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Minimum Aberration 3^{n-k} Designs †

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Abstract

The minimum aberration criterion is commonly used for selecting good fractional factorial designs. In this paper we give some necessary conditions for 3^{n-k} fractional factorial designs. We obtain minimum aberration 3^{n-k} designs for $k = 2$ and any n. For $k > 2$, minimum aberration designs have not found yet. As an alternative, we select a design with minimum aberration among minimum-variance designs.

Key Words: Defining Relation; Generator; Minimum Aberration Design; Minimum-variance Design; Resolution; Word.

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1. INTRODUCTION

Factorial experiments are conducted for investigating k factors simultaneously. If total runs consist of all possible combinations of the levels of the different factors, the experiment is called a complete factorial experiment. Often the total runs are so large that it is neither economical nor feasible to carry out the complete factorial experiment. For these cases, fractional factorial experiments are used to enable to choose a fraction of the possible factorial combinations. Fractional factorial designs s^{n-k} — especially for $s = 2$ or $s = 3$ — are useful in a variety of experimental situations. A $1/s^k$ fraction of an s^n factorial design is called an s^{n-k} fractional factorial designs, or briefly, an s^{n-k} design.

There is lots of ways for selecting s^{n-k} designs. A key question in selecting designs is to develop a good criterion of a design. It had been a standard practice to choose a fractional factorial designs with maximum resolution. An s^{n-k} design in which no t-factor or lower order interaction is aliased with another u-factor or lower order interaction is called an s^{n-k} design of resolution $t + u + 1$. For example, a design of resolution III does not confound main effects with one another but does confound main effects with two-factor interactions, and a design of resolution IV does not confound main effects with two-factor interactions but does confound two-factor interactions with one another.

Since designs with the same resolution are not equally good, a more refined criterion called minimum aberration was introduced by Fries and Hunter(1980). When the experimenter has little knowledge about the relative sizes of the factorial effects,the minimum aberration criterion selects designs with good overall properties.

When $s = 2$, lots of works has been done. Minimum aberration 2^{n-k} designs were obtained for $k \leq 5$ [for $k \leq 4$, Chen and Wu(1991) ; $k = 5$, Chen(1992)] and several interesting properties were investigated. However, there are a few theoretical results on 3^{n-k} designs. In section 2 we give some necessary conditions for 3^{n-k} fractional factorial designs. In section 3 we obtain minimum aberration designs 3^{n-k} for $k = 2$ and any n. For $k > 2$, minimum aberration designs have not found yet. As an alternative, we select a design with minimum aberration among minimum-variance designs. The algorithm for selecting minimum variance 3^{n-k} designs was given by Chen and Wu(1991). Except few cases(see Chen, 1992), minimum aberration designs have minimum variance among the defining contrast lengths. It implies that the selected designs are fairly good for minimum aberration criteria.

Generally, we are used to prove the proposed designs have minimum aberration by ruling out possibilities of less aberration. For better understanding, consider 3^{4-3} fractional factorial designs. By the trial-and-error method, we can find that a feasible solution (0,3,10,0,...), which satisfies all the conditions in section 3, is not wordlength pattern of any design. It says that finding more necessary conditions is going to be a first step for constructing minimum aberration designs in future works for $k > 2$. The problem is not occurred for $k = 2$ because the feasible solutions for proposed designs given in section 4 have minimum aberration among all possible feasible solutions and are the wordlength patterns of the proposed designs.

2. AN EXAMPLE AND DEFINITIONS

In this section we use a simple example to explain and motivate the definitions described before.

A 3^{n-k} fractional factorial design has n variables each of 3 levels and 3^{n-k} runs. When $k = 0$, it is a complete factorial design, in which the 3^{n-k} runs consist of all possible level combinations of the n variables. These combinations form a $3^n \times n$ matrix with n independent columns.

Suppose in an experiment, we have five factors at three levels to be studied and only 27 selected runs are performed. Denote the five factors (so five independent columns) by capital letters A,B,C,D and E (or letters 1,2,3,4 and 5). Firstly, arrange 3^3 complete factorial combinations into first three $(= n - k)$ columns. A fractional factorial 3^{5-2} design can be characterized by

$$I = ABCD = BC^2DE, \tag{2.1}$$

where I is the column 0's. The equation (2.1) means that 0 is formed by adding columns A, B, C, D (modulo 3) or adding columns B, D, E and two times of column C (modulo 3). The equation (2.1) can be rewrite as

$$D^2 = ABC, \quad E^2 = BC^2D. \tag{2.2}$$

All 3^3 possible level combinations of the 5 variables can be formed as follows; two times of levels of factor D is formed by adding levels of factors

A, B, C (modulo 3) and two times of E is form by adding B, D and two times of column C (modulo 3). For example, 00122 which satisfies the equations (2.1) and (2.2) is one of the selected 3^3 level combinations of the 5 factors.

Another relation among the factors(columns) is

$$\begin{aligned} I &= ABCD \times BC^2DE = AB^2D^2E, \\ I &= ABCD \times (BC^2DE)^2 = AC^2E^2. \end{aligned}$$

All together we have

$$I = ABCD = BC^2DE = AB^2D^2E = AC^2E^2,$$

which forms a defining contrast subgroup with I being its identity of the 3^{5-2} design. The elements $ABCD$, BC^2DE, AB^2D^2E and AC^2E^2 are called words. All $4(=\frac{3^k-1}{2})$ words can be generated from $2(=k)$ independent generators $ABCD$ and BC^2DE. The symbols A, B and so on are called letters. The number of letters in a word is called the length of the word or wordlength.

Formally, a word is written as an $n-$ dimensional vector with components in a finite field of 3 elements F_3. From Chen(1992), we know that the following relations hold when s is a prime power.

For two words

$$w_1 = (a_1, a_2, \ldots, a_n) \quad and \quad w_2 = (b_1, b_2, \ldots, b_n),$$

with $a_i, b_i \in F_3$, their product is a word defined by

$$w_1 \times w_2 = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

The word w_1 and all its constant multiples

$$\lambda w_1 = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \quad for \quad any \quad \lambda \neq 0, \lambda \in F_3$$

are considered to be the same in the subgroup. The length of a word is the number of its nonzero components.

In general, an s^{n-k} fractional factorial design at s levels, s prime power, is defined by its defining contrasts subgroup, which is a subgroup of words generated under the multiplication $\times$.

Denote the above design by d_1. Let $A_i(d_1)$ be the number of words of length i in the defining contrasts subgroup for d_1 and the vector

$$W(d_1) = (A_1(d_1), A_2(d_1), \ldots) = (0, 0, 1, 3, 0, \ldots)$$

be its wordlength pattern. The resolution of d_1 is the smallest i with positive $A_i(d_1)$. Thus, d_1 has resolution III.

Given n and k, a 3^{n-k} design is not uniquely determined by its resolution. Consider another 3^{5-2} design :

$$d_2 : I = ABD = BC^2E = AB^2C^2DE = ACDE^2.$$

The design d_2 also have resolution III, but have different wordlength patterns

$$W(d_2) = (0, 0, 2, 1, 1 \ldots).$$

Since the first unequal components of $W(d_1)$ and $W(d_2)$ (third component) have the relation

$$W(d_2)[3] = 2 > W(d_1)[3] = 1,$$

we say that d_1 has less aberration than d_2. If one assumes that three-factor and high-order interactions are negligible, there is the greater amount of confounding between main effect and two-factor interaction in design d_2 than d_1. When comparing two designs using resolution as the criterion, one considers the lengths of the shortest word in each defining relation. If these lengths are equal, the two designs are regarded as being equivalent. With aberration as the criterion, however, one continues to examine the length of the next shortest word in each defining relation until one design is ranked superior to the other. Obviously, the resolution of a design is the smallest r such that $A_r(d) \neq 0$.

Definition 1. An s^{n-k} design has maximum resolution, if no other s^{n-k} design has larger resolution.

Let d_1 and d_2 be two s^{n-k} designs and r be the smallest i such that $A_i(d_1) \neq A_i(d_2)$. d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. An s^{n-k} design has less minimum aberration, if no other s^{n-k} design has less aberration.

Now, we define the ith moment as $M_i(d) = \sum_{j=1}^{\infty} j^i A_j(d)$.

Definition 2. Let m be the first i such that $M_i(d_1) \neq M_i(d_2)$. If m is odd and $M_m(d_1) < M_m(d_2)$, d_2 has better moments; if m is even and $M_m(d_1) < M_m(d_2)$, d_1 has better moments. A s^{n-k} fractional factorial design has optimal moments, if no other s^{n-k} design has better moments. A s^{n-k} design is minimum-variance design if it has maximum first moment and minimizes the second moment $M_2(d)$.

Without loss of generality, we assume throughout the paper that each of the n letters in an s^{n-k} design must appear in at least one word of its defining relation. Indeed, this is equivalent to maximizing the first moment.

3. PROPERTIES OF s^{n-k} FRACTIONAL FACTORIAL DESIGNS

Before proceeding with our main argument, we need necessary conditions for existence of an identity relationship, for which the lengths of the generators and their products are given. Brownlee, Kelly and Loraine(1948) and Chen(1992) gave the following conditions for 2^{n-k} designs.

Condition 1 : $\sum A_i = 2^{k-1}$.
Condition 2 : $\sum iA_i = n2^{k-1}$.
Condition 3 : Either all the words have even length or 2^{k-1} words have odd length.
Condition 4 : For any 2^{n-k} fractional factorial design, its second moment is divisible by $2^k - 1$.

The condition 1, 2 are easily extended to 3^{n-k} designs as follows :

Condition 1 : $\sum A_i = \frac{3^k-1}{2}$.
Condition 2 : $\sum iA_i = n3^{k-1}$.

Condition 3 can be extended by using different wordlength concepts. Define new wordlength, say power-wordlength, be the total number of powers of letters in a word. For example, a word ABC^2 has power-wordlength 4. Then condition 2 is extended to :

Condition 3 : Either all the words have power-wordlength which is divisible by 3, i.e., 0 (modulo 3)
or $(3^{k-1})/2$ of them are 0 (modulo 3),
$[3^{k-1}/2] + 1$ of them are 1 (modulo 3) and
$[3^{k-1}/2]$ of them are 2 (modulo 3),
where $[x]$ is the greatest integer not exceeding x.

Condition 3 may not useful, since power-wordlength patterns does not apply to conditions 1, 2 and 4.

Franklin(1984) gave a formula for second moments M_2 of a 3^{n-k} design as follows :

$$M_2 = 3^{k-2}[2n^2 + n] + 3^{k-2} \sum_j f_j^2, \tag{3.1}$$

We'll define nonnegative integers f_j in section 4. From the equation (3.1), we have the following necessary condition.

Condition 4 : For any 3^{n-k} fractional factorial design, its second moment is divisible by 3^{k-2}.

4. MINIMUM ABERRATION 3^{n-k} FRACTIONAL FACTORIAL DESIGNS

For $k = 2$, minimum aberration 3^{n-2} designs are characterized by the following defining relations for two generators. Let $n - 3 = 4m + r$, where $0 \le r < 4$. Now, use letters $1, 2, 3, \ldots$ and subscripts instead of capital letters $A, B, C, \ldots$ and superscripts. For example, $1_{(2)}$ means A^2.

For $r = 0$, define

$$\begin{aligned} I &= H_1(1)H_1(2)\cdots H_1(m)(4m+1)(4m+2) \\ &= H_2(1)H_2(2)\cdots H_2(m)(4m+2)(4m+3) \end{aligned} \tag{4.1}$$

for $r = 1$, define

$$\begin{aligned} I &= H_1(1)H_1(2)\cdots H_1(m)(4m+1)(4m+2)(4m+3) \\ &= H_2(1)H_2(2)\cdots H_2(m)(4m+2)(4m+3_{(2)})(4m+4), \end{aligned} \tag{4.2}$$

for $r = 2$, define

$$\begin{aligned} I &= H_1(1)H_1(2)\cdots H_1(m)(4m+1)(4m+2)(4m+3)(4m+4) \\ &= H_2(1)H_2(2)\cdots H_2(m)(4m+2)(4m+3_{(2)})(4m+4)(4m+5), \end{aligned} \tag{4.3}$$

for $r = 3$, define

$$\begin{aligned} I &= H_1(1)H_1(2)\cdots H_1(m)(4m+1)(4m+2)(4m+3)(4m+4)(4m+5) \\ &= H_2(1)H_2(2)\cdots H_2(m)(4m+2)(4m+3_{(2)})(4m+4)(4m+5_{(2)})(4m+6), \end{aligned} \quad (4.4)$$

where $H_i(p)$ is a letter which satisfies

$$H_1(p) = \begin{cases} (4p-3)(4p-2)(4p-1) & \text{for} \quad p = 1, 2, \ldots, m \\ 0 & \text{for} \quad p = 0. \end{cases}$$

and

$$H_2(p) = \begin{cases} (4p-2)(4p-1)_{(2)}(4p) & \text{for} \quad p = 1, 2, \ldots, m \\ 0 & \text{for} \quad p = 0. \end{cases}$$

We will prove that the proposed designs have minimum aberration and therefore maximum resolution. Their resolution can be summarized by the formula

$$R_3(n, 2) = \left[\frac{3n}{4}\right],$$

where $R_s(n, k)$ denote the maximum resolution of an s^{n-k} design.

First we consider the case of $n = 3$. For $n = 3$, the design defined by (4.1) has wordlength pattern $(0, 3, 1, 0, \ldots)$. Since we do not allow any words of length 1, we have

$$\text{Condition} \quad 1: A_2 + A_3 = 4, \quad (4.5)$$

$$\text{Condition} \quad 2: 2A_2 + 3A_3 = 9, \quad (4.6)$$

which make $A_2 = 3$, $A_3 = 1$. The solution is the unique and obviously the proposed design has minimum aberration.

For $n = 4$, all feasible solutions are $(0, 2, 0, 2, \ldots)$, $(0, 1, 2, 1, \ldots)$ and $(0, 0, 4, 0, \ldots)$. The last one has less aberration and the proposed design has that pattern. For $n = 5, 6$, the proofs are similar. Table 1 give the wordlength patterns of the minimum aberration 3^{n-2} designs for $3 \leq n \leq 6$. We see that the minimum aberration 3^{n-2} designs are quite different from the designs which is presented in the National Bureau of Standards 1959 tables.

To find optimal designs of large size through those of small size, the following Lemma 1, which was given by Chen and Wu (1991), will be used.

Lemma 1. For any 3^{n-2} design d_1 with wordlength pattern $W(d_1)$, there exists an $3^{(n+4)-2}$ design d_2 with wordlength pattern $W(d_2)$, such that $W(d_2) = lag(W(d_1), 3)$, where $lag(W, m) = (\underbrace{0, \ldots, 0}_{m}, W)$.

Table 1. Minimum aberration 3^{n-2} designs for $3 \leq n \leq 6$

n	Wordlenth pattern W_n
3	$W_3 = (0, 3, 1, 0, \ldots)$
4	$W_4 = (0, 0, 4, 0, \ldots)$
5	$W_5 = (0, 0, 1, 3, 0, \ldots)$
6	$W_6 = (0, 0, 0, 2, 2, 0, \ldots)$

For $n \geq 7$, the proofs are essentially the same as those for $3 \leq n \leq 6$ because, from Lemma 1, they involve the same type of equations. Now, we divide $n = 4m + q, \quad q = 3, 4, 5, 6$.

Start with $m = 1$. When $n = 7$, the equations become

$$C1 : A_5 + A_6 + A_7 = 4, \tag{4.7}$$

$$C2 : 5A_5 + 6A_6 + 7A_7 = 21. \tag{4.8}$$

By subtracting 3 $\times$ (4.7) from (4.8), we get

$$2A_5 + 3A_6 + 4A_7 = 9, \tag{4.9}$$

which forces $A_7 = 0$. Therefore the proof for $n = 7$ reduces to that $n = 3$. Thus, the wordlength pattern of 3^{7-2} design becomes

$$W_7 = (0, 0, 0, W_3, 0, \ldots).$$

When $n = 8$, the equations become

$$C1 : A_5 + A_6 + A_7 + A_8 = 4, \tag{4.10}$$

$$C2 : 5A_5 + 6A_6 + 7A_7 + 8A_8 = 24. \tag{4.11}$$

By subtracting $3\times$ (4.10) from (4.11), we get

$$2A_5 + 3A_6 + 4A_7 + 5A_8 = 12. \tag{4.12}$$

Suppose $A_8 \neq 0$. Then only solution is $(A_5 = 2, A_6 = 1, A_7 = 0, A_8 = 1)$ and achieve resolution V. However, this is not the maximum resolution from lemma 1. Thus, by letting $A_8 = 0$, the proof for $n = 8$ reduces to that for $n = 4$. And the wordlength pattern

$$W_8 = (0, 0, 0, W_4, 0, \ldots).$$

We can easily show that the same relations hold for $n = 9, 10$. That is,

$$W_9 = (0, 0, 0, W_5, 0, \ldots),$$
$$W_{10} = (0, 0, 0, W_6, 0, \ldots).$$

By continuing this ways for $m > 1$, we conclude that for $n = 4m + q$, $3 \leq q \leq 6$, the minimum aberration wordlength patterns are

$$W_{(4m+q)} = (\mathbf{0}_3, \mathbf{0}_3, \ldots, \mathbf{0}_3, W_q, 0, \ldots),$$

where $\mathbf{0}_3$ denotes $(0, 0, 0)$.

Two designs are said to be equivalent if one can be obtained from the other via sign changes in columns, rearrangement of runs and rearrangement of columns. Therefore lots of equivalent designs may exist. The design which is given in the equations (4.1) $\sim$ (4.4) is one of equivalent designs with wordlength patterns, in Table 1 for $3 \leq n \leq 6$ for example.

Generally, we are used to prove the proposed designs have minimum aberration by ruling out possibilities of less aberration. For better understanding, consider 3^{4-3} fractional factorial designs. By the trial-and-error method, we can find that a feasible solution $(0, 3, 10, 0, \ldots)$, which satisfies all the conditions in section 3, is not wordlength pattern of any design. It says that finding more necessary conditions is necessary for constructing minimum aberration designs for $k > 2$. As we saw, the problem is not occurred for $k = 2$ because the feasible solutions for proposed designs given in section 4 have minimum aberration among all possible feasible solutions and are the wordlength patterns of the proposed designs.

As an alternative, we select a design with minimum aberration among minimum-variance designs. The algorithm for choosing minimum-variance 3^{n-k} designs was given by Chen and Wu(1991). Except few cases (see Chen,1992), minimum aberration designs have minimum variance among the defining contrast lengths. It implies that the selected designs are fairly good for minimum aberration criteria.

Now, we sketch the algorithm for choosing optimal moments designs. Let us construct a matrix

$$H = \begin{bmatrix} I_k & B \\ B^t & B^t B \end{bmatrix},$$

where B is a $k \times \left(\frac{3^k-1}{2} - k\right)$ matrix which contains all distinct and nonzero linear combinations (modulo 3) of column vectors of I_k.

To define a 3^{n-k} fractional factorial design, let us divide the n letters into $\frac{3^k-1}{2}$ subsets. Let f_i be the number of letters in the ith subsets, such that $\sum_{i=1}^{(3^k-1)/2} f_i = n$. For each row vector $\mathbf{u_j}$ of H, form a word by combining all the letters in those subsets for which the component of $\mathbf{u_j}$ is 1. Hereafter, we regard $(H, \mathbf{f})$ as a design, where $\mathbf{f} = (\mathbf{f_1}, \ldots, \mathbf{f_{(3k-1)/2}})$ is the frequency vector of the design. Chen and Wu(1991) showed that for any minimum-variance 3^{n-k} design, f_i can only take two neighboring values q or $q+1$, where q is determined by $n = q\left[\frac{3^k-1}{2}\right] + r$, where $0 \le r < (3^k - 1)/2$. Then, there are $\begin{pmatrix} (3^k-1)/2 \\ n \end{pmatrix}$ ways of choosing f_i. Then select a design with minimum aberration among them.

5. CONCLUDING REMARKS

We give minimum aberration 3^{n-2} designs. For $k > 2$ minimum aberration designs have not found yet. As an alternative, we select a design with minimum aberration among minimum variance. This algorithm, however, rapidly becomes inefficient whenever k grows because there are lots of ways of choosing minimum-variance designs. In future works, we need to find exact minimum aberration designs.

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