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A Game with N Players

DaeHyeon Cho [1]

Abstract

In this paper we consider the gambler's ruin problem with N players and derive the formula for computing the expected ruin time when the initial fortunes of all N players are the same. And we present an example for the case of $N = 5$.

Key Words : Gambler's ruin problem; Martingale; Stopping time.

1. INTRODUCTION

In the gambler's ruin problem, we are mainly interested in the expected ruin time and the probability of the gambler's ultimate ruin. In a fair game of two players A_1 and A_2 with initial fortunes a_1 and a_2 units respectively, they continue playing the game until one of them is ruined, and it is well known that the expected ruin time is $E(T(a_1, a_2)) = a_1 a_2$, where $T(a_1, a_2)$ is the ruin time depending on the initial fortunes a_1 and a_2. We are interested in a generalization of this problem, from a game with two players to a game with

[1] Department of statistics, Inje University, Kim-Hae, Kyongnam, 621-170, Korea.

N players. Consider a game with N players such that each A_j has an initial fortune $a_j, 1 \leq j \leq N$. In each round, N cards containing one special card and $(N-1)$ common cards are distributed randomly to N players and the player who obtains the special card collects one unit from each of the others. We can see that, in each round, the probability that A_j will collect $N-1$ units from the others is $\frac{1}{N}$ and the probability that A_j will lose one unit is $\frac{N-1}{N}$. They continue playing until at least one of the N players is ruined. In a game with three players, Sandell(1989) showed that the expected ruin time equals $\frac{a_1a_2a_3}{(a_1+a_2+a_3-2)}$. Chang(1995) considered the case that $N=4$ and all four players have the same initial fortunes and he derived the expected ruin time of the game.

In this paper we consider a game with N players. Similar to the procedure of Chang(1995), we drive the computing formula of the expected ruin time of the special case that the initial fortunes of all players are the same, that is, when $a_1 = a_2 = a_3 = a_4 = \cdots = a_N = r$ for some positive integer r. In Section 2 we derive the explicit formula of the expected ruin time using the martingale theory and the optional stopping theorem. Our main result is a generalization of the result of Chang(1995). In Section 3 we present examples when $N = 5$ and $r = 5, 6, 7$.

2. THE EXPECTED RUIN TIME IN A GAME WITH N PLAYERS

For $n \geq 0$, let $Z_i(n)$ denote the fortune of player A_i, after n rounds, $i = 1, 2, 3, \cdots, N$, where the initial fortunes are $Z_i(0) = a_i, 1 \leq i \leq N$, and the total fortune is $a = a_1 + a_2 + a_3 + \cdots + a_N$. Let σ_n denote the $\sigma-$ algebra generated by random variables $\{Z_i(m), 1 \leq i \leq N, 0 \leq m \leq n\}$. Then we have the following theorem.

Theorem 1. Let

$$\begin{aligned} U_n &= \sum_{1\leq i<j}^{N} Z_i(n)Z_j(n) + \binom{N}{2}n, \\ V_n &= \sum_{1\leq i<j<k}^{N} Z_i(n)Z_j(n)Z_k(n) + \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}n. \end{aligned} \tag{2.1}$$

Then $\{U_n, \sigma_n, n \geq 0\}$ and $\{V_n, \sigma_n, n \geq 0\}$ are martingales.

proof. Clearly $E(|U_n|) < \infty$ and $E(|V_n|) < \infty$. So, it is sufficient to show that $E(U_{n+1}|\sigma_n) = U_n$ and $E(V_{n+1}|\sigma_n) = V_n$. Let

$$X_i(n+1) = Z_i(n+1) - Z_i(n), 1 \leq i \leq N. \tag{2.2}$$

Then we can easily see that

$$\text{for } 1 \leq i \leq N, \qquad P\{X_i(n) = N-1\} = \frac{1}{N}, P\{X_i(n) = -1\} = \frac{N-1}{N},$$

$$\text{for } 1 \leq i \neq j \leq N, \qquad P\{X_i(n+1) = -1, X_j(n+1) = N-1\} = \frac{1}{N},$$

$$P\{X_i(n+1) = -1, X_j(n+1) = -1\} = \frac{N-2}{N},$$

and for $1 \leq i \neq j \neq k \neq i \leq N$,

$$P\{X_i(n+1) = -1, X_j(n+1) = -1, X_k(n+1) = N-1\} = \frac{1}{N},$$

$$P\{X_i(n+1) = -1, X_j(n+1) = -1, X_k(n+1) = -1\} = \frac{N-3}{N}.$$

From the above facts we can obtain the following expected values

$$\begin{aligned} &E[X_i(n+1)] = 0, \ 1 \leq i \leq N, \\ &E[X_i(n+1)X_j(n+1)] = -1, \ 1 \leq i \neq j \leq N, \\ &E[X_i(n+1)X_j(n+1)X_k(n+1)] = 2, \ 1 \leq i \neq j \neq k \neq i \leq N. \end{aligned} \tag{2.3}$$

Using (2.1) and noting that $Z_i(n) \in \sigma_n$ and $Z_i(n)$ and $X_j(n+1)$ are independent for $1 \leq i \neq j \leq N$, and that $Z_i(n)Z_j(n)$ and $X_k(n+1)$, $1 \leq i \neq j \neq k \neq i \leq N$, are independent, we have

$$\begin{aligned} E(U_{n+1}|\sigma_n) &= \sum_{1 \leq i < j}^{N} E[(X_i(n+1) + Z_i(n))(X_j(n+1) + Z_j(n))|\sigma_n] \\ &+ \binom{N}{2}(n+1) \\ &= \sum_{1 \leq i < j}^{N} \{E[X_i(n+1)X_j(n+1)] + Z_i(n)Z_j(n)\} + \binom{N}{2}(n+1) \\ &= -\binom{N}{2} + \sum_{1 \leq i < j}^{N} Z_i(n)Z_j(n) + \binom{N}{2}(n+1) = U_n, \end{aligned}$$

$$
\begin{aligned}
E(V_{n+1}|\sigma_n) &= \sum_{1\le i<j<k}^{N} E[(X_i(n+1)+Z_i(n))(X_j(n+1)+Z_j(n)) \\
&\quad (X_k(n+1)+Z_k(n))|\sigma_n] + \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}(n+1) \\
&= \sum_{1\le i<j<k}^{N} E[(X_i(n+1)X_j(n+1)X_k(n+1) \\
&+ X_i(n+1)X_j(n+1)Z_k(n) + X_i(n+1)Z_j(n)X_k(n+1) \\
&+ X_i(n+1)Z_j(n)Z_k(n) + Z_i(n)X_j(n+1)X_k(n+1) \\
&+ Z_i(n)X_j(n+1)Z_k(n) + Z_i(n)Z_j(n)X_k(n+1) \\
&+ Z_i(n)Z_j(n)Z_k(n)|\sigma_n] + \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}(n+1) \\
&= \left\{-\frac{3}{N}\binom{N}{3}a + 2\binom{N}{3}\right\} + \sum_{1\le i<j<k}^{N} Z_i(n)Z_j(n)Z_k(n) \\
&+ \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}(n+1) \\
&= V_n.
\end{aligned}
$$

Therefore $\{U_n, \sigma_n, n \ge 0\}$ and $\{V_n, \sigma_n, n \ge 0\}$ are martingales.

Let $T = \inf\{n :$ at least one of $Z_i(n)$ is zero, $i = 1, 2, 3, \cdots, N\}$. Then T is a stopping time for both martingales $\{U_n, \sigma_n, n \ge 0\}$ and $\{V_n, \sigma_n,\ n \ge 0\}$ since the event

$$
\{T = t\} = \left\{\bigcap_{1\le j\le N}\ \bigcap_{0\le n\le t-1} [Z_j(n) > 0]\right\} \cap \left\{\bigcup_{1\le j\le N} [Z_j(t) = 0]\right\} \in \sigma_t. \tag{2.4}
$$

To apply the optional stopping theorem in Grimmet and Stirzaker(1982) it is essential to show that the sufficient conditions of that theorem are satisfied.

Theorem 2. For the above two martingales $\{U_n\}$, $\{V_n\}$ and stopping time T the followings hold.
(a) $P(T < \infty) = 1$.
(b) $E(|U_T|) < \infty$ and $E(|V_T|) < \infty$.
(c) $\lim_{n\to\infty} E(|U_n||T > n)P(T > n) = 0$.
(d) $\lim_{n\to\infty} E(|V_n||T > n)P(T > n) = 0$.

Proof. Let $D_k = Z_1(ak) - Z_1(a(k-1)) = X_1(a(k-1)+1) + \cdots + X_1(ak)$ for $k = 1, 2, \cdots$. Then

$$\{T > ai\} \subseteq \bigcap_{k=1}^{i}\{0 < Z_1(ak) < a\} \subseteq \bigcap_{k=1}^{i}(D_k < a). \qquad (2.5)$$

Since $(X_1(i) = N-1, for\ i = a(k-1)+1, \cdots, ak) \subset (D_k \geq a)$, we have $P(D_k < a) \leq 1 - P(X_1(a(k-1)+1) = \cdots = X_1(ak) = N-1) = 1 - (\frac{1}{N})^a$. Noting the fact that $D_1, D_2, \cdots$ are independent, it follows from (2.5) that

$$P(T > ai) \leq \left\{1 - \left(\frac{1}{N}\right)^a\right\}^i. \qquad (2.6)$$

From (2.6) we can see that (a) holds and T has finite moments of all orders.

Since $0 \leq U_T = \binom{N}{2}T$ and $0 \leq V_T = \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}T$, so clearly $E(|U_T|) < \infty$ and $E(|V_T|) < \infty$, that is, (b) holds.

It therefore only remains to be shown that (c) and (d) hold. If $T > n$ then $0 < Z_i(n) < a$, for $i = 1, 2, \cdots, N$. From (2.1) and (2.6) it follows that

$$\begin{aligned}
0 \leq \quad & E(|U_n||T > n)P(T > n) \leq \left\{\binom{N}{2}a^2 + \binom{N}{2}n\right\}\left\{1 - \left(\frac{1}{N}\right)^a\right\}^{\frac{n}{a}}, \\
0 \leq \quad & E(|V_n||T > n)P(T > n) \\
\leq \quad & \left[\binom{N}{3}a^3 + \left\{\frac{3}{N}\binom{N}{3}a - 2\binom{N}{3}\right\}n\right]\left\{1 - \left(\frac{1}{N}\right)^a\right\}^{\frac{n}{a}}.
\end{aligned}$$

Letting $n \to \infty$, we can obtain the result of (c) and (d).

Applying the optional stopping theorem in Grimmet and Stirzaker(1982), we obtain the following expressions for the expected ruin time.

Theorem 3.

$$E(T) = \frac{\sum_{1\leq i<j}^{N} a_i a_j - E_1}{\binom{N}{2}},$$

where

$$E_1 = E\left(\sum_{1\leq i<j}^{N} Z_i(T)Z_j(T)\right).$$

$$E(T)=\frac{\sum_{1\le i<j<k}^{N}a_ia_ja_k-E_2}{\frac{3}{N}\binom{N}{3}a-2\binom{N}{3}},$$

where

$$E_2=E(\sum_{1\le i<j<k}^{N}Z_i(T)Z_j(T)Z_k(T)).$$

proof. Now we easily see that

$$E(U_0)=\sum_{1\le i<j}^{N}a_ia_j,\quad E(V_0)=\sum_{1\le i<j<k}^{N}a_ia_ja_k.$$

And

$$\begin{aligned}E(U_T)=&\quad E(\sum_{1\le i<j}^{N}Z_i(T)Z_j(T))+\binom{N}{2}E(T),\\E(V_T)=&\quad E(\sum_{1\le i<j<k}^{N}Z_i(T)Z_j(T)Z_k(T))\\&+\left\{\frac{3}{N}\binom{N}{3}a-2\binom{N}{3}\right\}E(T).\end{aligned}$$

Applying the optional stopping theorem $E(U_T)=E(U_0)$ and $E(V_T)=E(V_0)$, we obtain the above expressions for the expected ruin time.

In order to compute the expected ruin time, it is necessary to find out the value of E_1 or E_2. Consider the special case that the initial fortunes of all players are the same, that is, $a_i=r, i=1,2,\cdots,N$. Then $a=Nr$, and $(Z_1(T),Z_2(T),\cdots,Z_N(T))=N(r_1,r_2,\cdots,r_N)$, where $r_1+r_2+\cdots+r_N=r$, $\min\{r_1,r_2,\cdots,r_N\}=0$. For any positive integer r, let $R(r)$ denote the set of all distinct N-tuples $R_i=(r_{1i},\cdots r_{(N-1)i},0)$, with $r_{1i}+\cdots+r_{(N-1)i}=r$, $r_{1i}\ge\cdots\ge r_{(N-1)i}\ge 0$, where $1\le i\le u$ and $u=u(r)$ is the number of elements in the set $R(r)$. Let $B(r)$ denote the set of all distinct N-tuples $B_k=(b_{1k},b_{2k},\cdots,b_{Nk})$ with $b_{1k}+b_{2k}+\cdots+b_{Nk}=r$, $b_{1k}\ge b_{2k}\ge\cdots\ge b_{Nk}\ge 1$, $1\le k\le v$, and $v=v(r)$ is the number of elements in the set $B(r)$. And let S_N denote the set of all permutations of $\{1,2,\cdots,N\}$. For each $R_i=(r_{1i},\cdots r_{(N-1)i},0)\in R(r)$, let $[R_i]=[r_{1i},\cdots,r_{(N-1)i},0]$ denote the union over all permutations $s \in S_N$ of the events that after T rounds, the fortune of player $A_{s(j)}$ will be $Nr_{ji}, 1 \leq j \leq N$, where $r_{Ni} = 0$.

By Theorem 3 we have the following expressions for $E(T)$:

Corollary 4.

$$E(T) = \frac{\binom{N}{2} r^2 - E_1}{\binom{N}{2}},$$

where

$$E_1 = N^2 \sum_{i=1}^{u} \Big(\sum_{1 \leq j < k}^{N-1} r_{ji} r_{ki} \Big) P([R_i]).$$

$$E(T) = \frac{\binom{N}{3} r^3 - E_2}{\frac{3}{N}\binom{N}{3} a - 2\binom{N}{3}},$$

where

$$E_2 = N^3 \sum_{i=1}^{u} \Big(\sum_{1 \leq j < k < l}^{N-1} r_{ji} r_{ki} r_{li} \Big) P([R_i]).$$

To obtain $E(T)$, we must calculate the value of E_1 or E_2. For any N-tuple $Q = (q_1, q_2, \cdots, q_N)$ with $q_1 + q_2 + \cdots + q_N = q$ and all $q_i \geq 0$, let Q^* denote the event that in q rounds, the player A_i will win q_i times, $1 \leq i \leq N$. Then, we have

$$P(Q^*) = \{q!/(q_1! q_2! \cdots q_N!)\}/N^q. \tag{2.7}$$

If $q_i < 0$ for some i, let Q^* be the empty set. Let Q^{**} denote the union over all permutations $s \in S_N$ of the events that in q rounds, the player $A_{s(i)}$ will win q_i times, $1 \leq i \leq N$, that is, the event that the N players, arranged in any particular order, will win $q_1, q_2, \cdots, q_N$ times, respectively. Then, it can be seen that

$$P(Q^{**}) = \{q!/(q_1! q_2! \cdots q_N!)\}\{N!/f(q_1, q_2, \cdots, q_N)\}/N^q, \tag{2.8}$$

where $f(q_1, q_2, \cdots, q_N) = f(Q) = f_0! f_1! \cdots f_q!$, and f_m denotes the number of $m's$ in N-tuple Q. For $1 \leq i, j \leq v$, define the conditional probabilities $a_{ij} = P((B_j + (1, \cdots, 1))^{**} | B_i^{**})$. And let $w = (w_1, \cdots w_v), d = (d_1, \cdots, d_v)$, where $w_k = P(B_k^{**})$ and $d_k = P(((r_1, \cdots, r_{N-1}, 0) + (1, \cdots, 1))^{**} | B_k^{**})$, for $1 \le k \le v$. Then with the similar arguments in Chang(1995) we have the following theorem.

Theorem 5.

$$\begin{aligned} P([r_1, \cdots, r_{N-1}, 0]) &= P((r_1, \cdots, r_{N-1}, 0)^{**}) + wd^t + wAd^t + wA^2d^t + \cdots \\ &= P((r_1, \cdots, r_{N-1}, 0)^{**}) + w(I_v + A + A^2 + \cdots)d^t \\ &= P((r_1, \cdots, r_{N-1}, 0)^{**}) + w(I_v - A)^{-1}d^t, \end{aligned} \tag{2.9}$$

where d^t denote the transpose of the vector d and

$$\begin{aligned} d_k &= P(((r_1, \cdots, r_{N-1}, 0) + (1, \cdots, 1))^{**} | B_k^{**}) \\ &= \sum_{s \in S_N} P((r_{s(1)} + 1 - b_{1_k}, \cdots, r_{s(N)} + 1 - b_{N_k})^*) / f(r_1, \cdots, r_{N-1}, 0), \end{aligned} \tag{2.10}$$

$$\begin{aligned} a_{km} &= P((B_m + (1, \cdots, 1))^{**} | B_k^*) \\ &= \sum_{s \in S_N} P((b_{s(1)_m} + 1 - b_{1_k}, \cdots, b_{s(N)_m} + 1 - b_{N_k})^*) / f(b_{1_m}, \cdots, b_{N_m}). \end{aligned} \tag{2.11}$$

Thus we obtain the value of $P([R_i]), 1 \le i \le u$, and $E(T)$. And if we use Corollary 4 and take $N = 4$, we can get the results of Chang(1995).

3. EXAMPLES

Using the results in Section 2, consider the computation of the expected ruin time when $N = 5$ and $r = 5, 6, 7$. In this case, Corollary 4 can be represented as follows.

Corollary 4 ′.

$$E(T) = \frac{(10r^2 - E_1)}{10}, \tag{3.1}$$

where

$$E_1 = \sum_{i=1}^{u} 5^2 (\sum_{1 \le j < k}^{4} r_{ji} r_{ki}) P([R_i]).$$

$$E(T) = \frac{(10r^3 - E_2)}{(30r - 20)}, \tag{3.2}$$

where

$$E_2 = \sum_{i=1}^{u} 5^3 \Big(\sum_{1 \le j < k < l}^{4} r_{ji} r_{ki} r_{li} \Big) P([R_i]).$$

To obtain $E(T)$, it is necessary to find the value of $P[R_i]'s$. If $r = 5$, $R_i's$ are (5,0,0,0,0), (4,1,0,0,0), (3,2,0,0,0), (3,1,1,0,0),(2,2,1,0,0), (2,1,1,1,0). And $P[5,0,0,0,0] = \frac{1}{601}$, $P[4,1,0,0,0] = \frac{20}{601}$, $P[3,2,0,0,0] = \frac{40}{601}$, $P[3,1,1,0,0] = \frac{120}{601}$, $P[2,2,1,0,0] = \frac{180}{601}$, $P[2,1,1,1,0] = \frac{240}{601}$. $E_1 = \frac{25 \cdot 4760}{601}$, $E_2 = \frac{125 \cdot 2760}{601}$.

So it follows from (3.1) or (3.2) that $E(T) = \frac{3125}{601} = 5.19967$.

If $r = 6$, $P[6,0,0,0,0] = \frac{1}{2765}$, $P[5,1,0,0,0] = \frac{24}{2765}$, $P[4,2,0,0,0] = \frac{60}{2765}$, $P[4,1,1,0,0] = \frac{180}{2765}$, $P[3,3,0,0,0] = \frac{40}{2765}$, $P[3,2,1,0,0] = \frac{720}{2765}$, $P[3,1,1,1,0] = \frac{480}{2765}$, $P[2,2,2,0,0] = \frac{180}{2765}$, $P[2,2,1,1,0] = \frac{1080}{2765}$, $E_1 = \frac{811500}{2765}$ and $E_2 = \frac{3030000}{2765}$.

So from (3.1) or (3.2), $E(T) = \frac{1}{10}(10 \cdot 6^2 - \frac{811500}{2765}) = 6.651$

And if $r = 7$, then $P[7,0,0,0,0] = 0.00032$, $P[6,1,0,0,0] = 0.002392$, $P[5,2,0,0,0] = 0.00887$, $P[5,1,1,0,0] = 0.021028$, $P[4,3,0,0,0] = 0.01216$, $P[4,2,1,0,0] = 0.10494$, $P[4,1,1,1,0] = 0.0736$, $P[3,3,1,0,0] = 0.06966$, $P[3,2,2,0,0] = 0.10414$, $P[3,2,1,1,0] = 0.40186$, $P[2,2,2,1,0] = 0.20676$. So it follows from (3.1) or (3.2) that $E(T) = 8.3057$.

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