Journal of the Korean
Statistical Society
Vol. 25, No. 3, 1996

Conditions for the Non-ergodicity of Some Markov Chains †

Oesook Lee[1]

Abstract

We consider the discrete time randomly perturbed systems on separable Banach space given by $X_{n+1} = \Gamma_{n+1}(X_n) + \mathcal{E}_{n+1}$ where $\{\Gamma_n\}$ is a sequence of random functions and $\{\mathcal{E}_n\}$ is a sequence of disturbances. Sufficient conditions for non-ergodicity of $\{X_n\}$ are obtained.

Key Words : Randomly Perturbed Systems; Markov Process; Non-ergodicity; Transience.

1. INTRODUCTION

Let S be a separable Banach space with norm $\| \cdot \|$. By $\mathcal{B}(S)$ we denote the $\sigma-$algebra of Borel subsets of S. Let $\Gamma = \Gamma(S, S)$ be a collection of Borel measurable functions on S. Let $(\Omega, \mathcal{F}, P)$ be a sufficiently rich probability space on which are defined a sequence of random maps $\Gamma_1, \Gamma_2, \cdots$ taking values on Γ , a sequence of S-valued random elements $\mathcal{E}_1, \mathcal{E}_2, \cdots$ with $E \| \mathcal{E}_n \| < \infty$, and a random variable X_0 with values in S. Assume that the

† This research is supported by Ewha Womans University Faculty Research Fund,1994.
[1]Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea.

tuples $(\Gamma_n, \mathcal{E}_n), n \geq 1$ are independent and identically distributed and X_0 is independent of $\{(\Gamma_n, \mathcal{E}_n)\}$.

We consider the following randomly perturbed systems $\{X_n\}$ on S

$$X_0, \quad X_{n+1} = \Gamma_{n+1}(X_n) + \mathcal{E}_{n+1} \quad (n \geq 1). \tag{1.1}$$

Then $\{X_n : n \geq 0\}$ generated under the above assumptions is *Markovian* with one-step transition probability function

$$p(x, A) \;=\; P(\Gamma_{n+1}(X_n) + \mathcal{E}_{n+1} \in A | X_n = x), \quad (x \in S, A \in \mathcal{B}(S)).$$

Since analysis of time series models usually assumes that stationality (ergodicity) holds, it is of great importance to know whether the process is ergodic . Sufficient and/or necessary conditions for ergodicity of various types of Markov chains have been studied by a number of authors, for example, Tweedie(1983), Petruccelli and Woolford(1984), Feigen and Tweedie(1985), Tjøstheim(1986, 1990), Tong(1990), Burton and Rösler(1995), Lee(1996a, 1996b).

In this paper we consider a class of Markov processes including doubly stochastic dynamical systems and find some conditions for non-ergodicity, which improve the results given by Tjøstheim(1990) and Lee(1995). We follow the notations adopted in Nummelin(1984).

2. MAIN RESULTS

We shall write $X_n(x)$ for X_n with $X_0 = x$. Note that all equalities and inequalities which are connected to conditional probabilities hold with probability 1.

Theorem 1. Suppose there exist positive integer m_0 and $M \geq 0$ such that

$$\inf_{\|x\| \geq M} \frac{\| \Gamma_{m_0} \cdots \Gamma_1 x \|}{\| x \|} > 1, \quad a.s. \tag{2.1}$$

and

$$K := \sup_{\|x\| \geq M} E \| X_{m_0}(x) - \Gamma_{m_0} \cdots \Gamma_1 x \| < \infty. \tag{2.2}$$

Then $\{X_n\}$ is not ergodic if for $r \geq 0$, $P(\| X_{n_0} \| \geq r | X_0) > 0$ for some n_0.

Proof. For simplicity of notation, consider the case of $m_0 = 2$. Choose $\alpha > \beta > 1$ such that $\| \Gamma_2\Gamma_1 x \| \geq \alpha \| x \|$ a.s. for all $\| x \| \geq M$. For $\| X_n \| \geq M$,

$$\begin{aligned} & P(\| X_{n+2} \| \geq \beta \| X_n \| \,|X_n) \\ \geq\ & P(\| \mathcal{E}_{n+2} \| \leq -\beta \| X_n \| + \| \Gamma_{n+2}(\Gamma_{n+1}(X_n) + \mathcal{E}_{n+1}) \| \,|X_n) \\ \geq\ & P(\| \mathcal{E}_{n+2} \| + \| \Gamma_{n+2}(\Gamma_{n+1}(X_n) + \mathcal{E}_{n+1}) - \Gamma_{n+2}\Gamma_{n+1}(X_n) \| \\ & \qquad \leq (\alpha - \beta) \| X_n \| \,|X_n) \\ \geq\ & 1 - \frac{E \| \mathcal{E}_1 \| + K}{(\alpha - \beta) \| X_n \|}, \end{aligned}$$

by Markov inequality. Now choose $M' \geq M$ such that $(E \| \mathcal{E}_1 \| + K)/(\alpha - \beta)M' = c < 1$. Then for $\| X_n \| > M'$,

$$P(\| X_{n+2} \| \geq \beta \| X_n \| \,|X_n) \geq 1 - c,$$

which in turn implies

$$P(\| X_{n+4} \| \geq \beta \| X_{n+2} \|, \| X_{n+2} \| \geq \beta \| X_n \| \,|X_n) \geq (1 - c)(1 - \beta^{-1}c)$$

and hence if $\| X_0 \| \geq M'$

$$\begin{aligned} & P(\| X_{2n} \| \geq \beta^n \| X_0 \| \,|X_0) \\ \geq\ & P(\| X_{2l} \| \geq \beta \| X_{2(l-1)} \|,\ l = 1, 2, ..., n \,|X_0) \\ \geq\ & \prod_{i=1}^{n} (1 - \beta^{-i+1}c) \\ \geq\ & (1 - c)^{(1-\beta^{-1})^{-1}} \end{aligned}$$

for all n. Hence

$$P(\| X_{2n+n_0} \| \to \infty \text{ as } n \to \infty | X_0) \geq (1-c)^{(1-\beta^{-1})^{-1}} P(\| X_{n_0} \| \geq M' | X_0) > 0,$$

which implies the non-ergodicity of $\{X_n\}$.

Note that if $\{X_n\}$ is a ϕ-irreducible Markov chain, then we have, by definition of ϕ-irreducibility, for every $x \in S$, $P(\| X_n(x) \| \geq r) > 0$ for some n if $\phi(B_r) > 0$ where $B_r = \{x : \| x \| \geq r\}$.

Following theorem 2 is for the process $\{X_n\}$ which is obtained recursively by iteration of random maps

$$X_{n+1} = \Gamma_{n+1}(X_n) \quad (i.e. \mathcal{E}_n \equiv 0) \tag{2.3}$$

where $\Gamma_n, n \geq 1$ are independent and identically distributed with values in Γ.

Theorem 2. Consider the process $\{X_n\}$ given by (2.3). Assume that there exists a positive measurable function L on Γ such that

$$\forall \gamma \in \Gamma, \| \gamma(x) \| \geq L(\gamma) \| x \| \qquad \textit{for} \quad \forall x \in S \tag{2.4}$$

and

$$E[\log L(\Gamma_1)] > 0. \tag{2.5}$$

Then $\{X_n\}$ is not ergodic if $P(\| X_0 \| = 0) < 1$.

Proof. Since $L(\Gamma_1), L(\Gamma_2), ...$ is a sequence of independent and identically distributed random variables, by the strong law of large numbers,

$$\frac{1}{n} \log \prod_{i=1}^{n} L(\Gamma_i) \to E \log L(\Gamma_1) \quad a.s. \quad \text{as} \quad n \to \infty. \tag{2.6}$$

Let $a := E[\log L(\Gamma_1)]$ and assume $0 < a < \infty$. By Egoroff's theorem, for $\epsilon = a/2 > 0$, there exists a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) \geq (1 - \epsilon)$ such that $\frac{1}{n} \log \prod_{i=1}^{n} L(\Gamma_i) \to E \log L(\Gamma_1)$ uniformly on Ω_0, and hence choose N such that if $n \geq N$,

$$e^{\frac{na}{2}} \leq \prod_{i=1}^{n} L(\Gamma_i) \leq e^{\frac{3na}{2}}, \qquad \forall \omega \in \Omega_0.$$

Therefore we have, whenever $\| X_0 \| > 0$,

$$\begin{aligned} & P(\| X_n \| \to \infty | X_0) \\ \geq \; & P(\prod_{i=1}^{n} L(\Gamma_i) \| X_0 \| \to \infty | X_0) \\ \geq \; & P(\prod_{i=1}^{n} L(\Gamma_i) \| X_0 \| \geq e^{\frac{na}{2}} \| X_0 \|, n = N, N+1, ... | X_0) \\ \geq \; & P(\Omega_0), \end{aligned}$$

which implies the non-ergodicity of $\{X_n\}$.

For the case $a = \infty$, we let $\alpha > 1$ be fixed. Then by (2.6), we may choose N such that for $n \geq N$,

$$\prod_{i=1}^{n} L(\Gamma_i) \geq e^{n\alpha} \qquad a.s.$$

and from which, by similar manner, we get the conclusion.

Remark 1. For the process $\{X_n\}$ given by (2.3), following is proved in Lee(1996b): Let Γ be a set of continuous functions on S and $E \parallel \Gamma_1(x_0) \parallel < \infty$ for some $x_0 \in S$. If there exists a nonnegative measurable function $L : \Gamma \longrightarrow R$ such that $\forall \gamma \in \Gamma, \parallel \gamma(x) - \gamma(y) \parallel \leq L(\gamma) \parallel x - y \parallel$, for $\forall x, y \in S$ and $E[L(\Gamma_1)] < 1$, then $\{X_n\}$ is ergodic.

Let $\Gamma^{(m)}$ be the set of all compositions $\gamma_m \cdots \gamma_1$ with $\gamma_i \in \Gamma, 1 \leq i \leq m$.

Remark 2. Non-ergodicity of X_n can be shown if the hypotheses (2.4) and (2.5)of theorem 2 are replaced by the following alternatives: If there exist positive integer m_0 and a positive measurable function L on $\Gamma^{(m_0)}$ such that

$$\forall \gamma \in \Gamma^{(m_0)}, \parallel \gamma(x) \parallel \geq L(\gamma) \parallel x \parallel \qquad \text{for} \quad \forall x \in S$$

and

$$E[\log L(\Gamma_1^{(m_0)})] > 0,$$

where $\Gamma_i^{(m_0)} = \Gamma_{im_0} \cdots \Gamma_{(i-1)m_0+1}$, the compositions of m_0 random maps $\Gamma_{(i-1)m_0+1}, \ldots, \Gamma_{im_0}$.

Corollary 1. Let $\{X_n\}$ be a Markov chain generated by the difference equation

$$X_{n+1} = f(X_n) + \mathcal{E}_{n+1}, \tag{2.7}$$

where f is a measurable function on S into S. If

$$\inf_{\|x\| \geq M} \frac{\parallel f(x) \parallel}{\parallel x \parallel} > 1 \quad \text{for some} \quad M \geq 0.$$

Then $\{X_n\}$ is not ergodic if for $r \geq 0$, $P(\parallel X_{n_0} \parallel \geq r | X_0) > 0$ for some n_0.

Proof. This is for the case of $m_0 = 1, \Gamma_1 \equiv f$ in theorem 1 .

3. EXAMPLES

Example 1. Consider the process

$$X_{n+1} = A_{n+1} X_n + b_{n+1} \quad (n \geq 0) \tag{3.1}$$

where A_n is a random linear operator on S and b_n is a random vector in S. Suppose $\{(A_n, b_n) : n \geq 1\}$ is independent and identically jointly distributed, and independent of X_0. $\{X_n\}$ generated by above recursion is often called an iterated function systems and bilinear processes or generalized random coefficient autoregressive processes can be represented as of the form of (3.1), where $S = R^k$ and $\{A_n\}$ is a random matrix process. Suppose $E\log^+ \| A_n \| < \infty$, $E\log^+ \| b_n \| < \infty$. Then it is shown that(Berger(1992))

$$\lim_{n\to\infty}(1/n)\log \| A_n\cdots A_1 \| = \lim_{n\to\infty}(1/n)E\log \| A_n\cdots A_1 \| = \alpha, \qquad (3.2)$$

where α is almost everywhere constant and if $\alpha < 0$, then $\{X_n\}$ is ergodic. If $E \| A_n \| < \infty$, $E \| b_n \| < \infty$ and $\alpha > 0$, then by (3.2), $\| A_{m_0}\cdots A_1 \| > 1$ a.s. for some m_0 and (2.2) hold and hence by theorem 1, $\{X_n\}$ is not ergodic if for $r \geq 0, P(\| X_n \| \geq r|X_0) > 0$ for some n.

Example 2. Let $\{X_n\}$ be given by (2.7) with state space $S = R$ and continuous f. Assume the common distribution of $\mathcal{E}_n$ has an absolutely continuous component with an almost everywhere positive dendity with respect to the Lebesgue measure λ. Also $E|\mathcal{E}_1| < \infty$. Then $\lambda-$irreducible chain $\{X_n\}$ is transient if one of the following (a)-(d) holds. $(a)\inf_{x>r}\frac{f(x)}{x} > 1$ for some $r > 0$. $(b)\inf_{x<-r}\frac{f(x)}{x} > 1$ for some $r > 0$. $(c)\sup_{|x|>r}\frac{f(x)}{x} < -1$ for some $r > 0$. $(d)\beta < 0$ and $\alpha\beta > 1$ when $\alpha = \lim_{x\to-\infty}\frac{f(x)}{x}$ and $\beta = \lim_{x\to\infty}\frac{f(x)}{x}$ exist. The proof for the cases (a) , (b) and (c) are similar to that of theorem 1. To prove (d) , by theorem 1, it is enough to show the existence of $M \geq 0$ satisfying

$$(i)\ \inf_{|x|\geq M}\frac{|f(f(x))|}{|x|} > 1$$

and

$$(ii)\ \sup_{|x|\geq M} E|f(f(x)+\mathcal{E}_1) - f(f(x))| < \infty.$$

For (ii) we refer to Lee,C.(1995).To show (i), first choose $\theta > 0$ such that $\beta+\theta < 0$, and $(\alpha+\theta)(\beta+\theta) > 1$. There exist $M_1 > 0$ and $M_2 > 0$ such that $(\alpha+\theta)z < f(z) < (\alpha-\theta)z$ if $z < -M_1$ and $(\beta-\theta)x < f(x) < (\beta+\theta)x$ if $x > M_2$. Now choose $M_1' > M_1$ so that $(\alpha+\theta)z > M_2$ if $z < -M_1'$. Hence if $z < -M_1'$, then $f(z) > M_2$ and $(\beta-\theta)f(z) < f(f(z)) < (\beta+\theta)f(z)$,and therefore $\frac{|f(f(z))|}{|z|} > (\alpha+\theta)(\beta+\theta) > 1$. Similarly, choose $M_2' > M_2$ so that if $x > M_2'$,then $(\beta+\theta)x < -M_1$. For $x > M_2'$, we have $(\alpha+\theta)f(x) < f(f(x)) < (\alpha-\theta)f(x)$, from which $\frac{f(f(x))}{x} > (\alpha+\theta)(\beta+\theta) > 1$ is obtained. The proof of (i) is completed if we take $M = \max\{M_1', M_2'\}$.

Example 3. Let $\{X_n\}$ be given by (2.7) with $S = R^k$ and

$$f(x) = \begin{cases} A_1 x & \text{if } x \in B \\ A_2 x & \text{otherwise} \end{cases}$$

where B is a bounded region of R^k, and A_1 and A_2 are k-dimensional matrices. If $E \parallel \mathcal{E}_n \parallel_u < \infty$ and $\rho(A_2) < 1$, then it is shown that $\{X_n\}$ is geometically ergodic provided that $\{X_n\}$ is $\phi-$irreducible and aperiodic. Here $\parallel . \parallel_u$ is Euclidean norm on R^k and $\rho(A_2)$ denotes the maximum eigenvalue of A_2 in absolute value. Let $E \parallel \mathcal{E}_n \parallel_u < \infty$ and let A_2 be a diagonalizable matrix with $\rho(A_2) > 1$. Then there are $k \times k$ matrices S and D such that $SA_2S^{-1} = D$ where D is the diagonal matrix whose diagonal entries are eigenvalues of A_2. Moreover R^k has a basis of normalized eigenvectors of A_2, say $\{a_i : 1 \leq i \leq k\}$ and any $x \in R^k$ can be uniquely represented as

$$x = \sum_{i=1}^{k} b_i(x) a_i.$$

We now let $\rho(A_2) = |\lambda_j| > 1$ and let $\parallel x \parallel = |b_j(Sx)|$. Then for $x \in B^c$, $\parallel f(x) \parallel = |b_j(Sf(x))| = |b_j(SA_2x)| = |b_j(DSx)| = |\lambda_j||b_j(Sx)|$ and hence $\parallel f(x) \parallel / \parallel x \parallel = |\lambda_j| > 1$. Corollary 1 together with $|b_j(Sx)| \leq K \parallel x \parallel_u$ for some constant $K < \infty$ ensures the non-ergodicity of $\{X_n\}$ provided for $r \geq 0, P(\parallel X_n \parallel \geq r | X_0) > 0$ for some n.

Example 4. We consider the k^{th} order equation

$$Y_{n+1} = h(Y_n, Y_{n-1}, \cdots, Y_{n-k+1}) + \mathcal{E}_{n+1}$$

where $h : R^k \longrightarrow R^1$ is measurable and $\{\mathcal{E}_n\}$ is a one-dimensional process with $E|\mathcal{E}_1| < \infty$. If we take $f(x) = (h(x), x_1, \cdots, x_{k-1})$ for $x = (x_1, \cdots, x_k)$ then $X_n = (Y_n, \cdots, Y_{n-k+1})$ is the form of (2.7). Suppose $|h(x)| \geq \sum_{i=1}^k a_i |x_i|$ for some $a_i \geq 0$ for large x. If $a_1 > 1$ or $a_k > 1$,then $\{X_n\}$ is not ergodic if any $r \geq 0$, $P(\parallel X_n \parallel \geq r) > 0$ for some n under the norm given below in each case. For the proof of the case $a_1 > 1$, we let $\parallel x \parallel = |x_1|$, then the relation $\parallel f(x) \parallel = |h(x)| \geq a_1 \parallel x \parallel$ induces the conclusion. For the case $a_k > 1$, choose $p_k > p_{k-1} > \cdots > p_1 > 0$ and $\theta > 1$ such that $p_{i+1}/p_i > \theta$ for $i = 1, 2, ..., k-1$ and $a_k p_1 / p_k > \theta$. Let

$$\parallel x \parallel = \max_{1 \leq i \leq k} \{p_i |x_i|\}.$$

Then the conclusion follows from the fact

$$\begin{aligned} \parallel f(x) \parallel &= \max\{p_1|h(x)|, p_2|x_1|, ..., p_k|x_{k-1}|\} \\ &\geq \max\{p_1 a_k |x_k|, \theta p_1 |x_1|, ..., \theta p_{k-1} |x_{k-1}|\} \\ &\geq \theta \parallel x \parallel . \end{aligned}$$

REFERENCES

(1) Berger, M. A.(1992), Random affine iterated function systems: mixing and encoding in Mark,A.P. and Volker, M.(eds.) *Diffusion processes and related problems in analysis, VolII. Stochastic flows.*, Birkhoüser, Boston.

(2) Bhattacharya, R. N. and Lee, C.(1995),Ergodicity of nonlinear first order autoregressive models, *Journal of Theoretical Probability*, **8**, No.1, 207-219.

(3) Bhattacharya, R. N. and Lee, O.(1988), Ergodicity and central limit theorems for a class of Markov processes, *Journal Multivariate Analysis*, **27**, No.1, 80-90.

(4) Bhattacharya, R. N. and Waymire,E.C.(1990), *Stochastic processes with applications*, Wiley, New York.

(5) Burton, R. M. and Rösler,U.(1995), An L_2 convergence theorem for random affine mappings, *Journal of Applied Probability*, **32**, 183-192.

(6) Feigen, D. F. and Tweedie,R.L.(1985), Random coefficient autoregressive processes:a markov chain analysis of stationarity and finiteness of monents, *Journal of Time Series Analysis*, **6**, 1-14.

(7) Lee, C.(1995), Some necessary conditions for ergodicity of nonlinear first order autoregressive models. (To appear)

(8) Lee, O.(1995), Geometric ergodicity and transience for nonlinear autoregressive models *Communications in Korean Mathematical Society*, **10**, No. 2, 409-417.

(9) Lee, O.(1996a), Limit theorems for some doubly stochastic processes, *Statistics and Probability Letters* (To appear).

(10) Lee, O(1996b), Limit theorems for Markov processes generated by iterations of random maps (submitted for publication).

(11) Meyn, S. P.(1989), Ergodicity theorem for discrete time stochastic systems using a stochastic Lyapunov function, *Siam Journal Control and Optimization*, **27**, No.6, 1409-1439.

(12) Nummelin, E.(1984), *General irreducible Markov chains and nonnegative operators*, Cambridge Univ. Press, Cambridge.

(13) Pham, D. T.(1986), The mixing property of bilinear and generalized coefficient autoregressive models. *Stochastic Processes and their Applications*, **23**, 291-300.

(14) Tjøstheim, D.(1986), Some doubly stochastic time series model, *Journal of Time Series Analysis*, **7**, 51-72.

(15) Tjøstheim, D.(1990), Nonlinear time series and Markov chains. *Advanced Applied Probability*, **22**, 587-611.

(16) Tong, H.(1990), *Nonlinear time series, a dynamical system approach.* Oxford Univ. Press, New York.

(17) Tweedie, R. L.(1983), Criteria for rates of convergence of Markov chains with applications to queueing theory, in J.F.C.Kingman and G.E.H. Reuter(eds.) *Papers in probability, Statistics and nonnegative operators*, Cambridge Univ. Press, Cambridge.