Journal of the Korean
Statistical Society
Vol. 25, No. 4, 1996

Exact Tests for Variance Ratios in Unbalanced Random Effect Linear Models

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Abstract

In this paper, we propose a method for an exact test of $H : \rho_i = r_i$ for all i against $K : \rho_i \neq r_i$ for some i in an unbalanced random effect linear model, where ρ_i denotes the ratio of the i-th variance component to the error variance. Then we present a method to test $H : \rho_i \leq r$ against $K : \rho_i > r$ for some specific i by applying orthogonal projection on the model. We also show that any test statistic that follows an $F-$ distribution on the boundary of the hypotheses is equal to the one given here.

Key Words : Exact variance ratio test; MINQUE; Wald's test.

1. INTRODUCTION

When the design is unbalanced, it is well known that AOV(Analysis of Variance) procedure fails to decompose the total sum of squares into independently distributed sums of squares. Hence the AOV procedure can not

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The second author was supported in part by Hanshin University Research Fund, 1995.

be directly applied to the problem of hypothesis testing on variance components of a linear model. Many authors have attempted to obtain alternative procedures for the problem. One of the classical approaches is employing Satterthwaite's argument to obtain an approximate test. However in some cases the nominal significance level of an approximate test is highly unreliable (see, e.g., Khuri and Little 1987; Kleffe and Seifert 1988). Thus an exact test procedure is desirable.

Wald (1940, 1941, 1947) constructed an exact confidence interval for variance ratio, which is closely related to the testing problem. After Wald, many researchers derived exact tests on null variance components in specific models; see, for example, Spjøtvoll (1967, 1968), Thomsen (1975), etc. These results were generalized by Seely and El–Bassiouni (1983). They obtained the Wald test for testing a null variance ratio by applying reductions in sum of squares in a general linear model. They also provided the necessary and sufficient conditions for the existence of the Wald test and showed that the test statistics derived by Spjøtvoll and many others are identical to the test statistic of Wald. Lin and Harville (1991) carried out a simulation study on the performance of the Wald test relative to the locally best test and the Neyman–Pearson test, and showed that the Wald test is comparable to the latter two tests.

Let ρ_i denote the ratio of the i-th variance component to the error variance in a random effect model. We will consider, in section 2, a problem of testing $H : \rho_i = r_i$ for all $i = 1, 2, \dots, k$ against $K : \rho_i \neq r_i$ for some i in a general random effect linear model. A test statistic for this problem will be derived by applying the decomposition method to a variant of Rao's MINQUE(Minimum Norm Quadratic Unbiased Estimator) introduced in 1971. The resulting test statistic has an F–distribution under the null hypothesis. In section 3, the approach is extended to obtain an exact test of a single variance ratio by applying orthogonal projection on the model being considered. Section 4 contains concluding remarks and examples.

2. THE DERIVATION OF SIMULTANEOUS TEST

Consider the following general linear model:

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{X}_1\xi_1 + \cdots + \mathbf{X}_k\xi_k + \epsilon \tag{2.1}$$

where $\mathbf{y}$ is a vector of n observations, μ is a fixed unknown constant, $\mathbf{1}$ is a vector of ones of dimension n, $\mathbf{X}_i$ is an $n \times b_i$ design matrix, ξ_i is a vector of b_i uncorrelated random effects, and ϵ is a vector of n random errors. Further we assume that ξ_i and ϵ are statistically independent multivariate normal random vectors with $E(\xi_i) = \mathbf{0}, \text{Var}(\xi_i) = \sigma_i^2\mathbf{I}$, for $i = 1, 2, \ldots, k$ and $E(\epsilon) = \mathbf{0}, \text{Var}(\epsilon) = \sigma_{k+1}^2\mathbf{I}$.

Let $\mathbf{C}$ be a full row rank matrix of order $(n-1) \times n$ such that $\mathbf{C1} = \mathbf{0}, \mathbf{CC}' = \mathbf{I}_{n-1}$ and $\mathbf{C}'\mathbf{C} = \mathbf{I}_n - \frac{1}{n}\mathbf{11}'$. Then multiplying both sides of equation (2.1) by $\mathbf{C}$ yields

$$\mathbf{z} = \mathbf{CX}_1\xi_1 + \cdots + \mathbf{CX}_k\xi_k + \mathbf{C}\epsilon \tag{2.2}$$

where $\mathbf{z} = \mathbf{Cy}$.

It can be shown (see Rao, section 9, 1971, for example) that the MINQUE of $\sigma = (\sigma_1^2, \cdots, \sigma_k^2, \sigma_{k+1}^2)$ for model (2.1) is the same as that for model (2.2) and it is a solution to the equation

$$\mathbf{S}\hat{\sigma} = \mathbf{u} \tag{2.3}$$

where $\mathbf{S} = \{tr\,(\mathbf{RV}_i\mathbf{RV}_j)\}, \mathbf{u} = \{\mathbf{z}'\mathbf{RV}_i\mathbf{Rz}\}, \mathbf{V}_i = \mathbf{CX}_i\mathbf{X}_i'\mathbf{C}'$ for $i, j = 1, \cdots, k$, $\mathbf{V}_{k+1} = \mathbf{I}_{n-1}$, and $\mathbf{R} = (\sum_{i=1}^{k+1} r_i\mathbf{V}_i)^{-1} = (I + \sum_{i=1}^{k} r_i\mathbf{V}_i)^{-1}$ with r_i denoting the a-priori values of $\rho_i = \sigma_i^2/\sigma_{k+1}^2$, for $i = 1, \cdots, k$, and $r_{k+1} = 1$.

To derive a test statistic for testing $H : \rho_i = r_i$ for $i = 1, 2, \ldots, k$ against $K : \rho_i \neq r_i$ for some i, we consider the linear combination $\sum_{i=1}^{k+1} r_iu_i$ where u_i is the i-th element of $\mathbf{u}$ of equation (2.3). $\sum_{i=1}^{k+1} r_iu_i$ can be rewritten as follows:

$$\sum_{i=1}^{k+1} r_iu_i = \mathbf{z}'\mathbf{R}(\mathbf{I} + \sum_{i=1}^{k} r_i\mathbf{V}_i)\mathbf{Rz} = \mathbf{z}'\mathbf{Rz} \tag{2.4}$$

The quadratic form $\mathbf{z}'\mathbf{Rz}$ can be decomposed into two parts, each of which follows independently a central chi-square distribution under the null hypothesis. For this decomposition, we will decompose matrix $\mathbf{R}$ into two parts which are orthogonal to each other.

Since $\sum_{i=1}^{k} r_i\mathbf{V}_i$ in matrix $\mathbf{R}$ is symmetric, it can be expressed as

$$\sum_{i=1}^{k} r_i\mathbf{V}_i = \mathbf{PD}(\mathbf{r})\mathbf{P}'$$

where $\mathbf{D}(\mathbf{r})$ is a diagonal matrix of eigenvalues of $\sum_{i=1}^{k} r_i\mathbf{V}_i$ and $\mathbf{P}$ is the matrix of corresponding eigenvectors. Let g be the rank of $\sum_{i=1}^{k} r_i\mathbf{V}_i$, and the first g diagonal elements of $\mathbf{D}(\mathbf{r})$ be nonzero. Then, noting that the rank of partitioned matrix $(\mathbf{V}_1, \ldots, \mathbf{V}_k)$, which will be denoted by m, is greater than or equal to g, *i.e.*, $m = \text{rank}(\mathbf{V}_1, \ldots, \mathbf{V}_k) \geq g$, and that $\mathbf{P}$ is orthogonal, we have

$$\mathbf{R} = \left(\mathbf{I} + \sum_{i=1}^{k} r_i \mathbf{V}_i\right)^{-1} = (\mathbf{I} + \mathbf{P}\mathbf{D}(\mathbf{r})\mathbf{P}')^{-1}$$
$$= \mathbf{P}(\mathbf{I} + \mathbf{D}(\mathbf{r}))^{-1}\mathbf{P}' = \mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}' + \mathbf{P}\mathbf{D}_2\mathbf{P}' \qquad (2.5)$$

where

$$\mathbf{D}_1^-(\mathbf{r}) = diag(1/(1+\lambda_1(\mathbf{r})), \ldots, 1/(1+\lambda_g(\mathbf{r})), \overbrace{1, \ldots, 1}^{m-g}, \overbrace{0, \ldots, 0}^{n-1-m}), \qquad (2.6)$$
$$\mathbf{D}_2 = diag(0, \ldots, 0, \underbrace{1, \ldots, 1}_{n-1-m}),$$

and $\lambda_i(\mathbf{r})$ are nonzero eigenvalues of $\sum_{i=1}^{k} r_i \mathbf{V}_i$.

Combining the results of (2.4) and (2.5), we decompose $\sum_{i=1}^{k+1} r_i u_i$ into two parts as follows:

$$\sum_{i=1}^{k+1} r_i u_i = \mathbf{z}'\mathbf{R}\mathbf{z} = \mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z} + \mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}. \qquad (2.7)$$

Note that $m - g > 0$ when some of r_i's are specified to be zeros. Write $\mathbf{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_{n-1})$. If $\mathbf{p}_{g+1}, \ldots, \mathbf{p}_m$ are selected so that the column space of $(\mathbf{p}_1, \ldots, \mathbf{p}_m)$ is equal to the column space of $\sum_{i=1}^{k} \mathbf{V}_i$, then it is straightforward to see that $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}/\sigma_{k+1}^2$ is a chi-square random variable with m degrees of freedom when $\rho_i = r_i$ for all i. Also it is easy to check that, for all ρ_i, $\mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}/\sigma_{k+1}^2$ is a chi-square random variable with $n-1-m$ degrees of freedom and is independent of $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}/\sigma_{k+1}^2$. This suggests to us considering the following quantity as a test statistic for testing $H : \rho_i = r_i$ for $i = 1, 2, \ldots, k$ against $K : \rho_i \neq r_i$ for some i:

$$F = \frac{n-1-m}{m} \frac{\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}}{\mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}}.$$

The distribution of F is $F(m, n-1-m)$ under the null hypothesis and the null hypothesis is rejected if the observed value of F is too large or too small.

Remark 1. Note that $\sum_{i=1}^{k+1} r_i u_i$ is a linear combination of the MINQUE of ρ_i's. From the decomposition of $\sum_{i=1}^{k+1} r_i u_i$ in (2.7), it is clear that $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}$ is a linear combination of the MINQUE of ρ's. Hence $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}$ is locally minimum variance unbiased quadratic estimator under the hypothesized values; see Rao (1973), pp. 303–305.

3. THE DERIVATION OF SINGLE VARIANCE RATIO TEST

In this section, we will develop a test statistic for $H: \rho_1 \leq r$ against $K: \rho_1 > r$ under a certain condition on $\mathbf{X}_1$ in (2.1). For this, we rewrite model (2.2) as

$$\mathbf{z} = \mathbf{C}\mathbf{X}_1\xi_1 + \mathbf{C}\mathbf{X}\xi + \mathbf{C}\epsilon \tag{3.1}$$

where $\mathbf{X} = (\mathbf{X}_2, \ldots, \mathbf{X}_k)$ and $\xi' = (\xi_2', \ldots, \xi_k')$. In this model, we assume that the column space of $\mathbf{CX}_1$ is not a subset of the column space of $\mathbf{CX}$. This condition together with $n - 1 - m > 0$, which was implicitly assumed in the previous section, is identical to the condition that Seely and El–Bassiouni (1983) used to derive the Wald test statistic for testing $H: \rho_1 = 0$ against $K: \rho_1 > 0$.

Under the assumptions, we have

$$q = \text{rank}(\mathbf{CX}_1, \mathbf{CX}) - \text{rank}(\mathbf{CX}) > 0.$$

There are two situations under this assumption. The first one is that the column spaces of $\mathbf{CX}_1$ and $\mathbf{CX}$ are essentially disjoint, *i.e.*, $\text{col}(\mathbf{CX}_1) \cap \text{col}(\mathbf{CX}) = \{\mathbf{0}\}$, such as the case of balanced designs. The second one is that the column spaces intersect each other. The first situation is not considered here because it is easy to derive a test statistic and is essentially identical to the second one.

Let $\mathbf{M}$ be the orthogonal projection matrix on the orthogonal complement of the column space of $\mathbf{CX}$. Multiplying both sides of equation (3.1) by $\mathbf{M}$, we have

$$\mathbf{t} = \mathbf{Mz} = \mathbf{MCX}_1\xi + \mathbf{MC}\epsilon, \tag{3.2}$$

since $\mathbf{MCX} = \mathbf{0}$. For model (3.2), matrix $\mathbf{R}$ of MINQUE is given by

$$\mathbf{R} = (r\mathbf{MV}_1\mathbf{M} + \mathbf{M})^+$$

where $\mathbf{A}^+$ denotes the Moore–Penrose inverse or reflexive generalized inverse of $\mathbf{A}$ and r is the a-priori value of ρ_1.

For model (3.2), it is easy to check that $\text{rank}(\mathbf{MV}_1\mathbf{M}) = q$ and $\text{rank}(\mathbf{M}) = n - 1 - m + q$. Since $\mathbf{MV}_1\mathbf{M}$ and $\mathbf{M}$ are commuting each other, there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}'\mathbf{MV}_1\mathbf{MQ}$ and $\mathbf{Q}'\mathbf{MQ}$ are diagonal matrices with diagonal elements being the eigenvalues of the corresponding matrices.

Then, noting that the column space of $\mathbf{MV}_1\mathbf{M}$ is a proper subset of the column space of $\mathbf{M}$, matrix $\mathbf{R}$ can be decomposed as

$$\mathbf{R} = (r\mathbf{MV}_1\mathbf{M} + \mathbf{M})^+ = \mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}' + \mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'$$

where

$$\tilde{\mathbf{D}}_1^-(r) = \text{diag}(\frac{1}{1+r\lambda_1},\ldots,\frac{1}{1+r\lambda_q},0,\ldots,0)$$
$$\tilde{\mathbf{D}}_2 = \text{diag}(\underbrace{0,\ldots,0}_{q},\underbrace{1,\ldots,1}_{n-1-m},0,\ldots,0)$$

and λ_i's are nonzero eigenvalues of $\mathbf{MV}_1\mathbf{M}$. Thus,

$$\mathbf{t}'\mathbf{R}\mathbf{t} = \mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}'\mathbf{t} + \mathbf{t}'\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{t}. \tag{3.3}$$

Note that both terms in the right-hand side of (3.3) do not depend upon the unknown parameter ξ_1 or ξ in (3.1), and that the first term depends only upon the a-priori value r, which will be specified by the hypothesis.

Writing $\tilde{\mathbf{D}}_1^-(r) = \tilde{\mathbf{D}}_1^{-1/2}(r)\tilde{\mathbf{D}}_1^{-1/2}(r)$, it can be shown that

$$\tilde{\mathbf{D}}_1^{-1/2}(r)\mathbf{Q}'\mathbf{t}/\sigma_{k+1} \sim N\left(\mathbf{0},\tilde{\mathbf{D}}_1^-(r)\tilde{\mathbf{D}}_1(\rho_1)\right),$$
$$\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{t}/\sigma_{k+1} \sim N(\mathbf{0},\tilde{\mathbf{D}}_2),$$

and

$$\tilde{\mathbf{D}}_1^-(r)\text{Var}(\mathbf{Q}'\mathbf{t})\tilde{\mathbf{D}}_2 = \mathbf{0}.$$

Hence $\mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}'\mathbf{t}/\sigma_{k+1}^2$ and $\mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{t}/\sigma_{k+1}^2$ are independent and are distributed as $\sum_{i=1}^q(1+\rho_1\lambda_i)/(1+r\lambda_i)\chi_i^2$ and χ^2, where χ_i^2 and χ^2 are independent chi-square random variables with 1 and $n-1-m$ degrees of freedom, respectively.

Using the above results, we have the following test statistic,

$$F = \frac{n-1-m}{q}\frac{\mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}'\mathbf{t}}{\mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{t}}. \tag{3.4}$$

The distribution of F is $F_{q,n-1-m}$ on the the boundary of hypotheses. Since the numerator of F is increasing with ρ_1 while the denominator is independent of ρ_1, the rejection region of the test should be the upper tail of F-distribution. The power of the test is given by

$$\pi(\rho_1) = \Pr\left[\sum_{i=1}^q \frac{1+\rho_1\lambda_i}{1+r\lambda_i}\chi_1^2 > \frac{q}{n-1-m}c\chi^2\right]$$

where c is an appropriate constant so that the test is of desired size.

To compute the power of the test, we need to compute the probability of a linear combination of central chi-square random variables. Farebrother(1984) gives an algorithm for this problem. Lin and Harville (1991) did a simulation study for the power performance of the single variance component in the mixed model with one random component. They showed that the performance of the Wald exact test is comparable to the locally most powerful test and the Neyman–Pearson test.

Note that the denominator of F in (3.4), $\mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{t}$, is equal to the residual sum of squares of model (2.1) by the results of Propositions 3.3 of Seely and El–Bassiouni (1983). We have the following result regarding the uniqueness of our test.

Theorem 1. Assume that the column space of $\mathbf{CX}_1$ is not a subset of the column space of $\mathbf{CX}$. Let $\mathbf{A}$ be a symmetric matrix such that $\mathbf{A} \neq \mathbf{0}$. If $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma_{k+1}^2 \sim \chi_p^2$ when $\rho_1 = r$ and is independent of the error sum of squares of model (2.1), then $p \leq q$. Moreover, if $p = q$, then $\mathbf{A} = \mathbf{C}'\mathbf{M}\mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}'\mathbf{M}\mathbf{C}$, i.e., $\mathbf{y}'\mathbf{A}\mathbf{y} = \mathbf{t}'\mathbf{Q}\tilde{\mathbf{D}}_1^-(r)\mathbf{Q}'\mathbf{t}$. Any test statistic that follows $F_{q,n-1-m}$ distribution on the boundary of hypotheses is equal to the test statistic given in (3.4).

Proof. Because Var(y) is a positive definite matrix and $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma_{k+1}^2$ is a central chi–square random variable when $\rho_1 = r$, we have

$$\mathbf{A}(\mathbf{I} + r\mathbf{X}_1\mathbf{X}_1' + \sum_{i=2}^{k} \rho_i \mathbf{X}_i\mathbf{X}_i')\mathbf{A} = \mathbf{A} \quad \text{for all } \rho_i \geq 0$$

and

$$\mathbf{1}'\mathbf{A}\mathbf{1} = 0.$$

These mean that $\text{col}(\mathbf{A}) \subset \text{col}^{\perp}(\mathbf{1}, \mathbf{X}_2, \ldots, \mathbf{X}_k)$ where $\text{col}^{\perp}(\mathbf{B})$ denotes the orthogonal complement of the column space of $\mathbf{B}$. Since $\mathbf{C}'\mathbf{C}$ and $\mathbf{M}$ are orthogonal projection matrices on $\text{col}^{\perp}(\mathbf{1})$ and $\text{col}^{\perp}(\mathbf{CX}_2, \ldots, \mathbf{CX}_k)$, respectively, it can be concluded that $\mathbf{C}'\mathbf{C}\mathbf{A} = \mathbf{A}$ and $\mathbf{MCA} = \mathbf{CA}$. Hence

$$\begin{aligned} \mathbf{A} &= \mathbf{A}(\mathbf{I}_n + r\mathbf{X}_1\mathbf{X}_1')\mathbf{A} \\ &= \mathbf{A}\mathbf{C}'(\mathbf{I}_{n-1} + r\mathbf{C}\mathbf{X}_1\mathbf{X}_1'\mathbf{C}')\mathbf{C}\mathbf{A} \\ &= \mathbf{A}\mathbf{C}'(\mathbf{M} + r\mathbf{M}\mathbf{V}_1\mathbf{M})\mathbf{C}\mathbf{A} \\ &= \mathbf{A}\mathbf{C}'(\mathbf{Q}\tilde{\mathbf{D}}_1(r)\mathbf{Q}' + \mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}')\mathbf{C}\mathbf{A} \end{aligned} \tag{3.5}$$

The independence between $\mathbf{y}'\mathbf{A}\mathbf{y}$ and the error sum of squares of model (2.1), which is equal to $\mathbf{y}'\mathbf{C}'\mathbf{M}\mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{M}\mathbf{C}\mathbf{y}$, implies

$$\mathbf{A}\mathbf{C}'\mathbf{Q}\tilde{\mathbf{D}}_2\mathbf{Q}'\mathbf{C} = \mathbf{0},$$

and (3.5) is equivalent to

$$\mathbf{AC'Q\tilde{D}_1}(r)\mathbf{Q'CA} = \mathbf{A},$$

i.e., $\mathbf{C'Q\tilde{D}_1}(r)\mathbf{Q'C}$ is a generalized inverse of $\mathbf{A}$ and hence

$$\text{rank}(\mathbf{A}) \leq \text{rank}(\mathbf{C'Q\tilde{D}_1(r)Q'C}) = q.$$

Now assume that rank$(\mathbf{A}) = q$. From the decomposition of $\mathbf{R}$, it is clear col$(\mathbf{CA}) \subset$ col$(\mathbf{Q\tilde{D}_1(r)Q'})$. Note that $\mathbf{QD^*Q'}$ is the orthogonal projection matrix on col$(\mathbf{Q\tilde{D}_1}(r)\mathbf{Q'})$, where $\mathbf{D}^* = diag(\overbrace{1,\ldots,1}^{q},0,\ldots,0)\mathbf{Q'}$. Thus,

$$\begin{aligned}\mathbf{AC'} &= \mathbf{AC'QD^*Q'} \\ &= \mathbf{AC'Q\tilde{D}_1}(r)\mathbf{Q'Q\tilde{D}_1^-}(r)\mathbf{Q'} \\ &= \mathbf{AC'Q\tilde{D}_1}(r)\mathbf{Q'CC'Q\tilde{D}_1^-}(r)\mathbf{Q'}.\end{aligned}$$

This shows that

$$\mathbf{A} = \mathbf{AC'C} = \mathbf{AC'Q\tilde{D}_1}(r)\mathbf{Q'CC'Q\tilde{D}_1^-}(r)\mathbf{Q'C}$$

and

$$\begin{aligned}\mathbf{0} &= \mathbf{AC'Q\tilde{D}_1}(r)\mathbf{Q'C}\left(\mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C} - \mathbf{A}\right) \\ &= \left(\mathbf{C'Q\tilde{D}_1}(r)\mathbf{Q'CA}\right)'\left(\mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C} - \mathbf{A}\right). \qquad (3.6)\end{aligned}$$

Since col$(\mathbf{A}) \subset$ col$(\mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C})$ and rank$(\mathbf{A}) =$ rank$(\mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C})$ by assumption, we can conclude that col$(\mathbf{A}) =$ col$(\mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C})$. Hence the column space of the left-hand matrix in (3.6) is a subset of the column space of right-hand matrices. Therefore it must be the case that

$$\mathbf{A} = \mathbf{C'Q\tilde{D}_1^-}(r)\mathbf{Q'C} = \mathbf{C'MQ\tilde{D}_1^-}(r)\mathbf{MQ'C}.$$

The uniqueness of the test statistic follows by proposition 3.3 of Seely and El-Bassiouni (1983).

4. CONCLUSION AND EXAMPLES

The main object of this work is to develop an optimal test for $H : \rho_i \le r$ against $K : \rho_i > r$. Because the test described in section 3 is based on the locally minimum variance quadratic unbiased estimator, it can be expected that the test would yield a certain local optimal property in the class of tests based on quadratic forms of observations. However, for completeness, we should evaluate the power performance of the test. For this purpose, we need some referential tests to which the test can be compared. Unfortunately, it is hard to find such a referential test except the case that we wish to test $H : \rho_i = 0$ against $K : \rho_i > 0$. In this case, the test is identical to the Wald test which is described in Seely and El–Bassiouni (1983) and we can refer to Lin and Harville (1991) for the power performance, since the theoretical power of our test behaves similarly to that of the Wald test.

In this section, we provide an explicit form of the test statistic in the one–way model and numerical examples in the two–way model with some computational remarks.

Example 1. Consider the following one-way random model:

$$y_{ij} = \mu + \xi_i + \epsilon_{ij}$$

where $j = 1, \cdots, n_i; i = 1, \cdots, b$. Here μ is a constant, and ξ_i and ϵ_{ij} are independent normal random variables with a common mean 0, and variances σ_1^2 and σ_2^2, respectively. In matrix notation, the model can be written as

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{X}\xi + \epsilon$$

where $\mathbf{1}$ is a vector of 1's, $\mathbf{X}$ is an $n \times b$ design matrix with $n = \sum_{i=1}^{b} n_i$, and ξ and ϵ are multivariate normal random variables with a common mean vector $\mathbf{0}$ and variance-covariance matrices $\sigma_1^2\mathbf{I}$ and $\sigma_2^2\mathbf{I}$, respectively.

Let $r\mathbf{XX}' = \mathbf{QEQ}'$, where $\mathbf{E} = diag(rn_1, \ldots, rn_b, 0 \ldots, 0)$ is a diagonal matrix of eigenvalues of $r\mathbf{XX}'$ and $\mathbf{Q}$ is the matrix of corresponding eigenvectors. Then,

$$\begin{aligned}
(\mathbf{I} + r\mathbf{CXX}'\mathbf{C}')^{-1} &= (\mathbf{I} + \mathbf{CQEQ}'\mathbf{C}')^{-1} \\
&= \mathbf{I} - \mathbf{CQE}^{\frac{1}{2}} \left(\mathbf{I} + \mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{C}'\mathbf{CQE}^{\frac{1}{2}}\right)^{-1} \mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{C}' \\
&= \mathbf{I} - \mathbf{CQE}^{\frac{1}{2}} \left(\mathbf{I} + \mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\left(\mathbf{I} - \frac{1}{n}\mathbf{11}'\right)\mathbf{QE}^{\frac{1}{2}}\right)^{-1} \mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{C}' \\
&= \mathbf{I} - \mathbf{CQE}^{\frac{1}{2}} \left(\mathbf{I} + \mathbf{E} - \frac{1}{n}\mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{11}'\mathbf{QE}^{\frac{1}{2}}\right)^{-1} \mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{C}'.
\end{aligned}$$

Noting

$$\left(\mathbf{I}+\mathbf{E}-\frac{1}{n}\mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{1}\mathbf{1}'\mathbf{Q}\mathbf{E}^{\frac{1}{2}}\right)^{-1} = (\mathbf{I}+\mathbf{E})^{-1} + \frac{(\mathbf{I}+\mathbf{E})^{-1}E^{\frac{1}{2}}\mathbf{Q}'\mathbf{1}\mathbf{1}'\mathbf{Q}\mathbf{E}^{\frac{1}{2}}(\mathbf{I}+\mathbf{E})^{-1}}{n-\mathbf{1}'\mathbf{Q}\mathbf{E}^{\frac{1}{2}}(\mathbf{I}+\mathbf{E})^{-1}\mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{1}}$$

and

$$n-\mathbf{1}'\mathbf{Q}\mathbf{E}^{\frac{1}{2}}(\mathbf{I}+\mathbf{E})^{-1}\mathbf{E}^{\frac{1}{2}}\mathbf{Q}'\mathbf{1} = \sum_{i=1}^{b} n_i/(1+rn_i),$$

the following result can be extracted after some algebra:

$$\mathbf{z}'\mathbf{R}\mathbf{z} = \sum_{i=1}^{b}\sum_{j=1}^{n_i}(y_{ij}-\overline{y}_{i.})^2 + \sum_{i=1}^{b}\frac{n_i(\overline{y}_{i.}-\overline{y}_{..})^2}{1+rn_i} - \frac{\left(\sum_{i=1}^{b}\frac{rn_i^2(\overline{y}_{i.}-\overline{y}_{..})}{1+rn_i}\right)^2}{\sum_{i=1}^{b}\frac{n_i}{1+rn_i}} \tag{4.1}$$

where $\overline{y}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$ and $\overline{y}_{..} = \frac{1}{n}\sum_{i=1}^{b}\sum_{j=1}^{n_i} y_{ij}$. It can be shown that the first term in the right-hand side of (4.1) is equal to $\mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}$. Thus, the test statistic can be written as

$$F = \frac{n-b}{b-1}\,\frac{\sum_{i=1}^{b}\frac{n_i(\overline{y}_{i.}-\overline{y}_{..})^2}{1+rn_i} - \frac{\left(\sum_{i=1}^{b}\frac{rn_i^2(\overline{y}_{i.}-\overline{y}_{..})}{1+rn_i}\right)^2}{\sum_{i=1}^{b}\frac{n_i}{1+rn_i}}}{\sum_{i=1}^{b}\sum_{j=1}^{n_i}(y_{ij}-\overline{y}_{i.})^2}. \tag{4.2}$$

This test statistic was derived by Spjøtvoll (1967). He considered the hypothesis $H:\rho\le r$ against $K:\rho=r_1$ where $r_1>r$. For this testing problem, he derived test statistic $w(r,r_1)$ which depends on r_1. His test is known to be most powerful similar invariant test. $w(r,r_1)$ has the property of maximizing minimum power over the set of alternatives with $\rho\ge r_1$. For testing $K:\rho\le r$ against $H:\rho>r$, he used $w(r,\infty)$ which is equal to (4.2).

Example 2. Suppose that we wish to test $H:\rho_i=r_i$ for all $i=1,\ldots,k$ against $K:\rho_i\ne r_i$ for some i. As we noted earlier, if some of r_i's are specified to be zeros, $m-g$ terms of 1's should appear in (2.6). In this case, $\mathbf{R}$ in (2.5) essentially should be decomposed into three parts. Without loss of generality we assume that first s r_i's are nonzeros and the others are zeros. The decomposition can be obtained by the following steps:

1. Let $\mathbf{P}_1=(\mathbf{p}_1,\ldots,\mathbf{p}_g)$ where $\mathbf{p}_i, i=1,\ldots,g$, is the orthonormal eigenvector corresponding to the nonzero eigenvalue $\lambda_i(\mathbf{r})$ of $\sum_{i=1}^{s} r_i\mathbf{V}_i$.

2. Let $\mathbf{M}_{\mathbf{P}_1} = \mathbf{I} - \mathbf{P}_1\mathbf{P}_1'$ and $\mathbf{P}_2$ be a matrix of the eigenvectors corresponding to nonzero eigenvalues of $\mathbf{M}_{\mathbf{P}1}(\sum_{i=s+1}^{k} \mathbf{V}_i)\mathbf{M}_{\mathbf{P}1}$.
3. Let $\mathbf{P}_{12} = (\mathbf{P}_1, \mathbf{P}_2)$ and $\mathbf{M}_{\mathbf{P}12} = \mathbf{I} - \mathbf{P}_{12}\mathbf{P}_{12}'$.
4. Let $\mathbf{P}_3$ be a matrix of the eigenvectors corresponding to nonzero eigenvalues of $\mathbf{M}_{\mathbf{P}12}(\sum_{i=1}^{k} \mathbf{V}_i)\mathbf{M}_{\mathbf{P}12}$.
5. Let $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ and arrange $\lambda_i(\mathbf{r})$'s, 1's and 0's to form $\mathbf{D}_1^-(\mathbf{r})$ and $\mathbf{D}_2$ in (2.6).
6. Compute the quadratic forms, $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}$ and $\mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}$.

However, above-mentioned steps have only theoretical interest. In practice, it does not need to follow the steps because $\mathbf{z}'\mathbf{P}\mathbf{D}_2\mathbf{P}'\mathbf{z}$ is the error sum of squares and it is easy to compute in most cases. $\mathbf{z}'\mathbf{P}\mathbf{D}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z}$ can be obtained easily by subtracting the error sum of squares from the $\mathbf{z}'\mathbf{R}\mathbf{z}$ in (2.7).

Table 1. A simulated two-way model data with $\rho_1 = \rho_2 = \sigma_4^2 = 1$ and $\rho_3 = 0$.

	Main factor B				
Main Factor A	1	2	3	4	5
1	5.51977	4.08265	3.03083	6.61678	7.08004
	4.87296	5.49475	6.26167		
	4.68780	6.98880	4.60492		
		7.77430			
2	7.63796	5.19060	4.02078	7.85110	7.38554
	5.83436	7.77191	4.40675		6.70512
	6.09709	6.82135			
	5.78771	6.73037			
3	5.84260	5.17246	3.16696	6.96194	5.73017
	6.56457	5.36890	3.93901	8.88334	4.95029
	5.60145	5.37305	3.84190	5.67665	
	6.19103		2.75668		
4	7.91403	6.38014	3.63475	7.53400	5.51005
	7.19771	5.30325	4.63748		
	5.55758		2.08913		
	4.95710		5.94035		

Using this strategy, we compute the test statistic for testing $H : \rho_1 = \rho_2 = 1$ and $\rho_3 = 0$ where ρ_1, ρ_2 and ρ_3 are the ratios of the main variance components and the interaction variance component to the error variance component, respectively, in a two–way model. A simulated data, shown in Table 1, is generated under the null hypothesis with $\sigma_4^2 = 1$.

For the data, $\mathbf{z}'\mathbf{PD}_1^-(\mathbf{r})\mathbf{P}'\mathbf{z} = 14.23614, \mathbf{z}'\mathbf{PD}_2\mathbf{P}'\mathbf{z} = 40.91588, m = 19$ and $n - 1 - m = 33$, we get $F = 0.60431$. Since $F_{0.025,19,33} < F < F_{0.975,19,33}$, the null hypothesis could not be rejected at the 5% significance level. Also for the hypotheses, $H : \rho_3 = 0$ and $H : \rho_3 \leq 0.5$, $F = 0.58996$ and $F = 0.26714$ are computed. Both of the null hypotheses could not be rejected at the 5% significance level.

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