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On Fixed Width Confidence Bounds for the Difference of the Means of Two Linear Processes †

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Abstract

In this article we consider a sequential procedure for the fixed width interval estimation of the means of two mutually independent linear processes. It is shown that the proposed stopping rule is asymptotically efficient as in iid samples (cf. Robbins, Simons and Starr(1967)).

Key Words : Sequential procedure; Fixed width interval estimation; Linear processes; Stopping rule; Asymptotically efficient.

1. INTRODUCTION

Consider two mutually independent linear processes

$$X_t - \mu_1 = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}, \quad \{\varepsilon_t\} \sim \text{iid } (0, \sigma_1^2) \tag{1.1}$$

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and

$$Y_t - \mu_2 = \sum_{j=0}^{\infty} b_j \eta_{t-j}, \quad \{\eta_t\} \sim \text{iid } (0, \sigma_2^2), \tag{1.2}$$

where the parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ are unknown and the coefficients a_i and b_j are real numbers satisfying $\sum_{i=0}^{\infty} |a_i| < \infty$ and $\sum_{j=0}^{\infty} |b_j| < \infty$. Robbins, Simons and Starr (1968) proposed a sequential procedure for the fixed width interval estimation of the difference of the means of two iid populations. In this article we consider an analogous procedure for estimating the parameter $\Delta = \mu_1 - \mu_2$.

Since the work of Robbins (1959) there have been a large number of articles and developments on both sequential point estimation and interval estimation for iid random variables. Sequential estimation in time series emerged lately compared to iid cases. See Sriram (1987), Fakhre-Zakeri and Lee (1992, 1993), and the references cited in these papers.

Suppose that one wishes to find a confidence interval for $\Delta = \mu_1 - \mu_2$ of width $2d$ and with coverage probability $1 - \alpha$ $(0 < \alpha < 1)$, based on the random sequences $\{X_t\}$ and $\{Y_t\}$ in (1.1)-(1.2). To get the confidence interval, the following central limit theorem is useful:

$$\frac{\overline{X}_r - \overline{Y}_s - (\mu_1 - \mu_2)}{(\tau^2/r + \omega^2/s)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad \text{as} \quad r, s \to \infty, \tag{1.3}$$

where

$\overline{X}_r = r^{-1} \sum_{t=1}^{r} X_t, \overline{Y}_s = s^{-1} \sum_{t=1}^{s} Y_t, \tau^2 = (\sum_{i=0}^{\infty} a_i)^2 \sigma_1^2$ and $\omega^2 = (\sum_{j=0}^{\infty} b_j)^2 \sigma_2^2$.

Based on this, if

$$I = I_{r,s} = [\overline{X}_r - \overline{Y}_s - d, \overline{X}_r - \overline{Y}_s + d] \tag{1.4}$$

is the interval of width $2d$ centered at $\overline{X}_r - \overline{Y}_s$, then for all sufficiently large r, s,

$$P(\Delta \in I) \sim 2\Phi\left(\frac{d}{(\tau^2/r + \omega^2/s)^{1/2}}\right) - 1, \tag{1.5}$$

where $\Phi(\cdot)$ is the standard normal distribution. Hence, I has an approximate coverage probability $1 - \alpha$, providing that

$$\frac{\tau^2}{r} + \frac{\omega^2}{s} \le \frac{1}{b}, \tag{1.6}$$

where $b = (\frac{z_{\alpha/2}}{d})^2$ and $\Phi(z_\alpha) = 1 - \alpha$.

Regarding r and s as continuous variables and using the Lagrange multiplier method, one can see that (r_0, s_0) satisfying (1.6) that minimizes $n = r+s$ is given by

$$r_0 = b\tau(\tau + \omega) \quad \text{and} \quad s_0 = b\omega(\tau + \omega). \tag{1.7}$$

For this pair,

$$r_0/s_0 = \tau/\omega \tag{1.8}$$

and the total sample

$$n_0 = b(\tau + \omega)^2. \tag{1.9}$$

Moreover, due to (1.5) and (1.6)

$$\lim_{d \to 0} P(\Delta \in I_{r_0, s_0}) \geq 1 - \alpha. \tag{1.10}$$

However, in real practice τ^2 and ω^2 are unknown, and in order to construct confidence intervals one has to estimate τ^2 and ω^2. In the following we give a sequential procedure for determining r, s in such a manner that (1.7)-(1.10) will hold. The procedure is similar to that of Robbins, Simons and Starr (1967) in iid cases.

In order to estimate τ^2 and ω^2 consider the random variables

$$\hat{\gamma}_r(h) = r^{-1} \sum_{t=1}^{r-h} (X_{t+h} - \overline{X}_r)(X_t - \overline{X}_r)$$

and

$$\hat{\delta}_s(h) = s^{-1} \sum_{t=1}^{s-h} (Y_{t+h} - \overline{Y}_s)(Y_t - \overline{Y}_s).$$

They are conventional estimates of the autocovariance functions $\gamma(\cdot)$ and $\delta(\cdot)$ of the processes $\{X_t\}$ and $\{Y_t\}$ at lag h, respectively. Then, in view of the fact

$$\tau^2 = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h) \quad \text{and} \quad \omega^2 = \delta(0) + 2\sum_{h=1}^{\infty} \delta(h),$$

we employ as estimates of τ^2 and ω^2

$$\hat{\tau}_r^2 = \tilde{\tau}_r^2 I(\tilde{\tau}_r^2 > 0) + c_r I(\tilde{\tau}_r^2 \leq 0)$$

and

$$\hat{\omega}_s^2 = \tilde{\omega}_s^2 I(\tilde{\omega}_s^2 > 0) + c_s I(\tilde{\omega}_s^2 \leq 0),$$

where $\{c_n; n \geq 1\}$ is any sequence of positive real numbers decaying to 0, and

$$\begin{aligned} \tilde{\tau}_r^2 &= \hat{\gamma}_r(0) + 2 \sum_{h=1}^{h_r} \hat{\gamma}_r(h), \\ \tilde{\omega}_s^2 &= \hat{\delta}_s(0) + 2 \sum_{h=1}^{h_s} \hat{\delta}_s(h) \end{aligned}$$

with $\{h_n; n \geq 1\}$ being a sequence of positive integers such that as $n \to \infty$,

$$h_n \to \infty \text{ and } h_n/n^\lambda \to 0 \text{ for all } \lambda > 0. \tag{1.11}$$

A typical example of such sequences is $h_n = [(\log n)^2]$, where $[x]$ denotes the largest integer that does not exceed x.

To start the sequential procedure we take m_0 observations on $\{X_t\}$ and $\{Y_t\}$, where m_0 is initial sample size, and take observations until we find the pair (r, s) satisfying

$$r \geq b\hat{\tau}_r(\hat{\tau}_r + \hat{\omega}_s) \text{ and } s \geq b\hat{\omega}_s(\hat{\tau}_r + \hat{\omega}_s), \tag{1.12}$$

which is motivated by (1.7). To reach such r and s the proposed sampling scheme is as follows: at the stage we have taken $r \geq m_0$ observations on $\{X_t\}$ and $s \geq m_0$ observations on $\{Y_t\}$, we take the next observation on $\{X_t\}$ or $\{Y_t\}$ according as

$$r/s \leq \hat{\tau}_r/\hat{\omega}_s \text{ or } r/s > \hat{\tau}_r/\hat{\omega}_s. \tag{1.13}$$

Based on (1.12), the stopping rule is defined as the pair $(R, S) \in T$ such that $N := R + S \leq r + s$ for all $(r, s) \in T$, where T is the set consisting of pairs (r, s) satisfying (1.12).

In Section 2 we show that the suggested stopping rule is asymptotically efficient (cf. Theorem 1), and provide detailed proofs.

2. INTERVAL ESTIMATION

Theorem 1. If $E|\varepsilon_1|^{2\lambda} < \infty$ and $E|\eta_1|^{2\lambda} < \infty$ for $\lambda \geq 2$, then as $d \to 0$, we have

$$N/n_0 \to 1 \quad \text{a.s.} \tag{2.1}$$
$$P(\Delta \in I) \to 1 - \alpha \tag{2.2}$$
$$EN/n_0 \to 1. \tag{2.3}$$

To establish the theorem we introduce a series of lemmas in the following.

Lemma 1. If $E|\varepsilon_1|^{2\lambda} < \infty$ and $|\eta_1|^{2\lambda} < \infty$, $\lambda \geq 2$, then for all $\zeta > 0$,

$$P(|\tilde{\tau}_r^2 - \tau^2| > \zeta) = O((h_r/r)^\lambda)$$

and

$$P(|\tilde{\omega}_s^2 - \omega^2| > \zeta) = O((h_s/s)^\lambda).$$

Proof. A slight modification of Fakheri-Zakeri and Lee (1992) yields the lemma, and we omit the proof for brevity. □

The following corollary is a direct result of Lemma 1.

Corollary 1. Under the same condition of Lemma 1,

$$P(|\hat{\tau}_r^2 - \tau^2| > \zeta) = O((h_r/r)^\lambda)$$

and

$$P(|\hat{\omega}_s^2 - \omega^2| > \zeta) = O((h_s/s)^\lambda).$$

Therefore, $\hat{\tau}_r^2$ and $\hat{\omega}_s^2$ converge to τ^2 and ω^2 almost surely as r, s go to infinity.

Lemma 2. Under the same condition of Lemma 1,

$$P(|\hat{\tau}_r(\hat{\tau}_r + \hat{\omega}_s) - \tau(\tau + \omega)| > \zeta) = O(\max\{(h_r/r)^\lambda, (h_s/s)^\lambda\}) \tag{2.4}$$

and

$$P(|\hat{\omega}_s(\hat{\tau}_r + \hat{\omega}_s) - \tau(\tau + \omega)| > \zeta) = O(\max\{(h_r/r)^\lambda, (h_s/s)^\lambda\}). \tag{2.5}$$

Proof. Note that

$$\begin{aligned} P(|\hat{\tau}_r(\hat{\tau}_r + \hat{\omega}_s) - \tau(\tau + \omega)| > \zeta) \quad &\leq \quad P(|\hat{\tau}_r^2 - \tau^2| > \zeta/3) && (2.6) \\ &+ \quad P\left(|\frac{\hat{\tau}_r}{\hat{\omega}_s + \omega}(\hat{\omega}_s^2 - \omega^2)| > \zeta/3\right) && (2.7) \\ &+ \quad P\left(|\frac{\omega}{\hat{\tau}_r + \tau}(\hat{\tau}_r^2 - \tau^2)| > \zeta/3\right). && (2.8) \end{aligned}$$

Since the argument in (2.8) is bounded by

$$P(|\tilde{\tau}_r^2 - \tau^2| > \tau\zeta/3\omega) + P(\tilde{\tau}_r^2 \leq 0) = O((h_r/r)^\lambda)$$

due to Lemma 1, and similarly the arguments in (2.6)-(2.7) are $O(\max\{(h_s/s)^\lambda, (h_r/r)^\lambda\})$, (2.4) is yielded. The proof of (2.5) is similar to that of (2.4). □

Lemma 3. Under the same condition of Lemma 1,

$$R/r_0 \to 1 \ \text{ and } \ S/s_0 \to 1 \ \text{ a.s. as } \ d \to 0.$$

Proof. Suppose that $R > m_0$ and just before Rth observation on $\{X_t\}$, there were j observations on $\{Y_t\}$. Then by our sampling scheme

$$(R-1)/j \leq \hat{\tau}_{R-1}/\hat{\omega}_j$$

and by the stopping rule

$$R - 1 \leq b\hat{\tau}_{R-1}(\hat{\tau}_{R-1} + \hat{\omega}_j),$$

which leads to

$$R \leq b\hat{\tau}_{R-1}(\hat{\tau}_{R-1} + \hat{\omega}_j) + m_0. \tag{2.9}$$

The above remains valid even when $R = m_0$. Since $R \to \infty$ and $R/j \to \infty$ a.s. as $d \to 0$ in view of the lemma of Robbins, Simons and Starr (1967, P. 1358), it follows that $j \to \infty$ a.s.. This together with (2.9) and Corollary 1 implies

$$\limsup_{d \to 0} R/b \leq \tau(\tau + \omega) \ \text{ a.s}$$

Meanwhile, the reverse inequality for the limit-inf follows from the definition. Thus, $R/r_0 \to 1$ as $d \to 0$. The convergence result for S is yielded by similar arguments. □

Lemma 4. Under the same condition of Lemma 1,

$$\frac{\overline{X}_R - \overline{Y}_S - (\mu_1 - \mu_2)}{(\hat{\tau}_R^2/R - \hat{\omega}_S^2/S)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \ \text{ as } \ d \to 0.$$

Proof. Without loss of generality, we assume $\mu_1 = \mu_2 = 0$. Since $\hat{\tau}_r^2$ and $\hat{\omega}_s^2$ go to τ^2 and ω^2 as $r, s \to \infty$ due to Lemma 2, and since R and S diverge to infinity almost surely as $d \to 0$ by Lemma 3, $\hat{\tau}_R^2$ and $\hat{\omega}_S^2$ converge to τ^2 and ω^2 in probability as d goes to 0. Therefore, in view of the following:

$$\frac{\overline{X}_R - \overline{Y}_S}{(\tau^2/r_0 + \omega^2/s_0)^{1/2}} = \frac{\overline{X}_{r_0} - \overline{Y}_{s_0}}{(\tau^2/r_0 + \omega^2/s_0)^{1/2}} + \frac{\overline{X}_R - \overline{X}_{r_0} - \overline{Y}_S + \overline{Y}_{s_0}}{(\tau^2/r_0 + \omega^2/s_0)^{1/2}},$$

it suffices to show that

$$r_0^{1/2}(\overline{X}_R - \overline{X}_{r_0}) \xrightarrow{P} 0 \quad \text{and} \quad s_0^{1/2}(\overline{Y}_S - \overline{Y}_{s_0}) \xrightarrow{P} 0.$$

Here we only provide the proof for the former because the latter can be handled similarly. Simple algebra shows that we only have to show

$$r_0^{-1/2}(S_R - S_{r_0}) \xrightarrow{P} 0$$

(cf. Lemma 3). To this end, consider the time series $X_{t,m} = \sum_{i=0}^{m} a_i\varepsilon_{t-i}$ and put $S_{r,m} = \sum_{t=1}^{r} X_{t,m}$. By the Beveridge-Nelson decomposition (cf. Phillips and Solo (1992) and Fakhre-Zakeri and Lee (1992, P. 193)) and Lemma 3, we can write for each $m \geq 1$,

$$r_0^{-1/2}|S_{R,m} - S_{r_0,m}| = |(\sum_{i=0}^{m} a_i)r_0^{-1/2}\sum_{t=r_{\min}}^{r_{\max}} \varepsilon_t| + o_P(1) \quad \text{as} \quad d \to 0, \tag{2.10}$$

where $r_{\min}$ and $r_{\max}$ denote $\max\{R, r_0\}$ and $\min\{R, r_0\}$, respectively. Now, following the arguments similar to those of Gut (1988, P. 16) we can show that the left hand side of the above equality is $o_P(1)$.

Note that for all $\theta > 0$,

$$\begin{aligned} & \limsup_{d\to 0} P(r_0^{-1/2}|S_R - S_{R,m}| > \theta) \\ \leq \quad & \limsup_{d\to 0} P(\sum_{i=m+1}^{\infty} |a_i||r_0^{-1/2}\sum_{t=1}^{R}\varepsilon_{t-i}| > \theta) \\ = \quad & O(\sum_{i=m+1}^{\infty} |a_i|), \end{aligned}$$

which can be yielded following essentially the same lines in the proof of Lemma 2 of Lee (1994). Also, similar to (2.11) we have

$$\limsup_{d\to 0} P(r_0^{-1/2}|S_{r_0} - S_{r_0,m}| > \theta) = O(\sum_{i=m+1}^{\infty} |a_i|). \tag{2.11}$$

Now, in view of (2.10)-(2.12) and Proposition 6.3.9 of Brockwell and Davis (1990) we establish the lemma. □

Lemma 5. Under the same condition of Lemma 1, $\{N/n_0\}$ is uniformly integrable.

Proof. Let $\zeta > 0$ and $K_d = [b\{(\tau+\omega)^2 + 2\zeta\}] + 1$. For all $n > K_d$ and any r_n and s_n such that $r_n + s_n \le n$,

$$\begin{aligned}
P(N > n) &\le P(R + S > r_n + s_n) \\
&\le P(R > r_n \text{ or } S > s_n) \\
&\le P\Big(r_n < b\hat{\tau}_{r_n}(\hat{\tau}_{r_n} + \hat{\omega}_{s_n}) \text{ or } s_n < b\hat{\omega}_{s_n}(\hat{\tau}_{r_n} + \hat{\omega}_{s_n})\Big) \\
&\le P\Big(|\hat{\tau}_{r_n}(\hat{\tau}_{r_n} + \hat{\omega}_{s_n}) - \tau(\tau+\omega)| > \zeta\Big) \\
&\quad + P\Big(|\hat{\omega}_{s_n}(\hat{\tau}_{r_n} + \hat{\omega}_{s_n}) - \omega(\tau+\omega)| > \zeta\Big) \\
&\le K\{(h_{r_n}/r_n)^\lambda + (h_{s_n}/s_n)^\lambda\} \quad (K > 0),
\end{aligned}$$

where the last inequality follows from Lemma 2, and thus $\sum_{n=1}^{\infty} P(N > n) < \infty$, for example, by setting $r_n = s_n = n/2$. Hence, if B is a positive real number such that $b(\tau+\omega)^2 B > K_d$,

$$\begin{aligned}
\int_{N/n_0 > 2B} N/n_0 dP &\le \int_{N \ge 2Bb(\tau+\omega)^2} N dP \\
&\le 2\int_{N \ge K_d} (N - K_d) dP \\
&\le 2 \sum_{n=K_d+1}^{\infty} P(N > n),
\end{aligned}$$

which goes to 0 as $d \to 0$. This completes the proof. □

Proof of Theorem 1. (2.1) and (2.2) are obvious in view of Lemmas 3 and 2.4. (2.3) is a direct result of (2.1) and Lemma 5. □

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