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Parameter Estimation for an Infinite Dimensional Stochastic Differential Equation

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Abstract

When we deal with a Hilbert space-valued Stochastic Differential Equation (SDE) (or Stochastic Partial Differential Equation (SPDE)), depending on some unknown parameters, the solution usually has a Fourier series expansion. In this situation we consider the maximum likelihood method for the statistical estimation problem and derive the asymptotic properties (consistency and normality) of the Maximum Likelihood Estimator (MLE).

Key Words : Stochastic Differential Equation; Maximum Likelihood Estimator; Radon-Nikodym Derivative.

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1. INTRODUCTION

In recent years there has been a great deal of attention devoted to parameter estimation for finite dimensional SDE (see e.g. Kutoyants(1984) ([4]) and Kutoyants(1994) ([3]) and references cited there). Parameter eastmation problems for SPDE's have been studied by Hübner and Rozovskii(1994) ([2]). They investigated the asymptotic properties of the MLE for a parameter occurring in parabolic SPDE's with an initial boundary condition,

$$du(t,x) = (A_0 + \theta A_1)u(t,x)dt + dW(t,x) \tag{1.1}$$

where A_0 and A_1 are partial differential operators, $\theta \in R$ is an unknown parameter to be estimated, and $W(t,x)$ is a cylindrical Brownian motion on $L^2(G)$, G being a bounded domain in R^d. They computed the MLE $\hat{\theta}_N$ based on a finite dimensional projection of the solution $(u(t))$ to the above SPDE where N denotes an index for a finite dimensional projection of the solution $(u(t))$. In particular, they establish the relation between the asymptotic properties of the MLE $\hat{\theta}_N$ as $N \to \infty$ and the conditions on the orders of partial differential operators A_0 and A_1.

In this paper we consider the problem of parameter estimation for the Hilbert space-valued SDE's, corresponding to an unknown parameter θ. Using Kakutani's Theorem, we will first prove that two probability measures, corresponding to two different parameters, are mutually absolutely continuous, and consider the asymptotic properties of the MLE $\hat{\theta}_T$, say, as the observation time $T \to \infty$.

Let H be a separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $(\mathcal{F}_t), t \in [0,T]$, satisfying the usual conditions. Let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion on H.

In the present work, we consider the parameter estimation for equations of type

$$\begin{aligned} dX_t &= -LX_t dt + \theta A X_t dt + B dW_t \quad \text{for} \quad 0 < t \le T \\ X_0 &= 0. \end{aligned} \tag{1.2}$$

for an H-valued process where L is an unbounded operator. The solution (X_t) to the SDE(1.2), depending on an unknown parameter, has a Fourier series expansion, i.e.,

$$X_t = \sum_{k=1}^{\infty} X_{t,k}\phi_k$$

where $X_{t,k} :=< X_t, \phi_k >$ and $\{\phi_k\}$ is a CONS(Complete Orthonormal System) in H. Let $X_t^N := \sum_{k=1}^N X_{t,k}\phi_k$, i.e., X_t^N is the projection of the solution X_t onto the subspace $\Phi^N := span\{\phi_1, \ldots, \phi_N\}$.

Let us introduce the following notations:

(1) $C[0,T]$ $((C[0,T], \Phi^N)$, $(C[0,T], H))$ denotes the set of all continuous functions defined on $[0,T]$, taking values in R (Φ^N, H).

(2) $\tilde{\mathcal{B}}_T$ $(\mathcal{B}_T^N, \mathcal{B}_T)$ is the Borel σ-algebra on the space $C[0,T]$ $((C[0,T], \Phi^N)$, $(C[0,T], H))$.

(3) $\tilde{P}_{\theta,k}$, (P_θ^N, P_θ) is the measure induced on the space of realizations $\tilde{\mathcal{B}}_T$ $(\mathcal{B}_T^N, \mathcal{B}_T)$ by the processes X_k (X^N, X) where $X = (X_t)$ is the solution to the SDE (1.2), and $X_k = (< X_t, \phi_k >)$ and $X^N = (X_t^N)$ are as given above.

(4) $E_{\tilde{P}_{\theta,k}}$ $(E_{P_\theta^N}, E_{P_\theta}, E)$ denotes the expectation with respect to the measure $\tilde{P}_{\theta,k}$ $(P_\theta^N, P_\theta, P)$.

2. ABSOLUTE CONTINUITY AND RADON-NIKODYM DERIVATIVE

In this section we consider the statistical estimation problem of unknown parameter θ in the SDE (1.2) and derive the Radon-Nikodym derivative for estimating θ. In the SDE (1.2) the operator L^{-1} is a bounded positive self-adjoint with discrete spectrum. Let $\{\phi_k\}$ be the eigenfunctions of L^{-1} which constitute a CONS in H and let $\{\lambda_k^{-1}\}$ be the corresponding eigenvalues. We also assume that A and B are self-adjoint operators in $\mathcal{L}(H, H)$.

Throughout what follows, it will be assumed that

(C1) $A\phi_k = a_k\phi_k$, $B\phi_k = b_k\phi_k$, $a_k > 0, b_k > 0$ for all k.

(C2) The sequences $\{a_k\}$ and $\{b_k\}$ given in (C1) satisfy

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{b_k^2}{\lambda_k^\gamma} < \infty, \quad \text{for some } \gamma, \quad 0 < \gamma < 1.$$

It follows by the assumptions (C1) and (C2) that there exists a unique, continuous H-valued solution (X_t) to the SDE (1.2) for fixed θ (see Theorem 2.7 in Bhatt et al.(1993) [1]). Note that the solution (X_t) to the SDE (1.2) can be written as

$$X_t = \sum_{k=1}^{\infty} < X_t, \phi_k > \phi_k$$

where the Fourier coefficients satisfy the equations:

$$d < X_t, \phi_k > = < \theta A X_t - L X_t, \phi_k > dt + < B\phi_k, \phi_k > dW_t(\phi_k)$$

Let $X_{t,k} := < X_t, \phi_k >$. Then for all k,

$$dX_{t,k} = (\theta a_k - \lambda_k) X_{t,k} dt + b_k dW_t(\phi_k) \tag{2.1}$$

where $W_t(\phi_k)$ is a sequence of independent real Brownian motions. Note that (2.1) has the explicit solution

$$X_{t,k} = b_k \int_0^t e^{-(\lambda_k - \theta a_k)(t-s)} \, dW_s(\phi_k). \tag{2.2}$$

From (C1), $\{\phi_k\}$ is a CONS for H consisting of eigenvectors for $L^{-1}A$, i.e.,

$$L^{-1} A \phi_k = \frac{a_k}{\lambda_k}.$$

Since the operators L and A are self-adjoint operators, $sp\{L^{-1}A\}$ is contained in the set of real numbers where $sp\{L^{-1}A\}$ denotes the spectrum of $L^{-1}A$. Then we further assume that $\theta \in R \setminus sp\{L^{-1}A\}$. Let $\Theta := R \setminus sp\{L^{-1}A\}$.

Let us fix $\theta_0 \in \Theta$. It is well known (see Liptser and Shiryayev (1977)([5])) that the likelihood ratio is

$$\begin{aligned} & \frac{d\tilde{P}_{\theta,k}}{d\tilde{P}_{\theta_0,k}}(X_k) \\ = \; & \exp\left\{ (\theta - \theta_0) \frac{a_k}{b_k^2} \int_0^T X_{t,k} \, dX_{t,k} - \frac{\theta^2 - \theta_0^2}{2} \left(\frac{a_k^2}{b_k^2} \right) \int_0^T (X_{t,k})^2 \, dt \right. \\ & \left. + (\theta - \theta_0) \frac{a_k \lambda_k}{b_k^2} \int_0^T (X_{t,k})^2 \, dt \right\}. \end{aligned} \tag{2.3}$$

Note that under P, (X_k) is a sequence of independent random variables with values in $C[0,T]$. Let

$$\tilde{P}_\theta^N = \otimes_{k=1}^N \tilde{P}_{\theta,k} \quad \text{and} \quad \tilde{X}^N = (X_{t,1},\dots,X_{t,N})_{t\in[0,T]} \in \otimes_{k=1}^N C[0,T].$$

Then

$$Z_N := \frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N) = \prod_{k=1}^N q_k(X_k)$$

where

$$q_k(X_k) = \frac{d\tilde{P}_{\theta,k}}{d\tilde{P}_{\theta_0,k}}(X_k).$$

Let us now prove the following simple Lemma.

Lemma 1. For $N = 1,2,\dots$, we have

$$\frac{dP_\theta^N}{dP_{\theta_0}^N}(X^N) = \frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N)$$

where $X^N = \left(X_t^N := \sum_{k=1}^N X_{t,k}\phi_k\right)_{t\in[0,T]} \in C([0,T],\Phi^N)$.

Proof. Define a mapping T of $(\otimes_{k=1}^N C[0,T], \otimes_{k=1}^N \tilde{\mathcal{B}}_T)$ into $(C([0,T],\Phi^N), \mathcal{B}_T^N)$ as follows:

$$T(X_1,\dots,X_N) = T(\tilde{X}^N) = \sum_{k=1}^N X_k\phi_k = X^N.$$

It is obvious that T is a measurable mapping. For every $A \in \mathcal{B}_T^N$, then

$$\begin{aligned}
P_\theta^N(A) &= \tilde{P}_\theta^N(T^{-1}A)\\
&= \int_{T^{-1}A} \frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N)d\tilde{P}_{\theta_0}^N\\
&= E_{\tilde{P}_{\theta_0}^N}\left[\mathbf{1}_{T^{-1}A}(\tilde{X}^N)\frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N)\right]\\
&= E_{\tilde{P}_{\theta_0}^N}\left[\mathbf{1}_A(T\tilde{X}^N)\frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N)\right]\\
&= E_{\tilde{P}_{\theta_0}^N}\left\{E_{\tilde{P}_{\theta_0}^N}\left[\mathbf{1}_A(T\tilde{X}^N)\frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^{N,\theta_0})\middle|T\tilde{X}^N\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= E_{\tilde{P}^N_{\theta_0}} \left\{ \mathbf{1}_A (T\tilde{X}^N) E_{\tilde{P}^N_{\theta_0}} \left[\frac{d\tilde{P}^N_\theta}{d\tilde{P}^N_{\theta_0}} (\tilde{X}^N) \Big| T\tilde{X}^N \right] \right\} \\
&= \int_{C([0,T],\Phi^N)} \left\{ \mathbf{1}_A (X^N) E_{\tilde{P}^N_{\theta_0}} \left[\frac{d\tilde{P}^N_\theta}{d\tilde{P}^N_{\theta_0}} (\tilde{X}^N) \Big| X^N \right] \right\} d\tilde{P}^N_{\theta_0} \circ T^{-1} \\
&= \int_A E \left[\frac{d\tilde{P}^N_\theta}{d\tilde{P}^N_{\theta_0}} (\tilde{X}^N) \Big| X^N \right] dP^N_{\theta_0}
\end{aligned}
$$

where E denotes averaging with respect to the original probability P. Hence we have

$$
\frac{dP^N_\theta}{dP^N_{\theta_0}}(X^N) = E \left[\frac{d\tilde{P}^N_\theta}{d\tilde{P}^N_{\theta_0}} (\tilde{X}^N) \Big| X^N \right] = \frac{d\tilde{P}^N_\theta}{d\tilde{P}^N_{\theta_0}} (\tilde{X}^N).
$$

It follows from Lemma 1 that the measure P^N_θ is absolutely continuous with respect to $P^N_{\theta_0}$, i.e., P_θ is locally absolutely continuous with respect to P_{θ_0}, and the likelihood ratio is

$$
\begin{aligned}
Z_N &:= \frac{dP^N_\theta}{dP^N_{\theta_0}}(X^N) \\
&= \exp \left\{ (\theta - \theta_0) \sum_{k=1}^N \frac{a_k}{b_k^2} \int_0^T X_{t,k}\, dX_{t,k} - \frac{\theta^2 - \theta_0^2}{2} \sum_{k=1}^N \frac{a_k^2}{b_k^2} \int_0^T (X_{t,k})^2\, dt \right. \\
&\quad \left. + (\theta - \theta_0) \sum_{k=1}^N \frac{a_k \lambda_k}{b_k^2} \int_0^T (X_{t,k})^2\, dt \right\}. \qquad (2.4)
\end{aligned}
$$

The Fisher information corresponding to $dP^N_\theta / dP^N_{\theta_0}$ is given by

$$
\begin{aligned}
I^N(\theta) &:= E \left[\left(\frac{\partial}{\partial \theta} \log \frac{dP^N_\theta}{dP^N_{\theta_0}} (X^N) \right)^2 \right] \\
&= E \left[\left(\sum_{k=1}^N \frac{a_k}{b_k^2} \int_0^T X_{t,k} (dX_{t,k} - (\theta a_k - \lambda_k) X_{t,k} dt) \right)^2 \right] \\
&= E \left[\left(\sum_{k=1}^N \frac{a_k}{b_k} \int_0^T X_{t,k} dW_t(\phi_k) \right)^2 \right] \quad \text{from (2.1)} \\
&= \sum_{k=1}^N \frac{a_k^2}{b_k^2} \int_0^T E(X_{t,k})^2\, dt.
\end{aligned}
$$

From (2.2), we get

$$E(X_{t,k})^2 = b_k^2 \int_0^t e^{-2(\lambda_k - \theta a_k)(t-s)}\, ds,$$

and hence

$$\begin{aligned} I^N(\theta) &= \frac{T}{2}\sum_{k=1}^{N} \frac{a_k^2}{(\lambda_k - \theta a_k)} \\ &\quad -\frac{1}{4}\sum_{k=1}^{N} \frac{a_k^2}{(\lambda_k - \theta a_k)^2}\left(1 - e^{-2(\lambda_k - \theta a_k)T}\right). \end{aligned}$$

Since $\lambda_k \to \infty$ as $k \to \infty$, it follows from the first condition in (C2) that

$$\frac{a_k}{\lambda_k} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, for fixed $\theta \in \Theta$, there exists an N_0 such that $a_k/\lambda_k < \delta$ and $|\theta|\delta < 1$ for some $\delta > 0$ if $k \quad N_0$. From $\sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k} < \infty$, we have the following Lemma.

Lemma 2. For all $\theta \in \Theta$,

$$I^\infty(\theta) := \sum_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \int_0^T E(X_{t,k})^2\, dt < \infty,$$

where $I^\infty(\theta) := \lim_{N\to\infty} I^N(\theta)$.

Let us now prove that the measure P_θ is absolutely continuous with respect to P_{θ_0}. Also P_θ and P_{θ_0} are mutually absolutely continuous.

Define

$$\begin{aligned} \tilde{\Omega} &= \otimes C[0,T] \\ \tilde{\mathcal{F}} &= \otimes \tilde{\mathcal{B}}_T, \quad \tilde{\mathcal{F}}_N = \sigma\{X_1, \ldots, X_N\} \quad \text{for} \quad N \geq 1, \\ \tilde{P}_\theta &= \otimes_k \tilde{P}_{\theta,k}, \quad \tilde{P}_{\theta_0} = \otimes_k \tilde{P}_{\theta_0,k}. \end{aligned}$$

Note that $\tilde{P}_\theta$ is the law of a sequence of independent random variables (X_k) with values in $C[0,T]$.

Theorem 1. The measure P_θ is absolutely continuous with respect to P_{θ_0}, and the Radon-Nikodym derivative of P_θ with respect to P_{θ_0} is

$$Z_\infty := \frac{dP_\theta}{dP_{\theta_0}}(X)$$

$$= \exp\left\{(\theta-\theta_0)\sum_{k=1}^{\infty}\frac{a_k}{b_k^2}\int_0^T X_{t,k}\,dX_{t,k} - \frac{\theta^2-\theta_0^2}{2}\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt \right.$$
$$\left. +(\theta-\theta_0)\sum_{k=1}^{\infty}\frac{a_k\lambda_k}{b_k^2}\int_0^T (X_{t,k})^2\,dt\right\}.$$

Proof. It follows from Lemma 1 that for all N,

$$Z_N = \frac{dP_\theta^N}{dP_{\theta_0}^N}(X^N) = \frac{d\tilde{P}_\theta^N}{d\tilde{P}_{\theta_0}^N}(\tilde{X}^N).$$

Hence it is enough to show that $\tilde{P}_\theta$ is absolutely continuous with respect to $\tilde{P}_{\theta_0}$. By Theorem 2 of Section 6, Chapter VII in Shiryayev (1984)([7]), we need to show that

$$E_{\tilde{P}_{\theta_0}}(Z_\infty) = 1$$

where $\tilde{P}_{\theta_0}(Z_\infty = \lim_{N\to\infty} Z_N) = 1$. For $M < N$, we have

$$\begin{aligned}
& E_{\tilde{P}_{\theta_0}}|Z_N - Z_M| \\
\le\ & \left(E_{\tilde{P}_{\theta_0}}\left|\sqrt{Z_N}+\sqrt{Z_M}\right|^2\right)^{1/2}\left(E_{\tilde{P}_{\theta_0}}\left|\sqrt{Z_N}-\sqrt{Z_M}\right|^2\right)^{1/2} \\
\le\ & 2\left(\int_{\tilde{\Omega}}\left|\sqrt{Z_N}-\sqrt{Z_M}\right|^2 d\tilde{P}_{\theta_0}\right)^{1/2} \\
=\ & 2\left(\int_{\tilde{\Omega}}\left[1-\sqrt{Z_N Z_M}\right] d\tilde{P}_{\theta_0}\right)^{1/2} \\
=\ & 2\left(\int_{\tilde{\Omega}}\left[1-\prod_{k=M+1}^{N}\sqrt{q_k(X_k)}\right] d\tilde{P}_{\theta_0}\right)^{1/2} \\
=\ & 2\left[1-\prod_{k=M+1}^{N}\int_{C[0,T]}\sqrt{q_k(X_k)}\,d\tilde{P}_{\theta_0,k}\right]^{1/2}
\end{aligned} \tag{2.5}$$

If we prove that (2.5) converges to 0 as $M, N \to \infty$, Z_N converges in L^1. Since Z_N converges to Z_∞ $\tilde{P}_{\theta_0}$-(a.s.) and $E_{\tilde{P}_{\theta_0}}(Z_N) = 1$ for all N, we have

$$E_{\tilde{P}_{\theta_0}}(Z_\infty) = 1.$$

To prove that (2.5) converges to 0 as $M, N \to \infty$, we only need to show that the product

$$\prod_{k=1}^{\infty} E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)}$$

converges, which is equivalent to the following condition

$$\sum_{k=1}^{\infty}\left[1 - E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)}\right] < \infty. \tag{2.6}$$

First notice that

$$E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)} \le \left(E_{\tilde{P}_{\theta_0,k}} q_k(X_k)\right)^{1/2} = 1. \tag{2.7}$$

Hence the series

$$\sum_{k=1}^{\infty}\left[1 - E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)}\right]$$

converges or diverges with the series

$$\sum_{k=1}^{\infty}\left|\log E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)}\right|.$$

Using Equation (2.1), (2.4) can be written as

$$\begin{aligned}\frac{dP_\theta^N}{dP_{\theta_0}^N}(X^N) = \exp\Bigg\{(\theta-\theta_0)\sum_{k=1}^{N}\frac{a_k}{b_k}\int_0^T X_{t,k}\,dW_t(\phi_k) \\ -\frac{(\theta-\theta_0)^2}{2}\sum_{k=1}^{N}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt\Bigg\}.\end{aligned} \tag{2.8}$$

By Jensen's inequality, (2.7) and (2.8), we have

$$\begin{aligned}\sum_{k=1}^{\infty} & \left|\log E_{\tilde{P}_{\theta_0,k}}\sqrt{q_k(X_k)}\right| \\ \le & \frac{1}{2}\sum_{k=1}^{\infty}\left|E_{\tilde{P}_{\theta_0,k}}\log q_k(X_k)\right| \\ = & \frac{(\theta-\theta_0)^2}{4}\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T E_{\tilde{P}_{\theta_0,k}}(X_{t,k})^2\,dt < \infty\end{aligned} \tag{2.9}$$

from Lemma 2. This completes the proof of the theorem.

3. ASYMPTOTIC PROPERTIES OF MLE

Let $l(X,\theta) := \log(dP_\theta/dP_{\theta_0}(X))$. From $\frac{\partial}{\partial\theta}l(\theta) = 0$, the MLE $\hat\theta_T$ is given by

$$\begin{aligned}\hat\theta_T &= \left(\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt\right)^{-1} \\ &\quad\times\left(\sum_{k=1}^{\infty}\left[\frac{a_k}{b_k^2}\int_0^T X_{t,k}\,dX_{t,k} + \frac{a_k\lambda_k}{b_k^2}\int_0^T (X_{t,k})^2\,dt\right]\right). \qquad (3.1)\end{aligned}$$

Now we investigate the asymptotic properties of $\hat\theta_T$ as $T\to\infty$. For this we need the following Lemmas given in Loges (1984)([6]).

Lemma 3. [Loges] Let $(G_{t,k}), t\in[0,\infty)$, be a family of nonanticipating (w.r.t. $\mathcal{F}_t^W$) real valued stochastic processes on $(\Omega,\mathcal{F},P)$ such that there exists an $N_0<\infty$ such that

$$\sum_{k=1}^{N_0}\int_0^\infty (G_{t,k})^2\,dt = \infty \quad P-(a.s.) \qquad (3.2)$$

noindent and

$$E\left[\sum_{k=1}^{\infty}\int_0^L (G_{t,k})^2\,dt\right] < \infty \quad for\ all \quad L\in(0,\infty). \qquad (3.3)$$

Define the stopping time τ_T, $T\in[0,\infty)$, by

$$\tau_T = \inf\{t\in[0,\infty) : \sum_{k=1}^{\infty}\int_0^t (G_{s,k})^2\,ds > T\}$$

Then

$$Z_T^N := \sum_{k=1}^{N}\int_0^{\tau_T} G_{t,k}\,dW_t(\phi_k)$$

converges in L^2 as $N\to\infty$ and the limit Z_T, $T\in[0,\infty)$, is a standard Wiener process.

Lemma 4. [Loges] Let $(G_{t,k}), t\in[0,\infty)$, be a family of nonanticipating processes satisfying (3.2) and (3.3) in Lemma . We further assume that

$$\frac{1}{T}\sum_{k=1}^{\infty}\int_0^T (G_{t,k})^2\,dt \to c > 0 \quad \text{in probability as} \quad T\to\infty.$$

Then

$$\frac{1}{\sqrt{T}}\sum_{k=1}^{\infty}\int_0^T G_{t,k}\,dW_t(\phi_k) \to N(0,c) \text{ in distribution as } T\to\infty.$$

Theorem 2. [Strong Consistency] The MLE $\hat{\theta}_T$ is strongly consistent, i.e.,

$$\lim_{T\to\infty}\hat{\theta}_T = \theta \quad P-(\text{a.s.}).$$

Proof. By substituting the observation (2.1) into (3.1), it follows that

$$\hat{\theta}_T = \theta + \left(\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt\right)^{-1}\left(\sum_{k=1}^{\infty}\frac{a_k}{b_k}\int_0^T X_{t,k}\,dW_t(\phi_k)\right). \tag{3.4}$$

First we show that the condition (3.2) is satisfied. Write

$$G_{t,k} := \frac{a_k}{b_k}X_{t,k}.$$

Define

$$\Psi_T^k(\theta) := E\left[\exp\left(-\int_0^T (G_{t,k})^2\,dt\right)\right].$$

Using Lemma 17.3 in Liptser and Shiryayev (1977)([5]), we can show that

$$\begin{aligned}
&\Psi_T^k(\theta)\\
=\ &\exp\left\{\left[-\frac{(\theta a_k-\lambda_k)}{2}-\frac{\sqrt{(\theta a_k-\lambda_k)^2+2a_k^2}}{2}\right]T\right\}\times\left(2\sqrt{(\theta a_k-\lambda_k)^2+2a_k^2}\right)^{1/2}\\
&\times\left(\left[\sqrt{(\theta a_k-\lambda_k)^2+2a_k^2}-(\theta a_k-\lambda_k)\right]+\left[\sqrt{(\theta a_k-\lambda_k)^2+2a_k^2}+(\theta a_k-\lambda_k)\right]\right.\\
&\left.\times\exp\left\{-2\sqrt{(\theta a_k-\lambda_k)^2+2a_k^2T}\right\}\right)^{-1/2}.
\end{aligned}$$

Hence we have $\lim_{T\to\infty}\Psi_T^k(\theta)=0$ for all k, which in turn implies that

$$\lim_{T\to\infty}\exp\left(-\int_0^T (G_{t,k})^2\,dt\right)=0 \quad P-(a.s.)$$

by the dominated convergence theorem. Further, we have

$$\lim_{T\to\infty}\int_0^T (G_{t,k})^2\,dt=\infty \quad P-(a.s.).$$

Hence the condition (3.2) in Lemma 3 is satisfied with $N_0 = 1$, and it follows from Lemma 2 that the condition (3.3) is satisfied. Using the notation $G_{t,k}$, we write (3.4) as follows:

$$\begin{aligned}\hat{\theta}_T &= \theta + \left(\sum_{k=1}^{\infty}\int_0^T (G_{t,k})^2\,dt\right)^{-1}\left(\sum_{k=1}^{\infty}\int_0^T G_{t,k}\,dW_t(\phi_k)\right)\\ &:= \theta + M_T\end{aligned}$$

From Lemma 3,

$$Z_{\tau_T} := \sum_{k=1}^{\infty}\int_0^{\tau_T} G_{t,k}\,dW_t(\phi_k)$$

is a standard Wiener process. Hence we have

$$\lim_{T\to\infty} M_{\tau_T} = \lim_{T\to\infty}\frac{Z_{\tau_T}}{< Z >_{\tau_T}} = 0 \;\; P-(a.s.)$$

where $< Z >_t$ is the quadratic variation process of Z_t. By applying the same arguments as those in proof of Theorem 3 in Loges(1984)([6]), we get $\lim_{T\to\infty} M_T = 0$ P-(a.s.)

Theorem 3. [Asymptotic Normality] Suppose that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt = c > 0 \text{ in probability.}$$

Then $\sqrt{T}(\hat{\theta}_T - \theta) \to N(1, 1/c)$ in distribution as $T \to \infty$.

Proof. From (3.4), we have

$$\sqrt{T}(\hat{\theta}_T - \theta) = \frac{\frac{1}{\sqrt{T}}\sum_{k=1}^{\infty}\frac{a_k}{b_k}\int_0^T X_{t,k}\,dW_t(\phi_k)}{\frac{1}{T}\sum_{k=1}^{\infty}\frac{a_k^2}{b_k^2}\int_0^T (X_{t,k})^2\,dt}. \tag{3.5}$$

Let

$$G_{t,k} := \frac{a_k}{b_k}X_{t,k}.$$

Then $(G_{t,k})$ satisfy all the assumptions in Lemma 4. Hence, according to Lemma 4,

$$\frac{1}{\sqrt{T}}\sum_{k=1}^{\infty}\frac{a_k}{b_k}\int_0^T X_{t,k}\,dW_t(\phi_k) \to N(1, c) \quad \text{in distribution as } T \to \infty.$$

Hence it follows from (3.5) that

$$\sqrt{T}(\hat{\theta}_T - \theta) \to N(1, 1/c) \quad \text{in distribution as} \quad T \to \infty.$$

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