Journal of the Korean
Statistical Society
Vol. 25, No. 3, 1996

On Weak Convergence of Some Rescaled Transition Probabilities of a Higher Order Stationary Markov Chain

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Abstract

In this paper we consider weak convergence of some rescaled transition probabilities of a real-valued, k-th order ($k \geq 1$) stationary Markov chain. Under the assumption that the joint distribution of $k+1$ consecutive variables belongs to the domain of attraction of a multivariate extreme value distribution, the paper gives a sufficient condition for the weak convergence and characterizes the limiting distribution via the multivariate extreme value distribution.

Key Words : Weak convergence; Transition probability; Higher order stationary Markov chain; Domain of attraction; Multivariate extreme value distribution.

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1. INTRODUCTION

Extreme value theory for Markov chains has a relatively short history, compared with the general theory for them. Although a basic development can be found in early papers like Berman (1962) and Anderson (1970), it was not until late 1980's that main contributions were made in the literature. Among others there are O'Brien (1987) and Rootzén (1988) who found similar characterizations for the extremal index of a stationary Markov chain. Extreme values of a stationary Markov chain tend to appear in clusters and the average size of each cluster is typically approximated by the reciprocal of the extremal index. The extremal index, together with the marginal distribution, also explains the tail distribution of a sample maximum of a stationary Markov chain. The extremal index is therefore a key parameter for studying the extremal behavior of a stationary Markov chain. For review on the extremal index, see Leadbetter, Lindgren and Rootzén (1983).

From a computational point of view, O'Brien's characterization for the extremal index is still intractable to apply naively. Smith (1992) used this to find a more practical method for computing the extremal index of a stationary Markov chain $\{X_n\}$. The principal assumption he used in his paper is that the joint distribution of (X_n, X_{n+1}) is in the domain of attraction of a bivariate extreme value distribution. If standard Gumbel marginals are assumed, a version of the domain of attraction assumption for densities is stated as: the limit

$$h(x) = \lim_{u \to \infty} f(u + x|u), \; x \in \Re, \tag{1.1}$$

exists, where $f(y|x)$ denotes the stationary transition density of the chain. In fact, Smith adopted this assumption. The $h(x)$ in (1.1) is not in general a proper density on $\Re$.

One would expect, from (1.1), that, as $u \to \infty$,

$$P(X_{n+1} \le u + x | X_n = u) \xrightarrow{w} H(x) \tag{1.2}$$

for some distribution function H on $\{-\infty\} \cup \Re$, where $\xrightarrow{w}$ denotes weak convergence. If $h(x)$ in (1.1) is a proper density, then (1.2) holds automatically, with $H(-\infty) = 0$; otherwise, there is no guarantee for (1.2). Instead of (1.1), Perfekt (1994) assumed (1.2) to derive an expression for the extremal index of $\{X_n\}$. He also extended the Gumbel marginals to more general marginals which of course belong to the domain of attraction of a univariate extreme value distribution. Perfekt (1993) again extended these results to multivariate stationary Markov chains.

On the other hand, Yun (1995) extended Smith's result to a k-th order ($k \geq 1$) stationary Markov chain $\{X_n\}$, assuming that the joint distribution of $k+1$ consecutive variables is in the domain of attraction of a multivariate extreme value distribution. Galambos (1987) and Resnick (1987) have a good review on the theory of multivariate extreme value distributions. Under standard Gumbel marginals, a version of this assumption for densities may be stated as: for each $j = 1, ..., k$, the limit

$$\begin{aligned} &h_j(x_{j+1} - x_j; x_2 - x_1, ..., x_j - x_{j-1}) \\ &\quad = \lim_{u \to \infty} f_{j+1}(u + x_{j+1} | u + x_1, ..., u + x_j), \; x_1, ..., x_{j+1} \in \Re, \end{aligned} \tag{1.3}$$

exists, where $f_{j+1}(x_{j+1}|x_1, ..., x_j)$ denotes the stationary conditional density of X_{n+j+1} given $(X_{n+1}, ..., X_{n+j}) = (x_1, ..., x_j)$. As before, this however does not imply in general that, as $u \to \infty$,

$$\begin{aligned} &P(X_{n+j+1} \leq u + x_{j+1} | (X_{n+1}, ..., X_{n+j}) = (u + x_1, ..., u + x_j)) \\ &\quad \xrightarrow{w} H_j(x_{j+1} - x_j; x_2 - x_1, ..., x_j - x_{j-1}) \end{aligned} \tag{1.4}$$

for some distribution function $H_j(\cdot\,; x_2 - x_1, ..., x_j - x_{j-1})$ on $\{-\infty\} \cup \Re$. Removing the restriction of standard Gumbel marginals, one may get more general forms of (1.3) and (1.4).

This paper gives a sufficient condition for convergence (1.4) under assumption (1.3). Instead of standard Gumbel marginals, we will work with more general marginals. The limiting distributions in (1.4) depend heavily on the structure of the considered multivariate extreme value distribution from which a possible atom of $H_j(\cdot\,; x_2 - x_1, ..., x_j - x_{j-1})$ at $-\infty$ is characterized. It is also shown that, if the joint distribution of $k+1$ consecutive variables in the chain is itself a multivariate extreme value distribution, convergences (1.3) and (1.4) hold automatically.

An interpretation of (1.4) is that, when the chain starts at a high level u (i.e., $X_1 = u$) tending to ∞, the increments $Y_1 = X_2 - X_1, Y_2 = X_3 - X_2, ...$ form asymptotically a $(k-1)$st order Markov chain which is completely determined by the limiting distributions $H_j(y_j; y_1, ..., y_{j-1})$, $j = 1, ..., k$. This implies that, for $k = 1$ in particular, the given chain in the tails looks like a random walk (see Smith (1992) also). This tail behavior is extremely helpful in computing the extremal index and is expected to be used effectively for characterizing the extremal properties of higher order stationary Markov chains. As a statistical application, the limiting distributions in (1.4) are also expected to play an important role in modeling the joint distribution of extreme values within a cluster of a high level u.

The rest of the paper is organized as follows. Section 2 discusses briefly the domain of attraction of a multivariate extreme value distribution. Section 3 contains main results. Section 4 contains three examples.

2. PRELIMINARIES

For $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \Re^p$ and for $\alpha, \beta \in \Re$, we write

$$\mathbf{ax} + \mathbf{b} = (a_1 x_1 + b_1, ..., a_p x_p + b_p),$$
$$\alpha \mathbf{x} + \beta = (\alpha x_1 + \beta, ..., \alpha x_p + \beta).$$

Let $F_p(\mathbf{x})$ be a p-dim. distribution function with equal univariate marginals $F_1(x)$. Assume that F_p belongs to the domain of attraction of a p-dim. extreme value distribution, i.e., there exist p-dim. vectors $\mathbf{a}^{(n)} > \mathbf{0}$ (with componentwise ordering) and $\mathbf{b}^{(n)} \in \Re^p$, $n = 1, 2, ...,$ such that

$$F_p^n(\mathbf{a}^{(n)}\mathbf{x} + \mathbf{b}^{(n)}) \xrightarrow{w} G_p(\mathbf{x}), \text{ as } n \to \infty,$$

for some p-dim. nondegenerate distribution function G_p, where $F_p^n(\mathbf{x})$ stands for $(F_p(\mathbf{x}))^n$. This assumption is usually written as $F_p \in \mathcal{D}(G_p)$. Taking marginals shows that F_1 belongs to the domain of attraction of a univariate extreme value distribution. This is equivalent to the condition that there exists a $\xi \in \Re$ such that

$$\lim_{u \uparrow x_{F_1}} \frac{1 - F_1(u + g(u)x)}{1 - F_1(u)} = (1 + \xi x)_+^{-1/\xi}, \quad x \in \Re, \tag{2.1}$$

where $x_{F_1} = \sup\{x \in \Re : F_1(x) < 1\}$, $x_+ = \max\{x, 0\}$, and

$$\begin{array}{lll} x_{F_1} = \infty \text{ and } g(u) = 1 + \xi u & \text{if} & \xi > 0; \\ g(u) \text{ is some strictly positive function} & \text{if} & \xi = 0; \\ x_{F_1} < \infty \text{ and } g(u) = -\xi(x_{F_1} - u) & \text{if} & \xi < 0. \end{array} \tag{2.2}$$

Condition (2.1) is, in fact, a reformulation of Theorem 1.6.2 of Leadbetter et al. (1983). For the left endpoint of F_1, we use the symbol $x_{F_1}^*$, i.e., $x_{F_1}^* = \inf\{x \in \Re : F_1(x) > 0\}$. Considering that G_p is unique up to vector normalizations, one may let G_p have equal univariate marginals G_1, say, and take $G_1 = \Omega_\xi$, where

$$\Omega_\xi(x) = \exp\left\{-(1 + \xi x)_+^{-1/\xi}\right\}, \quad x \in \Re.$$

Throughout the paper the case $\xi = 0$ is always interpreted as the limit $\xi \to 0$, i.e.,

$$\Omega_0(x) = \exp(-e^{-x}), \ x \in \Re,$$

the standard Gumbel distribution. A higher dimensional extension of (2.1) is due to Marshall and Olkin (1983). Under the condition that G_p has equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$, the assumption $F_p \in \mathcal{D}(G_p)$ is equivalent to the condition that

$$\lim_{u \uparrow x_{F_1}} \frac{1 - F_p(u + g(u)\mathbf{x})}{1 - F_1(u)} = -\log G_p(\mathbf{x}), \ 1 + \xi \mathbf{x} > \mathbf{0}, \tag{2.3}$$

where x_{F_1} and g satisfy (2.2). This convergence gives a clue for the existence of the limit of the form in (1.3). The details are referred to the following section.

The multivariate extreme value distribution G_p has a special representation, which is due to Pickands (1981). In fact, there exists a finite positive measure Q_p on the $(p-1)$-dim. unit simplex

$$S_p = \left\{ \mathbf{w} \in \Re^p : \mathbf{w} \geq \mathbf{0}, \ \sum_{i=1}^{p} w_i = 1 \right\}$$

satisfying

$$\int_{S_p} w_i \, dQ_p(\mathbf{w}) = 1, \ i = 1, ..., p$$

such that

$$G_p(\mathbf{x}) = \exp\left[-\int_{S_p} \max_{1 \leq i \leq p} \left\{ w_i (1 + \xi x_i)^{-1/\xi} \right\} dQ_p(\mathbf{w}) \right], \ 1 + \xi \mathbf{x} > \mathbf{0}. \tag{2.4}$$

Assuming G_p is absolutely continuous, Coles and Tawn (1991) derived an explicit form of the derivative of G_p in terms of the measure densities of Q_p on the interior, and on each of the lower dimensional boundaries, of S_p. Specifically, for each $i = 1, ..., p$, let $q_p^{i,C}$ be the (nonnegative) density of Q_p on the $(i-1)$-dim. interior of $S_p^{i,C} = \{\mathbf{w} \in S_p : w_m = 0 \ \forall m \notin C\}$, where C is a nonempty subset of $C_p = \{1, ..., p\}$ of size i. By the construction of $S_p^{i,C}$ it is clear that $S_p^{p,C_p} = S_p$ and that $S_p^{i,C}$ is isomorphic to the $(i-1)$-dim. unit simplex S_i. It is therefore convenient to take the domain of $q_p^{i,C}$ as S_i instead of $S_p^{i,C}$. Let $\{\mathcal{C}_p^1, ..., \mathcal{C}_p^p\}$ be the partition of the collection of all nonempty subsets of C_p such that $\mathcal{C}_p^i$ is the collection of all subsets of C_p of size i. Because Coles and Tawn (1991) worked with the standard Fréchet marginals for G_1, we here provide the following reformulation of their results, based on the representation Ω_ξ for G_1.

Lemma 1. Let G_p be a p-dim. extreme value distribution, which is absolutely continuous, with equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$, and let $q_p^{i,C}$ be the measure densities of Q_p defined above, where Q_p is the measure satisfying (2.4). Let $V_p(\mathbf{x}) = -\log G_p(\mathbf{x})$.

(a) $V_p(\mathbf{x})$ can be expressed as

$$V_p(\mathbf{x}) = \sum_{i=1}^{p} \sum_{C \in \mathcal{C}_p^i} \sum_{\substack{D \subset C \\ D \neq \emptyset}} (-1)^{|D|-1} I(C;D), \ 1 + \xi \mathbf{x} > \mathbf{0},$$

where

$$I(C;D) = \int_0^\infty \cdots \int_0^\infty \int_{u_{\pi(t_1)}=(1+\xi x_{t_1})^{1/\xi}}^\infty \cdots \int_{u_{\pi(t_r)}=(1+\xi x_{t_r})^{1/\xi}}^\infty q_p^{i,C}\left(\frac{u_1}{\sum_{s=1}^i u_s}, \dots, \frac{u_i}{\sum_{s=1}^i u_s}\right)\left(\sum_{s=1}^i u_s\right)^{-(i+1)} d\mathbf{u}$$

and $\pi : \{j_1 < \cdots < j_i\} \to \{1, \dots, i\}$ is a function such that $\pi(j_s) = s$, for $D = \{t_1 < \cdots < t_r\} \subset C = \{j_1 < \cdots < j_i\}$.

(b) For any $C = \{j_1 < \cdots < j_i\} \subset C_p = \{1, \dots, p\}$,

$$\frac{\partial^i V_p(\mathbf{x})}{\partial x_{j_1} \cdots \partial x_{j_i}} = -\frac{\prod_{s=1}^i (1+\xi x_{j_s})^{1/\xi - 1}}{\left\{\sum_{s=1}^i (1+\xi x_{j_s})^{1/\xi}\right\}^{i+1}} \cdot q_p^{i,C}\left(\frac{(1+\xi x_{j_1})^{1/\xi}}{\sum_{s=1}^i (1+\xi x_{j_s})^{1/\xi}}, \dots, \frac{(1+\xi x_{j_i})^{1/\xi}}{\sum_{s=1}^i (1+\xi x_{j_s})^{1/\xi}}\right)$$

on $\{\mathbf{x} \in \Re^p : 1 + \xi x_l > 0 \ \forall l \in C \text{ and } (1+\xi x_m)^{1/\xi} = 0 \ \forall m \notin C\}$.

3. WEAK CONVERGENCE OF RESCALED TRANSITION PROBABILITIES

Let $\{X_n\}$ denote a real-valued, k-th order stationary Markov chain. Here, $\{X_n\}$ being a k-th order Markov chain means that, for every n, the conditional distribution of X_n given the past depends only on the k immediate past values. Since $\{X_n\}$ is stationary, the distribution of the whole chain is clearly determined by the joint distribution F_{k+1}, say, of $(X_n, ..., X_{n+k})$ via its successive transition kernels. Assume that $F_{k+1} \in \mathcal{D}(G_{k+1})$ for some $(k+1)$-dim. extreme value distribution G_{k+1} with equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$. It is also assumed that both of F_{k+1} and G_{k+1} are absolutely continuous. This is quite reasonable since most multivariate distributions are specified by densities, not by distribution functions. We need some notation: for $i = 1, ..., k+1$, we write

$$\begin{aligned} \mathbf{x}_i &= (x_1, ..., x_i) \in \Re^i, \\ F_i(\mathbf{x}_i) &= F_i(x_1, ..., x_i) = P(X_{n+1} \le x_1, ..., X_{n+i} \le x_i), \\ f_i(\mathbf{x}_i) &= \frac{\partial^i F_i(\mathbf{x}_i)}{\partial x_1 \cdots \partial x_i}, \end{aligned}$$

and, for $j = 1, ..., k$, we write

$$\begin{aligned} \mathbf{x}_{j+1} &= (x_1, ..., x_j, x_{j+1}) = (\mathbf{x}_j, x_{j+1}) \in \Re^{j+1}, \\ F_{j+1}(x_{j+1}|\mathbf{x}_j) &= P(X_{n+j+1} \le x_{j+1} | (X_{n+1}, ..., X_{n+j}) = (x_1, ..., x_j)), \\ f_{j+1}(x_{j+1}|\mathbf{x}_j) &= \frac{f_{j+1}(\mathbf{x}_{j+1})}{f_j(\mathbf{x}_j)}. \end{aligned}$$

Since the assumption $F_{k+1} \in \mathcal{D}(G_{k+1})$ implies $F_i \in \mathcal{D}(G_i)$, $i = 1, ..., k+1$, where

$$G_i(\mathbf{x}_i) = G_{k+1}(\mathbf{x}_i, \underbrace{x_{G_1}, ..., x_{G_1}}_{k+1-i}),$$

we have, for each $i = 1, ..., k+1$,

$$\lim_{u \uparrow x_{F_1}} \frac{1 - F_i(u + g(u)\mathbf{x}_i)}{1 - F_1(u)} = -\log G_i(\mathbf{x}_i), \ 1 + \xi \mathbf{x}_i > \mathbf{0},$$

from (2.3) with p replaced by i. Taking partial derivatives in these convergences implies possibility of the convergences

$$\begin{aligned} & g(u) f_{j+1}(u + g(u)x_{j+1} | u + g(u)\mathbf{x}_j) \\ &= \left(\frac{(g(u))^{j+1} f_{j+1}(u + g(u)\mathbf{x}_{j+1})}{1 - F_1(u)} \right) \Big/ \left(\frac{(g(u))^j f_j(u + g(u)\mathbf{x}_j)}{1 - F_1(u)} \right) \\ &\to \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}} \right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j} \right), \ 1 + \xi \mathbf{x}_{j+1} > \mathbf{0}, \ j = 1, ..., k, \end{aligned}$$

as $u \uparrow x_{F_1}$, where $V_i(\mathbf{x}_i) = -\log G_i(\mathbf{x}_i)$, $i = 1, ..., k+1$. If $F_1 = \Omega_0$, then $g(u) = 1$ and so these rescaled transition densities are precisely those appearing in (1.3). We will work with these convergences.

Theorem 1. Let F_{k+1} be the joint distribution function of $(X_n, ..., X_{n+k})$ having a joint density function f_{k+1} such that $F_{k+1} \in \mathcal{D}(G_{k+1})$ with auxiliary function $g(u)$ satisfying (2.2), where G_{k+1} is some $(k+1)$-dim. extreme value distribution, which is absolutely continuous, with equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$. Suppose that, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_j \in \Re^j$ with $1 + \xi\mathbf{x}_j > \mathbf{0}$,

$$g(u) f_{j+1}(u + g(u)x_{j+1} | u + g(u)\mathbf{x}_j) \to \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right), \tag{3.1}$$

$1 + \xi x_{j+1} > 0$, as $u \uparrow x_{F_1}$, where $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero and $V_i(\mathbf{x}_i) = -\log G_i(\mathbf{x}_i)$, $i = 1, ..., k+1$. If, for each $j = 1, ..., k$, the measure Q_{j+1} in representation (2.4), with p replaced by $j+1$, for G_{j+1} has a positive, continuous density only on the interior, and zero densities on the lower dimensional boundaries, of S_{j+1}, then, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_j \in \Re^j$ with $1 + \xi\mathbf{x}_j > \mathbf{0}$, the conditional distribution of $(X_{n+j+1} - u)/g(u)$ given

$$((X_{n+1} - u)/g(u), ..., (X_{n+j} - u)/g(u)) = (x_1, ..., x_j)$$

converges weakly to a probability distribution on $\{x \in \Re : 1 + \xi x > 0\}$ as $u \uparrow x_{F_1}$ and the limiting distribution function is given by

$$\begin{aligned} &\lim_{u \uparrow x_{F_1}} P\left\{\frac{X_{n+j+1} - u}{g(u)} \le x_{j+1} \,\middle|\, \left(\frac{X_{n+1} - u}{g(u)}, ..., \frac{X_{n+j} - u}{g(u)}\right) = (x_1, ..., x_j)\right\} \\ &= \left(\frac{\partial^j V_{j+1}\left((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}), 0\right)}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j\left((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1})\right)}{\partial x_1 \cdots \partial x_j}\right), \\ &\quad 1 + \xi x_{j+1} > 0. \end{aligned} \tag{3.2}$$

Proof. Let $j = 1, ..., k$ be fixed, and let $1 + \xi\mathbf{x}_j > \mathbf{0}$. First of all observe that, from Lemma 1(a) with p replaced by $j+1$, we have, for $1 + \xi x_{j+1} > 0$,

$$\begin{aligned} &\frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{j+1})}{\partial x_1 \cdots \partial x_j} \\ &= (-1)^{j-1} \frac{\partial^j I(C_j; C_j)}{\partial x_1 \cdots \partial x_j} + (-1)^{j-1} \frac{\partial^j I(C_{j+1}; C_j)}{\partial x_1 \cdots \partial x_j} + (-1)^j \frac{\partial^j I(C_{j+1}; C_{j+1})}{\partial x_1 \cdots \partial x_j} \end{aligned}$$

$$= -\frac{\prod_{i=1}^{j}(1+\xi x_i)^{1/\xi-1}}{\left\{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}\right\}^{j+1}} \cdot q_{j+1}^{j,C_j}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right)$$
$$-\int_0^{(1+\xi x_{j+1})^{1/\xi}} \frac{\prod_{i=1}^{j}(1+\xi x_i)^{1/\xi-1}}{\left\{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}\right\}^{j+2}}$$
$$\cdot q_{j+1}^{j+1,C_{j+1}}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}, \frac{u}{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right) du, \qquad (3.3)$$

where $C_j = \{1, ..., j\}$, $C_{j+1} = \{1, ..., j+1\}$, and

$$(1+\xi \mathbf{x}_j)^{1/\xi} = ((1+\xi x_1)^{1/\xi}, ..., (1+\xi x_j)^{1/\xi}).$$

Using the fact that

$$V_j(\mathbf{x}_j) = V_{j+1}(\mathbf{x}_j, x_{G_1}) = \sum_{i=1}^{j+1} \sum_{C \in C_{j+1}^i} \sum_{\substack{D \subset C \setminus \{j+1\} \\ D \neq \emptyset}} (-1)^{|D|-1} I(C; D),$$

we also have

$$\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j} = \frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{G_1})}{\partial x_1 \cdots \partial x_j}$$
$$= (-1)^{j-1}\frac{\partial^j I(C_j; C_j)}{\partial x_1 \cdots \partial x_j} + (-1)^{j-1}\frac{\partial^j I(C_{j+1}; C_j)}{\partial x_1 \cdots \partial x_j}$$
$$= -\frac{\prod_{i=1}^{j}(1+\xi x_i)^{1/\xi-1}}{\left\{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}\right\}^{j+1}} \cdot q_{j+1}^{j,C_j}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right)$$
$$-\int_0^{\infty} \frac{\prod_{i=1}^{j}(1+\xi x_i)^{1/\xi-1}}{\left\{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}\right\}^{j+2}}$$
$$\cdot q_{j+1}^{j+1,C_{j+1}}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}, \frac{u}{u+\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right) du. \qquad (3.4)$$

Comparing (3.4) with (3.3), we therefore conclude that

$$\left(\frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right) \uparrow 1 \text{ as } x_{j+1} \uparrow x_{G_1}. \qquad (3.5)$$

We here note that $x_{G_1} = \infty$ for $\xi \geq 0$; $x_{G_1} = -1/\xi$ for $\xi < 0$. On the other hand, $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ has another representation as

$$\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j} = -\frac{\prod_{i=1}^{j}(1+\xi x_i)^{1/\xi-1}}{\left\{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}\right\}^{j+1}} \cdot q_j^{j,C_j}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right), \qquad (3.6)$$

which is a special case of Lemma 1(b), with p replaced by j. From (3.3) and (3.6), we also conclude that

$$\left(\frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right) \downarrow \frac{q_{j+1}^{j,C_j}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right)}{q_j^{j,C_j}\left(\frac{(1+\xi \mathbf{x}_j)^{1/\xi}}{\sum_{i=1}^{j}(1+\xi x_i)^{1/\xi}}\right)} \text{ as } x_{j+1} \downarrow x_{G_1}^*. \tag{3.7}$$

In fact, $x_{G_1}^* = -1/\xi$ for $\xi > 0$; $x_{G_1}^* = -\infty$ for $\xi \le 0$. Results (3.5) and (3.7) indicate that the ratio

$$\left(\frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right)$$

as a function of x_{j+1} is not in general a distribution function but a subdistribution function on $\{x_{j+1} \in \Re : 1 + \xi x_{j+1} > 0\}$. The assumption of zero mass of the measure Q_{j+1} on the lower dimensional boundaries of S_{j+1}, however, removes the possibility of this improper distribution. Thus, the ratio is a valid distribution function of x_{j+1} on $\{x_{j+1} \in \Re : 1 + \xi x_{j+1} > 0\}$ and its density function is given by the right hand side of (3.1). Therefore, condition (3.1) implies that

$$\begin{aligned}
&F_{j+1}(u + g(u)x_{j+1} | u + g(u)\mathbf{x}_j) \\
&= \int_{-\infty}^{x_{j+1}} g(u) f_{j+1}(u + g(u)t | u + g(u)\mathbf{x}_j)\, dt \\
&\to \left(\frac{\partial^j V_{j+1}(\mathbf{x}_j, x_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right), \; 1 + \xi x_{j+1} > 0, \text{ as } u \uparrow x_{F_1},
\end{aligned}$$

by the Scheffé theorem. Moreover, from representation (2.4) with p replaced by j, we have

$$V_j(\mathbf{x}_j) = (1 + \xi x_{j+1})^{-1/\xi} V_j((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1})).$$

In fact, this property is equivalent to the max-stability of G_j. Similarly,

$$V_{j+1}(\mathbf{x}_j, x_{j+1}) = (1 + \xi x_{j+1})^{-1/\xi} V_{j+1}((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}), 0).$$

Hence, we arrive at

$$\begin{aligned}
&P\left\{\frac{X_{n+j+1} - u}{g(u)} \le x_{j+1} \,\middle|\, \left(\frac{X_{n+1} - u}{g(u)}, \ldots, \frac{X_{n+j} - u}{g(u)}\right) = (x_1, \ldots, x_j)\right\} \\
&= F_{j+1}(u + g(u)x_{j+1} | u + g(u)\mathbf{x}_j) \\
&\to \left(\frac{\partial^j V_{j+1}((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}), 0)}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}))}{\partial x_1 \cdots \partial x_j}\right),
\end{aligned}$$

$1+\xi x_{j+1}>0$, as $u \uparrow x_{F_1}$. $\square$

Remark 1. Convergence (3.5) is generalized into: for any fixed $p=1,...,k$ and for any $C^*=\{j_1<\cdots<j_i\}\subset C_p=\{1,...,p\}$,

$$\frac{\partial^i V_{p+1}(\mathbf{x}_p,x_{p+1})}{\partial x_{j_1}\cdots\partial x_{j_i}} \to \frac{\partial^i V_p(\mathbf{x}_p)}{\partial x_{j_1}\cdots\partial x_{j_i}},\ 1+\xi\mathbf{x}_p>\mathbf{0},\ \text{as } x_{p+1}\uparrow x_{G_1}, \tag{3.8}$$

which is because, from Lemma 1(a) with p replaced by $p+1$, we have

$$\frac{\partial^i V_{p+1}(\mathbf{x}_p,x_{p+1})}{\partial x_{j_1}\cdots\partial x_{j_i}} = \sum_{s=i}^{p+1}\sum_{\substack{C\in C^s_{p+1}\\ C\supset C^*}}\sum_{\substack{D\subset C\\ D\supset C^*}}(-1)^{|D|-1}\frac{\partial^i I(C;D)}{\partial x_{j_1}\cdots\partial x_{j_i}}$$

$$=\sum_{s=i}^{p+1}\sum_{\substack{C\in C^s_{p+1}\\ C\supset C^*}}\left\{\sum_{\substack{D\subset C\setminus\{p+1\}\\ D\supset C^*}}(-1)^{|D|-1}\frac{\partial^i I(C;D)}{\partial x_{j_1}\cdots\partial x_{j_i}}+\sum_{\substack{D\subset C\\ D\supset C^*\cup\{p+1\}}}(-1)^{|D|-1}\frac{\partial^i I(C;D)}{\partial x_{j_1}\cdots\partial x_{j_i}}\right\}$$

so that

$$\lim_{x_{p+1}\uparrow x_{G_1}}\frac{\partial^i V_{p+1}(\mathbf{x}_p,x_{p+1})}{\partial x_{j_1}\cdots\partial x_{j_i}}$$

$$=\sum_{s=i}^{p+1}\sum_{\substack{C\in C^s_{p+1}\\ C\supset C^*}}\sum_{\substack{D\subset C\setminus\{p+1\}\\ D\supset C^*}}(-1)^{|D|-1}\frac{\partial^i I(C;D)}{\partial x_{j_1}\cdots\partial x_{j_i}}$$

$$=\frac{\partial^i}{\partial x_{j_1}\cdots\partial x_{j_i}}\left(\sum_{s=1}^{p+1}\sum_{C\in C^s_{p+1}}\sum_{\substack{D\subset C\setminus\{p+1\}\\ D\neq\emptyset}}(-1)^{|D|-1}I(C;D)\right)$$

$$=\frac{\partial^i V_{p+1}(\mathbf{x}_p,x_{G_1})}{\partial x_{j_1}\cdots\partial x_{j_i}}=\frac{\partial^i V_p(\mathbf{x}_p)}{\partial x_{j_1}\cdots\partial x_{j_i}}.$$

Coles and Tawn (1991) developed and summarized several flexible models for G_{k+1} in terms of the measure densities. The logistic model and the Dirichlet model, for instance, fall under the category of Theorem 1. If the right hand side of (3.2) is not a probability distribution on $\{x_{j+1}\in\Re: 1+\xi x_{j+1}>0\}$, convergence (3.2) is not guaranteed in general. If we however impose additional mild conditions on convergence (3.1), then convergence (3.2) can be made still valid. We are going to work with the concept of locally uniform integrability of the rescaled transition densities. A class of real-valued functions defined on $\Re$ is said to be locally uniformly integrable over an unbounded interval A if the class is uniformly integrable over any compact subset of A. For example, real-valued functions defined on $\Re$ are locally uniformly integrable if they are dominated by a continuous function. Using this concept, we provide the following theorem.

Theorem 2. Let F_{k+1} be the joint distribution function of $(X_n, ..., X_{n+k})$ having a joint density function f_{k+1} such that $F_{k+1} \in \mathcal{D}(G_{k+1})$ with auxiliary function $g(u)$ satisfying (2.2), where G_{k+1} is some $(k+1)$-dim. extreme value distribution, which is absolutely continuous, with equal univariate marginals $G_1 = \Omega_\xi$ for some $\xi \in \Re$. Suppose that, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_j \in \Re^j$ with $1 + \xi \mathbf{x}_j > \mathbf{0}$,

(a) $g(u) f_{j+1}(u + g(u) x_{j+1} | u + g(u)\mathbf{x}_j) \to \left(\dfrac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\dfrac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right)$, $1 + \xi x_{j+1} > 0$, as $u \uparrow x_{F_1}$;

(b) there exists a $u_j^*(\mathbf{x}_j)$ such that the class

$$\{g(u) f_{j+1}(u + g(u) x_{j+1} | u + g(u)\mathbf{x}_j) : u_j^*(\mathbf{x}_j) \le u < x_{F_1}\}$$

of functions of x_{j+1} is locally uniformly integrable over $\{x_{j+1} \in \Re : 1 + \xi x_{j+1} > 0\}$;

(c) $\lim_{L \uparrow x_{G_1}} \overline{\lim}_{u \uparrow x_{F_1}} P(X_{j+1} > u + g(u)L | \mathbf{X}_j = u + g(u)\mathbf{x}_j) = 0$,

where $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, $\mathbf{X}_j = (X_1, ..., X_j)$, and $V_i(\mathbf{x}_i) = -\log G_i(\mathbf{x}_i)$, $i = 1, ..., k+1$. Then, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_j \in \Re^j$ with $1 + \xi \mathbf{x}_j > \mathbf{0}$, convergence (**??**) holds, where the limiting distribution need not be necessarily a probability distribution on $\{x_{j+1} \in \Re : 1 + \xi x_{j+1} > 0\}$.

Proof. For a notational convenience, we write

$$\begin{aligned} l_j(x_{j+1}; \mathbf{x}_j) &= \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right), \\ L_j(x_{j+1}; \mathbf{x}_j) &= \left(\frac{\partial^j V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right). \end{aligned}$$

Now, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_j \in \Re^j$ with $1 + \xi \mathbf{x}_j > \mathbf{0}$, let $x_{G_1}^* < x_{j+1} < L < x_{G_1}$. Then

$$\overline{\lim}_{u \uparrow x_{F_1}} |F_{j+1}(u + g(u) x_{j+1} | u + g(u)\mathbf{x}_j) - L_j(x_{j+1}; \mathbf{x}_j)|$$

$$\begin{aligned}
&\leq \overline{\lim_{u \uparrow x_{F_1}}} \left| \int_{x_{j+1}}^{L} g(u) f_{j+1}(u + g(u)t | u + g(u)\mathbf{x}_j)\, dt - \int_{x_{j+1}}^{L} l_j(t; \mathbf{x}_j)\, dt \right| \\
&\quad + \overline{\lim_{u \uparrow x_{F_1}}} |F_{j+1}(u + g(u)L | u + g(u)\mathbf{x}_j) - L_j(L; \mathbf{x}_j)| \\
&= \overline{\lim_{u \uparrow x_{F_1}}} |F_{j+1}(u + g(u)L | u + g(u)\mathbf{x}_j) - L_j(L; \mathbf{x}_j)| \text{ (by assumptions (a) and (b))} \\
&\leq \overline{\lim_{u \uparrow x_{F_1}}} |1 - F_{j+1}(u + g(u)L | u + g(u)\mathbf{x}_j)| + |1 - L_j(L; \mathbf{x}_j)|.
\end{aligned}$$

Here, letting $L \uparrow x_{G_1}$, we conclude that

$$F_{j+1}(u + g(u)x_{j+1} | u + g(u)\mathbf{x}_j) \to L_j(x_{j+1}; \mathbf{x}_j) \text{ as } u \uparrow x_{F_1},$$

from (3.5) and assumption (c). The remaining part of the proof is the same as that of Theorem 1. This completes the proof. □

The functional form of the limiting distribution in (3.2) suggests the existence of H_j of the form in (1.4). Specifically, since

$$\frac{\partial^j V_j((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}))}{\partial x_1 \cdots \partial x_j} = (1 + \xi x_{j+1})^{-j} \frac{\partial^j V_j(\mathbf{z}_j)}{\partial z_1 \cdots \partial z_j}$$

and

$$\frac{\partial^j V_{j+1}((1 + \xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}), 0)}{\partial x_1 \cdots \partial x_j} = (1 + \xi x_{j+1})^{-j} \frac{\partial^j V_{j+1}(\mathbf{z}_j, 0)}{\partial z_1 \cdots \partial z_j}$$

where $z_i = (1 + \xi x_{j+1})^{-1}(x_i - x_{j+1})$, $i = 1, ..., j$, their ratio must be a function of

$$\frac{x_1 - x_{j+1}}{1 + \xi x_{j+1}}, ..., \frac{x_j - x_{j+1}}{1 + \xi x_{j+1}}.$$

Here, if $\xi \neq 0$, then, for each $i = 1, ..., j$,

$$\frac{x_i - x_{j+1}}{1 + \xi x_{j+1}} = \frac{1}{\xi} \left(\frac{1 + \xi x_i}{1 + \xi x_{j+1}} - 1 \right)$$

is again a function of

$$\frac{1 + \xi x_{i+1}}{1 + \xi x_i}, \frac{1 + \xi x_{i+2}}{1 + \xi x_{i+1}}, ..., \frac{1 + \xi x_{j+1}}{1 + \xi x_j}$$

or, equivalently, a function of

$$\frac{1}{\xi} \log \left(\frac{1 + \xi x_{i+1}}{1 + \xi x_i} \right), \frac{1}{\xi} \log \left(\frac{1 + \xi x_{i+2}}{1 + \xi x_{i+1}} \right), ..., \frac{1}{\xi} \log \left(\frac{1 + \xi x_{j+1}}{1 + \xi x_j} \right).$$

Therefore, we conclude that the right hand side of (3.2) is a function of

$$\frac{1}{\xi}\log\left(\frac{1+\xi x_2}{1+\xi x_1}\right), \ldots, \frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right),$$

which is valid even for $\xi = 0$ (recall that $\xi = 0$ is always interpreted as the limit $\xi \to 0$), and so we may denote it as

$$H_j\left(\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right); \nabla \mathbf{x}_j\right),$$

that is,

$$\begin{aligned} &H_j\left(\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right); \nabla \mathbf{x}_j\right) \\ &= \left(\frac{\partial^j V_{j+1}((1+\xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}), 0)}{\partial x_1 \cdots \partial x_j}\right) \Bigg/ \left(\frac{\partial^j V_j((1+\xi x_{j+1})^{-1}(\mathbf{x}_j - x_{j+1}))}{\partial x_1 \cdots \partial x_j}\right) \\ &= \left(\frac{\partial^j V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Bigg/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right), \ 1+\xi x_{j+1} > 0, \end{aligned}$$

where

$$\nabla \mathbf{x}_j = \left(\frac{1}{\xi}\log\left(\frac{1+\xi x_2}{1+\xi x_1}\right), \ldots, \frac{1}{\xi}\log\left(\frac{1+\xi x_j}{1+\xi x_{j-1}}\right)\right).$$

We here note that $H_j(y; \nabla \mathbf{x}_j)$ is well-defined on $y \in \Re$ since

$$\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right) \to \infty \text{ or } -\infty$$

as $x_{j+1} \uparrow x_{G_1}$ or $\downarrow x^*_{G_1}$ respectively. If we further denote the derivative of $H_j(y; \nabla \mathbf{x}_j)$ with respect to y as $h_j(y; \nabla \mathbf{x}_j)$, i.e.,

$$h_j(y; \nabla \mathbf{x}_j) = \frac{d}{dy} H_j(y; \nabla \mathbf{x}_j), \ y \in \Re,$$

then the right hand side of (3.1) is given by

$$\begin{aligned} &\left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Bigg/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right) \\ &= \frac{1}{1+\xi x_{j+1}} h_j\left(\frac{1}{\xi}\log\left(\frac{1+\xi x_{j+1}}{1+\xi x_j}\right); \nabla \mathbf{x}_j\right), \ 1+\xi x_{j+1} > 0. \end{aligned}$$

This means that

$$(1+\xi x_{j+1}) \cdot \left(\frac{\partial^{j+1} V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}} \right) \Big/ \left(\frac{\partial^{j} V_{j}(\mathbf{x}_{j})}{\partial x_1 \cdots \partial x_{j}} \right)$$

is necessarily a function of

$$\frac{1}{\xi} \log \left(\frac{1+\xi x_2}{1+\xi x_1} \right), \ldots, \frac{1}{\xi} \log \left(\frac{1+\xi x_{j+1}}{1+\xi x_j} \right).$$

It should be noted that, when $\xi = 0$, H_j and h_j are interpreted as $H_j(x_{j+1} - x_j; x_2 - x_1, \ldots, x_j - x_{j-1})$ and $h_j(x_{j+1} - x_j; x_2 - x_1, \ldots, x_j - x_{j-1})$ respectively.

According to the proof of Theorem 1, $H_j(y; \nabla \mathbf{x}_j)$ is not in general a distribution function of y on $\Re$, because its mass need not vanish as $y \to -\infty$. If we however add $-\infty$ to the domain of H_j, then $H_j(y; \nabla \mathbf{x}_j)$ may be considered as a distribution function of y on $\{-\infty\} \cup \Re$, where it has an atom at $y = -\infty$ and is absolutely continuous on $\Re$, having $h_j(y; \nabla \mathbf{x}_j)$ as its density there. The mass of the atom is given by the right hand side of (3.7). Theorem 1, therefore, says that the assumption $F_{k+1} \in \mathcal{D}(G_{k+1})$ implies convergence (3.2) under condition (3.1) when $H_j(y; \nabla \mathbf{x}_j)$ has zero mass at $y = -\infty$. Otherwise, that convergence is not in general guaranteed though the convergence can be made still valid with additional conditions as in Theorem 2. What happens if F_{k+1} is itself a multivariate extreme value distribution? The following theorem tells us that if F_{k+1} is itself a multivariate extreme value distribution, then both of convergences (3.1) and (3.2) are valid, provided that $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, whether $H_j(y; \nabla \mathbf{x}_j)$ has zero mass at $y = -\infty$ or not. This will be proved directly using the max-stability of F_{k+1}.

Theorem 3. Let F_{k+1} be the joint distribution function of $(X_n, \ldots, X_{n+k})$ having a joint density function f_{k+1}. If F_{k+1} is itself a multivariate extreme value distribution with equal univariate marginals $F_1 = \Omega_\xi$ for some $\xi \in \Re$, then, for each $j = 1, \ldots, k$ and for every fixed $\mathbf{x}_{j+1} \in \Re^{j+1}$ with $1 + \xi \mathbf{x}_{j+1} > \mathbf{0}$,

$$g(u) f_{j+1}(u + g(u) x_{j+1} | u + g(u)\mathbf{x}_j)$$

and

$$P\left\{ \frac{X_{n+j+1} - u}{g(u)} \le x_{j+1} \,\Bigg|\, \left(\frac{X_{n+1} - u}{g(u)}, \ldots, \frac{X_{n+j} - u}{g(u)} \right) = (x_1, \ldots, x_j) \right\}$$

are convergent as $u \uparrow x_{F_1}$; moreover, their limits are given by

$$\frac{1}{1+\xi x_{j+1}} h_j \left(\frac{1}{\xi} \log \left(\frac{1+\xi x_{j+1}}{1+\xi x_j} \right); \nabla \mathbf{x}_j \right) \text{ and } H_j \left(\frac{1}{\xi} \log \left(\frac{1+\xi x_{j+1}}{1+\xi x_j} \right); \nabla \mathbf{x}_j \right)$$

respectively, provided that $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, where $g(u) = 1 + \xi u$ and $V_i(\mathbf{x}_i) = -\log F_i(\mathbf{x}_i)$, $i = 1, ..., k+1$. Here, $H_j(y; \nabla \mathbf{x}_j)$ may have possibly a positive mass at $y = -\infty$.

Proof. First of all observe that the max-stability of F_{k+1} (or, equivalently, representation (2.4) with p, G_p replaced by $k+1, F_{k+1}$ respectively) implies that

$$F_{k+1}^n(n^\xi \mathbf{x}_{k+1} + (n^\xi - 1)/\xi) = F_{k+1}(\mathbf{x}_{k+1}), \; n = 1, 2, ...,$$

for every fixed $\mathbf{x}_{k+1} \in \Re^{k+1}$ with $1 + \xi \mathbf{x}_{k+1} > \mathbf{0}$, which means $F_{k+1} \in \mathcal{D}(F_{k+1})$ by the definition of multivariate domain of attraction. Thus (2.3) holds, with p, G_p replaced by $k+1, F_{k+1}$ respectively. When $\xi = 0$, the validity of the choice $g(u) = 1$ follows from the fact that $F_1 = \Omega_0$, the standard Gumbel distribution. In view of Theorem1, this implies plausibility of convergences of the proposed forms of rescaled transition kernels. In fact, notice that, for each $i = 1, ..., k+1$,

$$\begin{aligned} f_i(\mathbf{x}_i) &= \frac{\partial^i F_i(\mathbf{x}_i)}{\partial x_1 \cdots \partial x_i} = \frac{\partial^i e^{-V_i(\mathbf{x}_i)}}{\partial x_1 \cdots \partial x_i} \\ &= e^{-V_i(\mathbf{x}_i)} \left[-\frac{\partial^i V_i(\mathbf{x}_i)}{\partial x_1 \cdots \partial x_i} + \cdots + (-1)^i \prod_{s=1}^{i} \frac{\partial V_i(\mathbf{x}_i)}{\partial x_s} \right] \\ &= e^{-V_i(\mathbf{x}_i)} \sum_{m=1}^{i} (-1)^m \sum_{\{D_i^1, ..., D_i^m\}} \left(\frac{\partial^{|D_i^1|} V_i(\mathbf{x}_i)}{\prod_{s \in D_i^1} \partial x_s} \cdots \frac{\partial^{|D_i^m|} V_i(\mathbf{x}_i)}{\prod_{s \in D_i^m} \partial x_s} \right), \end{aligned}$$

where $\{D_i^1, ..., D_i^m\}$ varies on the partitions of $\{1, ..., i\}$ such that each of $D_i^1, ..., D_i^m$ contains at least one element. Thus, for each fixed $j = 1, ..., k$,

$$\begin{aligned} & g(u) f_{j+1}(u + g(u)\mathbf{x}_{j+1}) \\ &= g(u) \exp\{-V_{j+1}(u + g(u)\mathbf{x}_{j+1})\} \sum_{m=1}^{j+1} (-1)^m \\ &\cdot \sum_{\{D_{j+1}^1, ..., D_{j+1}^m\}} \left(\frac{\partial^{|D_{j+1}^1|} V_{j+1}(u + g(u)\mathbf{x}_{j+1})}{\prod_{s \in D_{j+1}^1} \partial(u + g(u)x_s)} \cdots \frac{\partial^{|D_{j+1}^m|} V_{j+1}(u + g(u)\mathbf{x}_{j+1})}{\prod_{s \in D_{j+1}^m} \partial(u + g(u)x_s)} \right). \end{aligned}$$

Here, notice that, for every $r = 1, ..., m$,

$$\frac{\partial^{|D_{j+1}^r|} V_{j+1}(u + g(u)\mathbf{x}_{j+1})}{\prod_{s \in D_{j+1}^r} \partial(u + g(u)x_s)} = (g(u))^{-|D_{j+1}^r|} \frac{\partial^{|D_{j+1}^r|} V_{j+1}(u + g(u)\mathbf{x}_{j+1})}{\prod_{s \in D_{j+1}^r} \partial x_s}$$

and that $|D_{j+1}^1| + \cdots + |D_{j+1}^m| = j+1$. Moreover, since F_{j+1} is also a multivariate extreme value distribution, its max-stability implies that, for any

$t > 0$,

$$V_{j+1}(t^{\xi}\mathbf{x}_{j+1} + (t^{\xi} - 1)/\xi) = t^{-1}V_{j+1}(\mathbf{x}_{j+1}),\ 1 + \xi\mathbf{x}_{j+1} > \mathbf{0}.$$

Taking $t = (1 + \xi u)^{1/\xi} > 0$, we have

$$\begin{aligned} V_{j+1}(u + g(u)\mathbf{x}_{j+1}) &= V_{j+1}(t^{\xi}\mathbf{x}_{j+1} + (t^{\xi} - 1)/\xi) = t^{-1}V_{j+1}(\mathbf{x}_{j+1}) \\ &= (1 + \xi u)^{-1/\xi}V_{j+1}(\mathbf{x}_{j+1}),\ 1 + \xi\mathbf{x}_{j+1} > \mathbf{0}, \end{aligned}$$

since $g(u) = 1+\xi u$. Applying these to the expression of $g(u)f_{j+1}(u + g(u)\mathbf{x}_{j+1})$, we have

$$\begin{aligned} &g(u)f_{j+1}(u + g(u)\mathbf{x}_{j+1}) \\ &= (g(u))^{-j}\exp\left\{-(1 + \xi u)^{-1/\xi}V_{j+1}(\mathbf{x}_{j+1})\right\}\sum_{m=1}^{j+1}(-1)^m(1 + \xi u)^{-m/\xi} \\ &\quad\cdot\sum_{\{D^1_{j+1},\ldots,D^m_{j+1}\}}\left(\frac{\partial^{|D^1_{j+1}|}V_{j+1}(\mathbf{x}_{j+1})}{\prod_{s\in D^1_{j+1}}\partial x_s}\cdots\frac{\partial^{|D^m_{j+1}|}V_{j+1}(\mathbf{x}_{j+1})}{\prod_{s\in D^m_{j+1}}\partial x_s}\right),\ 1 + \xi\mathbf{x}_{j+1} > \mathbf{0}. \end{aligned} \tag{3.9}$$

Similarly, one can show that

$$\begin{aligned} &f_j(u + g(u)\mathbf{x}_j) \\ &= (g(u))^{-j}\exp\left\{-(1 + \xi u)^{-1/\xi}V_j(\mathbf{x}_j)\right\}\sum_{m=1}^{j}(-1)^m(1 + \xi u)^{-m/\xi} \\ &\quad\cdot\sum_{\{D^1_j,\ldots,D^m_j\}}\left(\frac{\partial^{|D^1_j|}V_j(\mathbf{x}_j)}{\prod_{s\in D^1_j}\partial x_s}\cdots\frac{\partial^{|D^m_j|}V_j(\mathbf{x}_j)}{\prod_{s\in D^m_j}\partial x_s}\right),\ 1 + \xi\mathbf{x}_j > \mathbf{0}. \end{aligned} \tag{3.10}$$

The functional forms (3.9) and (3.10) guarantee the existence of the limit $l_j(x_{j+1};\mathbf{x}_j)$, say, of

$$g(u)f_{j+1}(u + g(u)x_{j+1}|u + g(u)\mathbf{x}_j) = \frac{g(u)f_{j+1}(u + g(u)\mathbf{x}_{j+1})}{f_j(u + g(u)\mathbf{x}_j)}$$

as $u \uparrow x_{F_1}$, where $l_j(x_{j+1};\mathbf{x}_j)$ could be ∞ identically. However, from the Fatou lemma,

$$\int_{x^*_{F_1}}^{x_{F_1}} l_j(x_{j+1};\mathbf{x}_j)\,dx_{j+1} \le \varliminf_{u\uparrow x_{F_1}}\int_{x^*_{F_1}}^{x_{F_1}} g(u)f_{j+1}(u+g(u)x_{j+1}|u+g(u)\mathbf{x}_j)\,dx_{j+1} = 1,$$

which implies that $l_j(x_{j+1};\mathbf{x}_j)$ is finite for almost every $x_{j+1} \in \Re$ with $1 + \xi x_{j+1} > 0$. In view of the almost everywhere uniqueness of the conditional density function, an appropriate choice of that makes $l_j(x_{j+1};\mathbf{x}_j)$ finite for every fixed $\mathbf{x}_{j+1} \in \Re^{j+1}$ with $1+\xi\mathbf{x}_{j+1} > \mathbf{0}$. Therefore, the rescaled transition densities

$$g(u)f_{j+1}(u + g(u)x_{j+1}|u + g(u)\mathbf{x}_j), \; j = 1, ..., k,$$

converge to finite limits as $u \uparrow x_{F_1}$. Moreover, if $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero, then it is obvious, from (3.9) and (3.10), that

$$g(u)f_{j+1}(u + g(u)x_{j+1}|u + g(u)\mathbf{x}_j) \to \left(\frac{\partial^{j+1}V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_{j+1}}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right)$$

as $u \uparrow x_{F_1}$. For the convergences of the rescaled transition probabilities, we can similarly show that, for each $j = 1, ..., k$,

$$\begin{aligned}
& P\left\{\frac{X_{n+j+1} - u}{g(u)} \le x_{j+1} \,\middle|\, \left(\frac{X_{n+1} - u}{g(u)}, ..., \frac{X_{n+j} - u}{g(u)}\right) = (x_1, ..., x_j)\right\} \\
&= F_{j+1}(u + g(u)x_{j+1}|u + g(u)\mathbf{x}_j) = \int_{x^*_{F_1}}^{x_{j+1}} g(u)f_{j+1}(u + g(u)t|u + g(u)\mathbf{x}_j)\,dt \\
&= \int_{x^*_{F_1}}^{u+g(u)x_{j+1}} f_{j+1}(t|u + g(u)\mathbf{x}_j)\,dt \\
&= \left(\frac{\partial^j e^{-V_{j+1}(u+g(u)\mathbf{x}_{j+1})}}{\partial(u + g(u)x_1) \cdots \partial(u + g(u)x_j)}\right) \Big/ \left(\frac{\partial^j e^{-V_j(u+g(u)\mathbf{x}_j)}}{\partial(u + g(u)x_1) \cdots \partial(u + g(u)x_j)}\right) \\
&\quad + c(u;\mathbf{x}_j) \\
&= \frac{A_1(u;\mathbf{x}_{j+1})}{A_2(u;\mathbf{x}_j)} + c(u;\mathbf{x}_j), \; 1 + \xi\mathbf{x}_{j+1} > \mathbf{0},
\end{aligned}$$

for some function $c(u;\mathbf{x}_j)$, where

$$\begin{aligned}
A_1(u;\mathbf{x}_{j+1}) &= \exp\left\{-(1 + \xi u)^{-1/\xi} V_{j+1}(\mathbf{x}_{j+1})\right\} \sum_{m=1}^{j} (-1)^m (1 + \xi u)^{-m/\xi} \\
&\quad \cdot \sum_{\{D_j^1,...,D_j^m\}} \left(\frac{\partial^{|D_j^1|}V_{j+1}(\mathbf{x}_{j+1})}{\prod_{s\in D_j^1} \partial x_s} \cdots \frac{\partial^{|D_j^m|}V_{j+1}(\mathbf{x}_{j+1})}{\prod_{s\in D_j^m} \partial x_s}\right)
\end{aligned}$$

and

$$\begin{aligned}
A_2(u;\mathbf{x}_j) &= \exp\left\{-(1 + \xi u)^{-1/\xi} V_j(\mathbf{x}_j)\right\} \sum_{m=1}^{j} (-1)^m (1 + \xi u)^{-m/\xi} \\
&\quad \cdot \sum_{\{D_j^1,...,D_j^m\}} \left(\frac{\partial^{|D_j^1|}V_j(\mathbf{x}_j)}{\prod_{s\in D_j^1} \partial x_s} \cdots \frac{\partial^{|D_j^m|}V_j(\mathbf{x}_j)}{\prod_{s\in D_j^m} \partial x_s}\right).
\end{aligned}$$

Here, notice that, by (3.8), $A_1(u;\mathbf{x}_{j+1}) \to A_2(u;\mathbf{x}_j)$ as $x_{j+1} \uparrow x_{F_1}$ from which $c(u;\mathbf{x}_j)$ must be zero. Also, the functional forms of A_1 and A_2 indicate that their ratio $A_1(u;\mathbf{x}_{j+1})/A_2(u;\mathbf{x}_j)$ is convergent as $u \uparrow x_{F_1}$, and moreover, its limit will be

$$\left(\frac{\partial^j V_{j+1}(\mathbf{x}_{j+1})}{\partial x_1 \cdots \partial x_j}\right) \Big/ \left(\frac{\partial^j V_j(\mathbf{x}_j)}{\partial x_1 \cdots \partial x_j}\right),$$

provided that $\partial^j V_j(\mathbf{x}_j)/(\partial x_1 \cdots \partial x_j)$ is not zero. □

4. EXAMPLES

Example 1 (Logistic model). Let $\{X_n\}$ be a k-th order stationary Markov chain in which F_{k+1} follows the law of the logistic model with parameter $r > 1$, i.e.,

$$F_{k+1}(\mathbf{x}_{k+1}) = \exp\left\{-\left(\sum_{s=1}^{k+1} e^{-rx_s}\right)^{1/r}\right\}, \ \mathbf{x}_{k+1} \in \Re^{k+1}. \tag{4.1}$$

Tawn (1988) and Smith (1992) dealt with applications of this multivariate extreme value distribution. Using the fact that every lower dimensional marginal distribution function $F_i(\mathbf{x}_i)$ ($i = 1, ..., k+1$) is of the same form as (4.1), one may get the explicit form of the corresponding density $f_i(\mathbf{x}_i)$. In fact, it can be shown by induction on i that $f_i(\mathbf{x}_i)$ is given by

$$f_i(\mathbf{x}_i) = \exp\left(-r\sum_{s=1}^{i} x_s\right) F_i(\mathbf{x}_i) \sum_{s=1}^{i} a_{is}(V_i(\mathbf{x}_i))^{i+1-s-ir}, \ \mathbf{x}_i \in \Re^i, \tag{4.2}$$

where $V_i(\mathbf{x}_i) = -\log F_i(\mathbf{x}_i)$ and the coefficients a_{is} are determined by the recursive formula:

$$\begin{aligned} &a_{t1} = 1, \ a_{t,t+1} = 0, \ t = 1, ..., i; \\ &a_{ts} = a_{t-1,s} + \{(t-1)r + s - t - 1\}a_{t-1,s-1}, \ s = 2, ..., t. \end{aligned}$$

Since $F_1 = \Omega_0$ and $g(u) = 1$, we examine the convergence of $f_{j+1}(u + x_{j+1}|u + \mathbf{x}_j)$, $\mathbf{x}_{j+1} \in \Re^{j+1}$, $j = 1, ..., k$, as $u \to \infty$. Observe first that, from (4.2), we have

$$\begin{aligned} f_{j+1}(x_{j+1}|\mathbf{x}_j) \ &= \ \exp\left[-V_j(\mathbf{x}_j)\left\{\left(1 + V_j^{-r}(\mathbf{x}_j - x_{j+1})\right)^{1/r} - 1\right\}\right] V_j^{-r}(\mathbf{x}_j - x_{j+1}) \\ &\cdot \frac{\sum_{s=0}^{j} a_{j+1,j+1-s} V_j^s(\mathbf{x}_j)\left(1 + V_j^{-r}(\mathbf{x}_j - x_{j+1})\right)^{(s+1)/r-(j+1)}}{\sum_{s=0}^{j-1} a_{j,j-s} V_j^s(\mathbf{x}_j)}. \end{aligned}$$

Thus,

$$\lim_{u\to\infty} f_{j+1}(u+x_{j+1}|u+\mathbf{x}_j)$$
$$= \lim_{u\to\infty}\Bigg[V_j^{-r}(\mathbf{x}_j - x_{j+1}) \cdot \frac{\sum_{s=0}^{j} a_{j+1,j+1-s} V_j^s(u+\mathbf{x}_j)\Big(1+V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big)^{(s+1)/r-(j+1)}}{\sum_{s=0}^{j-1} a_{j,j-s} V_j^s(u+\mathbf{x}_j)}\Bigg]$$
$$= \frac{a_{j+1,j+1}}{a_{jj}} \cdot V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big(1+V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big)^{1/r-j-1}$$
$$= \frac{jr-1}{\sum_{s=1}^{j} e^{r(x_{j+1}-x_s)}}\left(1+\frac{1}{\sum_{s=1}^{j} e^{r(x_{j+1}-x_s)}}\right)^{1/r-j-1} \tag{4.3}$$

since $a_{tt} = \prod_{s=1}^{t-1}(sr-1)$. Similarly, it can be checked that

$$P(X_{n+j+1} \le x_{j+1}|(X_{n+1},...,X_{n+j}) = (x_1,...,x_j))$$
$$= \exp\left[-V_j(\mathbf{x}_j)\left\{\Big(1+V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big)^{1/r} - 1\right\}\right] \cdot \frac{\sum_{s=0}^{j-1} a_{j,j-s} V_j^s(\mathbf{x}_j)\Big(1+V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big)^{(s+1)/r-j}}{\sum_{s=0}^{j-1} a_{j,j-s} V_j^s(\mathbf{x}_j)},$$

from which we have

$$\lim_{u\to\infty} P(X_{n+j+1} \le u + x_{j+1}|(X_{n+1},...,X_{n+j}) = (u+x_1,...,u+x_j))$$
$$= \Big(1+V_j^{-r}(\mathbf{x}_j - x_{j+1})\Big)^{1/r-j} = \left(1+\frac{1}{\sum_{s=1}^{j} e^{r(x_{j+1}-x_s)}}\right)^{1/r-j}. \tag{4.4}$$

The derivation of (4.3) and (4.4) is somewhat complicated. One may instead apply Theorem 3 to obtain the same results using the formula

$$\frac{\partial^i V_i(\mathbf{x}_i)}{\partial x_1 \cdots \partial x_i} = -\left\{\prod_{s=1}^{i-1}(sr-1)\right\}\exp\left(-r\sum_{s=1}^{i} x_s\right)\left(\sum_{s=1}^{i} e^{-rx_s}\right)^{1/r-i},$$

which is a lot easier.

Example 2 (Dirichlet model). This model was introduced recently by Coles and Tawn (1991) in terms of the measure density q_i^{i,C_i}, of Q_i on the interior of S_i ($i = 1,...,k+1$), defined by

$$q_i^{i,C_i}(\mathbf{w}_i) = \frac{i\Gamma(i\alpha)}{(\Gamma(\alpha))^i}\prod_{s=1}^{i} w_s^{\alpha-1},\ \ \mathbf{w}_i = (w_1,...,w_i) \in S_i,\ \ \alpha > 0,$$

where $C_i = \{1, ..., i\}$ and

$$V_i(\mathbf{x}_i) = -\log F_i(\mathbf{x}_i) = \int_{S_i} \max_{1\le s\le i} \left(w_s e^{-x_s}\right) dQ_i(\mathbf{w}_i), \ \mathbf{x}_i \in \Re^i.$$

Let $\{X_n\}$ be the corresponding k-th order stationary Markov chain. They derived the explicit form of $F_2(x_1, x_2)$, but it is intractable to obtain a compact form of $F_i(\mathbf{x}_i)$ for $i \geq 3$. However, since $F_{k+1}(\mathbf{x}_{k+1})$ is a multivariate extreme value distribution with equal univariate marginals $F_1 = \Omega_0$, the convergences of the corresponding rescaled transition kernels are guaranteed by Theorem 3 without knowing the explicit forms of $f_i(\mathbf{x}_i)$, $i = 1, ..., k+1$. In fact, from Lemma 1(b), for each $i = 1, ..., k+1$,

$$\begin{aligned}\frac{\partial^i V_i(\mathbf{x}_i)}{\partial x_1 \cdots \partial x_i} &= -\frac{\exp\left(\sum_{s=1}^i x_s\right)}{\left(\sum_{s=1}^i e^{x_s}\right)^{i+1}} \cdot q_i^{i,C_i}\left(\frac{e^{x_1}}{\sum_{s=1}^i e^{x_s}}, ..., \frac{e^{x_i}}{\sum_{s=1}^i e^{x_s}}\right) \\ &= -\frac{i\Gamma(i\alpha)}{(\Gamma(\alpha))^i} \cdot \frac{\exp\left(\alpha \sum_{s=1}^i x_s\right)}{\left(\sum_{s=1}^i e^{x_s}\right)^{i\alpha+1}}.\end{aligned}$$

Therefore, for each $j = 1, ..., k$ and for every fixed $\mathbf{x}_{j+1} \in \Re^{j+1}$,

$$\begin{aligned}&\lim_{u\to\infty} f_{j+1}(u + x_{j+1}|u + \mathbf{x}_j) \\ &= h_j(x_{j+1} - x_j; \nabla\mathbf{x}_j) \\ &= \left(1 + \frac{1}{j}\right) \frac{\Gamma(\alpha + j\alpha)}{\Gamma(\alpha)\Gamma(j\alpha)} \cdot \frac{\exp(\alpha x_{j+1})}{\left(\sum_{s=1}^j e^{x_s}\right)^\alpha} \left\{1 + \frac{e^{x_{j+1}}}{\sum_{s=1}^j e^{x_s}}\right\}^{-(j+1)\alpha - 1}\end{aligned}$$

and

$$\begin{aligned}&\lim_{u\to\infty} P(X_{n+j+1} \leq u + x_{j+1}|(X_{n+1}, ..., X_{n+j}) = (u + x_1, ..., u + x_j)) \\ &= H_j(x_{j+1} - x_j; \nabla\mathbf{x}_j),\end{aligned}$$

where $H_j(y; \nabla\mathbf{x}_j)$ is the distribution function of the random variable

$$Y = \log\left(\sum_{s=1}^j e^{x_s - x_j}\right) - \log\left(\frac{1}{Z} - 1\right),$$

with $Z \sim \text{Beta}(\alpha, j\alpha + 1)$.

Example 3 (Multivariate t-distribution). Let $\{X_n\}$ be a k-th order stationary Markov chain in which F_{k+1} is a multivariate t-distribution with parameter ν (ν: a positive integer), which is defined by its density f_{k+1} as (see Johnson and Kotz (1972), page 134):

$$f_{k+1}(\mathbf{x}_{k+1}) = \frac{\Gamma((\nu+k+1)/2)}{(\pi\nu)^{(k+1)/2}\Gamma(\nu/2)}\left(1+\nu^{-1}\sum_{s=1}^{k+1}x_s^2\right)^{-(\nu+k+1)/2}, \quad \mathbf{x}_{k+1}\in\Re^{k+1}.$$

Here, the univariate marginal F_1 is the t-distribution with ν degrees of freedom so that $F_1 \in \mathcal{D}(\Omega_{\nu^{-1}})$. Yun (1994) showed that $F_{k+1} \in \mathcal{D}(G_{k+1})$ where G_{k+1} is given by

$$\begin{aligned} V_{k+1}(\mathbf{x}_{k+1}) &= -\log G_{k+1}(\mathbf{x}_{k+1}) \\ &= \sum_{i=1}^{k+1}(1+\nu^{-1}x_i)^{-\nu} + \sum_{\substack{D\subset\{1,\dots,k+1\} \\ |D|\ge 2}} (-1)^{|D|-1}J_{k+1}(D;\mathbf{x}_{k+1}),\ 1+\nu^{-1}\mathbf{x}_{k+1} > \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} J_{k+1}(D;\mathbf{x}_{k+1}) &= \frac{\Gamma((\nu+r)/2)}{(\nu\sqrt{\pi})^{r-1}\Gamma((\nu+1)/2)} \\ &\quad \cdot \int_{x_{i_1}}^{\infty}\cdots\int_{x_{i_r}}^{\infty}\left\{\sum_{s=1}^{r}(1+\nu^{-1}y_s)^2\right\}^{-(\nu+r)/2} d\mathbf{y}_r \end{aligned}$$

for $D = \{i_1 < \cdots < i_r\} \subset \{1,\dots,k+1\}$ with $r \ge 2$. It should be noted that the auxiliary function $g(u) = 1+\xi u$ in (2.2), for $\xi > 0$, may be replaced by $g(u) = \xi u$ with no distortion in the whole story of this paper. In this example we work with $g(u) = \nu^{-1}u$, which then saves our labor for the complicated algebraic computations. First of all observe that, for each $j = 1,\dots,k$ and for every fixed $x_1,\dots,x_j > -\nu$,

$$\begin{aligned} &\lim_{u\to\infty} \nu^{-1}uf_{j+1}(u+\nu^{-1}ux_{j+1}|u+\nu^{-1}u\mathbf{x}_j) \\ &= \frac{\Gamma((\nu+j+1)/2)}{\nu\sqrt{\pi}\Gamma((\nu+j)/2)}\left\{\sum_{s=1}^{j}(1+\nu^{-1}x_s)^2\right\}^{-1/2}\left\{1+\frac{(1+\nu^{-1}x_{j+1})^2}{\sum_{s=1}^{j}(1+\nu^{-1}x_s)^2}\right\}^{-(\nu+j+1)/2} \\ &= \frac{1}{1+\nu^{-1}x_{j+1}}h_j\left(\nu\log\left(\frac{1+\nu^{-1}x_{j+1}}{1+\nu^{-1}x_j}\right);\nabla\mathbf{x}_j\right),\ x_{j+1} > -\nu. \end{aligned}$$

Here, notice that $2h_j(y;\nabla\mathbf{x}_j)$ is the density function of the random variable

$$Y = \frac{\nu}{2}\log\left\{\frac{\sum_{s=1}^{j}(x_s+\nu)^2}{(x_j+\nu)^2(1/Z-1)}\right\},$$

where $Z \sim \text{Beta}(1/2, (\nu+j)/2)$. In other words, $h_j(y; \nabla \mathbf{x}_j)$ gives a total mass 1/2 on $\Re$, and so we apply Theorem 2. In fact, for any fixed $u^* > 0$,

$$\nu^{-1}uf_{j+1}(u+\nu^{-1}ux_{j+1}|u+\nu^{-1}u\mathbf{x}_j)$$
$$\le \frac{\nu^{(\nu+j)/2-1}\Gamma((\nu+j+1)/2)}{\sqrt{\pi}\Gamma((\nu+j)/2)} \cdot \frac{\left\{(u^*)^{-2}+\nu^{-1}\sum_{s=1}^{j}(1+\nu^{-1}x_s)^2\right\}^{(\nu+j)/2}}{\left\{\sum_{s=1}^{j}(1+\nu^{-1}x_s)^2\right\}^{1/2}(1+\nu^{-1}x_{j+1})^{\nu+j}},$$
$$x_{j+1} > -\nu, \; u \ge u^*,$$

which verifies conditions (b) and (c). Therefore, by Theorem 2,

$$\lim_{u\to\infty} P\left\{\frac{X_{n+j+1}-u}{\nu^{-1}u} \le x_{j+1} \,\middle|\, \left(\frac{X_{n+1}-u}{\nu^{-1}u}, \ldots, \frac{X_{n+j}-u}{\nu^{-1}u}\right) = (x_1, \ldots, x_j)\right\}$$
$$= H_j\left(\nu \log\left(\frac{1+\nu^{-1}x_{j+1}}{1+\nu^{-1}x_j}\right); \nabla \mathbf{x}_j\right), \; x_{j+1} > -\nu,$$

where
$$H_j(y; \nabla \mathbf{x}_j) = 1 - \int_y^\infty h_j(t; \nabla \mathbf{x}_j)\,dt, \; y \in \{-\infty\} \cup \Re.$$

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