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Smoothed Perturbation Analysis for Performance Measures in a Markov Renewal Process

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Abstract

In this paper, we derive unbiased estimators for the sensitivities of expected performance measures in a Markov renewal process. We restrict our derivation to the performance measures during a busy cycle and apply smoothed perturbation analysis method to find those estimators. The results show all the terms in the derived estimators can be obtained from a single sample path.

Key Words : Smoothed perturbation analysis; Discrete event system; Simulation.

1. INTRODUCTION

To motivate the derivation of Smoothed Perturbation Analysis(SPA) estimates, we consider an $M/M/1$ queue with interarrival time distribution $F(x,\theta) = 1 - e^{-\frac{1}{\theta}x}$ and service time distribution $G(y,\mu) = 1 - e^{-\frac{1}{\mu}y}$. If $|D|$ is the number of departures during a busy cycle,

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$$\frac{dE\left[|D|\right]}{d\theta} = \frac{-\mu}{(\theta-\mu)^2} \neq 0 = E\left[\frac{d|D|}{d\theta}\right]$$

that is, infinitesimal perturbation analysis(IPA) does not work in general. It is now well known that the discontinuity of performance measures is the reason for this; for the above example, $|D|$ is discontinuous with respect to θ. To solve this problem, the method of smoothed perturbation analysis(SPA) was introduced. The basic idea behind SPA is to smooth the discontinuities through the use of conditional expectation. Zazanis and Suri(1986) first applied conditioning ideas in estimating the second derivative of the mean system time in a $GI/G/1$ queue. Gong and Ho(1987) then gave birth to SPA. Significant practical and theoretical advances in SPA were made in Glasserman and Gong(1990) and Fu and Hu(1992). Glasserman and Gong derived SPA estimators for performance measures in a Generalized Semi-Markov Process that are noninterruptive and satisfy the commuting condition. Fu and Hu extended this to more general performance measures and to a more general discrete-event system, ie., to a system which doesn't necessarily satisfy the commuting condition. In this article, we derive SPA estimators for performance measures in a Markov renewal process, which is interruptive as we see in the following.

2. PERTURBATION ANALYSIS

Throughout this article, we will consider a Markov renewal process defined by two continuous distribution functions $F(x,\theta)$ and $G(y,\mu)$ as follows.

At each state, except state 0, X and Y are generated according to $F(x,\theta)$ and $G(y,\mu)$ respectively. If $\min\{X,Y\} = X$, X becomes a sojourn time of the state, and the process jumps up one step. If $\min\{X,Y\} = Y$, Y becomes a sojourn time of the state and the process jumps down one step. If $X = Y$, we arbitrary choose X as a sojourn time, in which case the corresponding probability measure is zero. At state 0, X always becomes a sojourn time after which the process makes a transition to state 1. Thus, the embedded Markov chain is a random walk on the nonnegative integers with a reflecting barrier at zero. A typical sample path of the above process is given in [6].

2.1 Notations

We define some notations. $C_{0,0}(\theta)$ denotes the length of a busy cycle i.e. the recurrence time of state 0, $C_{i,0}(\theta)$ denotes the time for the first transition from state i to state 0.

$R_{i,0}(\theta)$ represents the area under the graph of the process from state i to state 0. We note that $R_{0,0}(\theta)$ is the sum of system times during a busy cycle for the above mentioned Markov renewal process. $W_{i,0}(\theta, s)$ represents the time spent in the state s during $C_{i,0}(\theta)$.

Let U denote the set of indices at which the process jumps up during $C_{0,0}(\theta)$, D be the set of indices at which the process jumps down during the same busy cycle $C_{0,0}(\theta)$. Let $|D|$ be the cardinal number of D, i.e. the number of customers served during a busy cycle $C_{0,0}(\theta)$.

Let $N_{i,0}(\theta)$ be the number of customers served during $C_{i,0}(\theta)$. We note that $N_{0,0}(\theta)$ is equal to $|D|$.

Let U_i be the set of indices which is greater than i and at which the process jumps up during $C_{0,0}(\theta)$, D_i be the set of indices which is greater than i and at which the process jumps down during $C_{0,0}(\theta)$.

Let $U(s)$ be the set of indices at which the state of the process is s and the process jumps up during $C_{0,0}(\theta)$.

Suppose that when the parameter θ is decreased to $\theta - \Delta\theta$, X is correspondingly decreased to $X - \Delta X$. In this case, we let A_i be the event of a corresponding change from jump-down to jump-up at the end of sojourn time Y_i and B_i be the event that any change from jump-down to jump-up does not occur, i.e.,

$$
\begin{aligned}
A_i &= \{Y_i < X_i, Y_i \geq X_i - \Delta X_i\}, \\
B_i &= \{Y_i < X_i, Y_i < X_i - \Delta X_i\}.
\end{aligned}
$$

IC denotes the set of indices at which interchanges occur when there are two or more interchanges in a sample path during $C_{0,0}(\theta)$.

For convenience, we introduce the notation W, as follows

$$
W_j = \begin{cases} A_j, & \text{if} \quad j \in IC \\ B_j, & \text{if} \quad j \notin IC \end{cases}
$$

$n(i)$ denotes the state of process at the i-th transition. Finally, let a_i denote the i-th jump in a sample path. Since our Markov renewal process begins at state 0, a_1 always means jump-up. For $i > 1$, a_i means jump-up or jump-down depending on $\{Y_i < X_i\}$ or $\{X_i \quad Y_i\}$ respectively.

2.2 SPA Estimate for the Mean Number of Departures during a Busy Cycle

In this section, using appropriate conditional expectations, we derive unbiased derivative estimators for performance measures under the above defined Markov renewal process. First, we need the following assumptions.

A1 $P[X < Y] < 1/2$ for all θ in a given open interval.
A2 $E[X] < \infty$, $E[Y] < \infty$.
A3 θ is a scale parameter of $F(x, \theta)$.
A4 $\frac{dF_X(x)}{dx} = f_X(x)$ exists and $|f_X(x)| \le M$ for all $x \ge 0$.

Under these assumptions, we will find an estimator for the derivative of the expected total number of deaprtures during a busy cycle with respect to a scale parameter θ . We will assume that this derivative exists and the detailed derivation will be done for the left-hand derivative estimator.

Theorem 1. Under the assumptions A1 through A4,

$$\frac{dE[|D|]}{d\theta^-} = \frac{E[f_X(Y)Y]}{\theta P(Y_i < X_i)} \; E[|D|] \;\; E[1 - N_{2,0}(\theta)]$$

Proof. Since θ is a scale parameter, $X(\theta)$ is an increasing function of θ. If θ is changed to $\theta - \Delta\theta$, X decreases by the amount of ΔX, but Y does not change. This decrease can affect the sample path in the following two ways. The first effect is simply a reduction in X. In this case, $N_{0,0}(\theta)$ does not change. The second, more serious, effect happens by interchanging; jumping up instead of jumping down. If only one interchange happens during a busy cycle, we can observe from the sample path that $N_{0,0}(\theta) - N_{0,0}(\theta - \Delta\theta)$ becomes $1 - N_{2,0}(\theta - \Delta\theta)$. From these argument, we have

$$\begin{aligned} N_{0,0}(\theta) - N_{0,0}(\theta - \Delta\theta) &= \sum_{i \in D} I(\text{only one interchange at } Y_i) \{1 - N_{2,0}(\theta - \Delta\theta)\} \\ &+ \; I(\text{two or more interchanges}) \{N_{0,0}(\theta) - N_{0,0}(\theta - \Delta\theta)\} \end{aligned}$$

Using notations in section 1, we have the following expectation from the above expression.

$$\frac{E[N_{0,0}(\theta) - N_{0,0}(\theta - \Delta\theta)]}{\Delta\theta} = \frac{1}{\Delta\theta} E\left[\sum_{i \in D} \prod_{j \in D - i} I(B_j) \{1 - N_{2,0}(\theta - \Delta\theta)\}\right]$$

$$+\frac{1}{\Delta\theta}E\left[\sum_{|IC|>1}\left\{\prod_{i\in D}I(W_i)\right\}\{N_{0,0}(\theta)-N_{0,0}(\theta-\Delta\theta)\}\right] \qquad (2.1)$$

When $\Delta\theta$ goes to zero, we compute the limit value of the first term in the right hand side of (2.1) and show the limit value of the second term is equal to zero. To do this, let us consider events of the form $\{a_1, a_2, \ldots, a_k\}$ where k denotes the total number of jumps during the first busy cycle. If k is given, by selecting all the possible choices of inequality signs, we have finite events of this form. Since we have finite events for each $k > 1$, total number of events of this form is countable. Let H be the smallest -algebra generated by all these countable events of this form.

Now, conditioning on H and noting that $I(A_i), I(B_j)$, and $N_{2,0}(\theta - \Delta\theta)$ are conditionally independent, we compute the limit value of the first term in the right hand side of (2.1)

$$\begin{aligned}
&E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}\{1-N_{2,0}(\theta-\Delta\theta\}\right]\\
&=E\left[E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}\{1-N_{2,0}(\theta-\Delta\theta)\}\,|H\right]\right]\\
&=E\left[\sum_{i\in D}E\left[I(A_i)|H\right]E\left[\prod_{j\in D-i}I(B_j)|H\right]E\left[1-N_{2,0}(\theta-\Delta\theta)\right]\right] \qquad (2.2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E\left[I(A_i)|H\right] &= E\left[I(Y_i<X_i, Y_i\geq X_i-\Delta X_i)\mid Y_i<X_i\right]\\
&= E\left[I\left(Y_i<X_i<\frac{Y_i}{1-\frac{\Delta\theta}{\theta}}\right)\mid Y_i<X_i\right]\\
&= \frac{E\left[F_X\left(\frac{Y_i}{1-\frac{\Delta\theta}{\theta}}\right)-F_X(Y_i)\right]}{P(Y_i<X_i)}\\
&= \frac{E\left[f_X(\xi)Y\frac{\Delta\theta}{\theta-\Delta\theta}\right]}{P(Y_i<X_i)}, \quad Y<\xi<\frac{Y}{1-\frac{\Delta\theta}{\theta}} \qquad (2.3)
\end{aligned}$$

Since $E\left[I(A_i)|H\right]$ is a constant for all $i \in D$, the expression (2.2) becomes

$$E\left[I(A_i)|H\right] E\left[1 - N_{2,0}(\theta - \Delta\theta)\right]\ E\left[E\left[\sum_{i\in D}\prod_{j\in D-i} I(B_j)|H\right]\right]$$
$$= \frac{\Delta\theta}{\theta - \Delta\theta}\frac{E\left[f_X(\xi)Y\right]}{P(Y_i < X_i)}\ E\left[1 - N_{2,0}(\theta - \Delta\theta)\right]\ E\left[\sum_{i\in D}\prod_{j\in D-i} I(B_j)\right] \quad (2.4)$$

Hence,

$$\lim_{\Delta\theta\to 0^+}\frac{1}{\Delta\theta}E\left[\sum_{i\in D} I(A_i)\left\{\prod_{j\in D-i} I(B_j)\right\}\{1 - N_{2,0}(\theta - \Delta\theta)\}\right]$$
$$= \lim_{\Delta\theta\to 0^+}\frac{E\left[f_X(\xi)Y\right]}{\theta P(Y_i < X_i)}E\left[1 - N_{2,0}(\theta - \Delta\theta)\right]E\left[\sum_{i\in D}\prod_{j\in D-i} I(B_j)\right]$$

Since $|f_X(\xi)Y| \le MY$, $\sum_{i\in D}\prod_{j\in D-i} I(B_j) \le |D|$, $E[Y] < \infty$, and $E[|D|] < \infty$, by the Lebesgue convergence theorem, the above expression becomes

$$\frac{E\left[\lim_{\Delta\theta\to 0^+} f_X(\xi)Y\right]}{\theta P(Y_i < X_i)}E\left[\lim_{\Delta\theta\to 0^+}\{1 - N_{2,0}(\theta - \Delta\theta)\}\right]\cdot E\left[\lim_{\Delta\theta\to 0^+}\sum_{i\in D}\prod_{j\in D-i} I(B_j)\right]$$
$$= \frac{E\left[f_X(Y)Y\right]}{\theta P(Y_i < X_i)}E\left[|D|\right]E\left[1 - N_{2,0}(\theta)\right] \quad (2.5)$$

On the other hand,

$$E\left[\sum_{|IC|>1}\left\{\prod_{i\in D} I(W_i)N_{0,0}(\theta - \Delta\theta) - N_{0,0}(\theta)\right\}\right]$$
$$\le E\left[\sum_{i\in D} I(A_i)\left\{\prod_{j\in D-i} I(W_j)\right\}N_{2|D|,0}(\theta - \Delta\theta)\right]$$
$$= E\left[E\left[\sum_{i\in D} I(A_i)\left\{\prod_{j\in D-i} I(W_j)\right\}N_{2|D|,0}(\theta - \Delta\theta)\mid H\right]\right]$$
$$= E\left[I(A_i)\mid H\right]\ E\left[N_{2,0}(\theta - \Delta\theta)\right]\ E\left[\sum_{i\in D}|D|\prod_{j\in D-i} I(W_j)\right]$$
$$= \frac{\Delta\theta}{\theta - \Delta\theta}\frac{E\left[f_X(\xi)Y\right]}{P(Y_i < X_i)}\ E\left[\sum_{i\in D}|D|\prod_{j\in D-i} I(W_j)\right]\ E\left[N_{2,0}(\theta - \Delta\theta)\right] \quad (2.6)$$

Hence

$$\lim_{\Delta\theta\to 0^+}\frac{1}{\Delta\theta}E\left[\sum_{|IC|>1}\left\{\prod_{i\in D}I(W_i)\right\}\{N_{0,0}(\theta-\Delta\theta)-N_{0,0}(\theta)\}\right]$$
$$\leq \lim_{\Delta\theta\to 0^+}\frac{E\left[f_X(\xi)Y\right]}{\theta P(Y_i<X_i)}\,E\left[\sum_{i\in D}|D|\prod_{j\in D-i}I(W_j)\right]\,E\left[N_{2,0}(\theta-\Delta\theta)\right]$$

Again by Lebesgue convergence theorem, the above expression becomes

$$\frac{E\left[\lim_{\Delta\theta\to 0^+}f_X(\xi)Y\right]}{\theta P(Y_i<X_i)}E\left[\lim_{\Delta\theta\to 0^+}\sum_{i\in D}|D|\prod_{j\in D-i}I(W_j)\right]E\left[\lim_{\Delta\theta\to 0^+}N_{2,0}(\theta-\Delta\theta)\right]=0 \tag{2.7}$$

From (2.1), (2.5) and (2.7), theorem 1 is proved.

2.3 SPA Estimate for other Performance Measures

In this section, using expression (2.3), we find SPA estimators for some more performance measures in a Markov renewal process. Since the proof is similar to the one in section 2.2, we omit the details.

Theorem 2. Under the additional assumption $E\left[|D|^2\right]<\infty$ to theorem 1,

$$\frac{dE\left[C_{0,0}(\theta)\right]}{d\theta^-}=\frac{1}{\theta}E\left[\sum_{j\in U}X_j\right]-\frac{E\left[f_X(Y)Y\right]}{\theta P(Y<X)}E\left[|D|\right]E\left[C_{2,0}(\theta)\right]$$

Proof. By obserbing the sample paths $C_{0,0}(\theta)$ and $C_{0,0}(\theta-\Delta\theta)$, we have

$$\begin{aligned}C_{0,0}(\theta)-C_{0,0}(\theta-\Delta\theta)&=I(\text{no interchange})\sum_{j\in U}\Delta X_j\\&\quad+\sum_{i\in D}I(\text{only one interchange at }Y_i)\\&\qquad[\sum_{j\in U}\Delta X_j+\{Y_i-(X_i-\Delta X_i)\}-C_{2,0}(\theta-\Delta\theta)]\\&\quad+I(\text{two or more interchanges})\{C_{0,0}(\theta)-C_{0,0}(\theta-\Delta\theta)\}\end{aligned}$$

Taking out the common factor $\sum_{j\in U}\Delta X_j$ and using notations in section 2.1, the above expression becomes

$$\sum_{j\in U}\Delta X_j+\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}[\{Y_i-(X_i-\Delta X_i)\}-C_{2,0}(\theta-\Delta\theta)]$$
$$+\sum_{|IC|>1}\left\{\prod_{i\in D}I(W_i)\right\}\left[C_{0,0}(\theta)-C_{0,0}(\theta-\Delta\theta)-\sum_{j\in U}\Delta X_j\right]$$

From this, we have

$$\begin{aligned}
&\frac{dE\left[C_{0,0}(\theta)\right]}{d\theta^-}\\
&=\lim_{\Delta\theta\to 0}\frac{E\left[C_{0,0}(\theta)-C_{0,0}(\theta-\Delta\theta)\right]}{\Delta\theta}\\
&=\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{j\in U}\frac{X_j}{\theta}\Delta\theta\right]\\
&\quad+\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}\{Y_i-(X_i-\Delta X_i)\}\right]\\
&\quad-\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}C_{2,0}(\theta-\Delta\theta)\right]\\
&\quad+\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{|CI|>1}\left\{\prod_{i\in D}I(W_i)\right\}\left[C_{0,0}(\theta)-C_{0,0}(\theta-\Delta\theta)-\sum_{j\in U}\Delta X_j\right]\right]
\end{aligned}\tag{2.8}$$

Using (2.3) and taking similar steps as in section 2.2, we can show that the second and fourth term of (2.8) are equal to zero, and the third term is equal to

$$\frac{E\left[f_X(Y)Y\right]}{\theta P(Y<X)}E\left[|D|\right]\ E\left[C_{2,0}(\theta)\right]$$

In this way, theorem 2 can be proved.

Theorem 3. Under the additional assumption $E\left[|D|^3\right]<\infty$ to theorem 1,

$$\begin{aligned}
&\frac{dE\left[W_{0,0}(\theta,s)\right]}{d\theta^-}\\
&=\frac{1}{\theta}E\left[\sum_{i\in U(s)}X_i\right]-\frac{E\left[f_X(Y)Y\right]}{\theta P(Y<X)}E\left[\sum_{i\in D}\left\{W_{n(i)+1,0}(\theta,s)-W_{n(i)-1}(\theta,s)\right\}\right]
\end{aligned}$$

and

$$\frac{dE\left[R_{0,0}(\theta)\right]}{d\theta^-}$$
$$= \frac{1}{\theta}E\left[\sum_{i\in U} n(i)X_i\right] - \frac{E\left[f_X(Y)Y\right]}{\theta P(Y < X)}\left\{E[\sum_{i\in D} 2C_{n(i)-1,0}(\theta)] + E\left[|D|\right]E\left[R_{2,0}(\theta)\right]\right\}$$

Proof. By obserbing areas under the sample paths $C_{0,0}(\theta)$ and $C_{0,0}(\theta - \Delta\theta)$, we have

$$\begin{aligned}
R_{0,0}(\theta) - R_{0,0}(\theta - \Delta\theta) &= I(\text{no interchange})\sum_{j\in U} n(j)X_j \\
&+ \sum_{i\in D} I(\text{only one interchange at } Y_i)\left[\sum_{j\in U} n(j)\Delta X_j - \sum_{t\in D_i} 2Y_t\right. \\
&\left. - \sum_{t\in U_i} 2(X_t - \Delta X_t) + n(i)\{Y_i - (X_i - \Delta X_i)\} - R_{2,0}(\theta - \Delta\theta)\right] \\
&+ I(\text{two or more interchanges})\{R_{0,0}(\theta) - R_{0,0}(\theta - \Delta\theta)\}
\end{aligned}$$

Taking out the common factor $\sum_{j\in U} n(j)\Delta X_j$ and using notations in section 2.1, the above expression becomes

$$\begin{aligned}
&\sum_{j\in U} n(j)\Delta X_j + \sum_{i\in D} I(A_i)\left\{\prod_{j\in D-i} I(B_j)\right\}\left[n(i)\left\{Y_i - (X_i - \Delta X_i)\right\}\right] \\
&+ \sum_{i\in D} I(A_i)\left\{\prod_{j\in D-i} I(B_j)\right\}\left[-\sum_{t\in D_i} 2Y_t - \sum_{t\in U_i} 2(X_t - \Delta X_t) - R_{2,0}(\theta - \Delta\theta)\right] \\
&+ \sum_{|IC|>1}\left\{\prod_{i\in D} I(W_i)\right\}\left[R_{0,0}(\theta) - R_{0,0}(\theta - \Delta\theta) - \sum_{j\in U} n(j)\Delta X_j\right]
\end{aligned}$$

From this,

$$\begin{aligned}
\frac{dE\left[R_{0,0}(\theta)\right]}{d\theta^-} &= \lim_{\Delta\theta\to 0}\frac{E\left[R_{0,0}(\theta) - R_{0,0}(\theta - \Delta\theta)\right]}{\Delta\theta} \\
&= \lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{j\in U} n(j)\frac{X_j}{\theta}\Delta\theta\right]
\end{aligned}$$

$$+\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}n(i)\left\{Y_i - X_i\left(1-\frac{\Delta\theta}{\theta}\right)\right\}\right]$$
$$-\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{i\in D}I(A_i)\left\{\prod_{j\in D-i}I(B_j)\right\}\right.$$
$$\left.\left[\sum_{t\in D_i}2Y_t + \sum_{t\in U_i}2X_t(1-\frac{\Delta\theta}{\theta}) + R_{2,0}(\theta-\Delta\theta)\right]\right]$$
$$+\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}E\left[\sum_{|CI|>1}\left\{\prod_{i\in D}I(W_i)\right\}\left[R_{0,0,}(\theta) - R_{0,0}(\theta-\Delta\theta) - \sum_{j\in U}n(j)\Delta X_i\right]\right] \tag{2.9}$$

Using (2.3) and taking similar steps as in section 2.2, we can show that the second and fourth term of (2.9) are equal to zero, and the third term is equal to

$$\frac{E\left[f_X(Y)Y\right]}{\theta P(Y<X)}\left\{E\left[\sum_{i\in D}2C_{n(i)-1,0}(\theta)\right] + E\left[|D|\right]E\left[R_{2,0}(\theta)\right]\right\}$$

In this way, the second part of theorem 3 can be proved. Since the proof for $W_{0,0}(\theta, s)$ follows almost the same steps as in the proof of $R_{0,0}(\theta)$, we omit the proof.

2.4 Verification for M/M/1 Queue

In this section, by taking $M/M/1$ queue as an example, we show the above theorems really work. First, we verify theorem 1. Let $F(x,\theta) = 1 - e^{-\frac{1}{\theta}x}$ and $G(y,\mu) = 1 - e^{-\frac{1}{\mu}y}$. Assume $\mu < \theta$. Trivially, this example satisfies all the assumptions in theorem 1.

$$E\left[|D|\right] = \frac{\theta}{\theta-\mu}$$
$$\frac{dE\left[|D|\right]}{d\theta} = -\frac{\mu}{(\theta-\mu)^2} \tag{2.10}$$

On the other hand,

$$E\left[f_X(Y)Y\right] = \frac{\theta\mu}{(\theta+\mu)^2},\quad P(Y<X) = \frac{\theta}{\theta+\mu}.$$
$$E\left[N_{2,0}(\theta)\right] = 2E\left[N_{1,0}(\theta)\right] = 2E\left[|D|\right]$$

Hence,

$$\frac{E\left[f_X(Y)Y\right]}{\theta P(Y_i < X_i)} E\left[|D|\right] E\left[1 - N_{2,0}(\theta)\right] = -\frac{\mu}{(\theta-\mu)^2} \tag{2.11}$$

From (2.10) and (2.11), the result follows.

Noting that $\frac{E[f_X(Y)Y]}{\theta P(Y_i < X_i)}$ equal to $\frac{\mu}{\theta(\theta+\mu)}$ for $M/M/1$ queue, we can find the verification for theorem 2 in section 2.4 of [5].

3. CONCLUSION

In this paper, we extended the estimation results for the derivative of expected performance measures in the $M/M/1$ queue to the results in a Markov renewal process. We also derived an estimator for the derivative, with respect to a scale parameter θ, of the expected number of customers served during a busy cycle. As we see, all the terms of the results can be estimated from a single sample path.

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