .427 | .075 | .019 | .003 |

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