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# Efficiency and Minimaxity of Bayes Sequential Procedures in Simple versus Simple Hypothesis Testing for General Nonregular Models

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**Abstract**

we consider the question of efficiency of the Bayes sequential procedure with respect to the optimal fixed sample size Bayes procedure in a simple vs. simple testing problem for data coming from a general nonregular density $b(\theta)h(x)I(x < \theta)$. Efficiency is defined in two different ways in these calculations. Also, the minimax sequential risk(and minimax sequential stratage) is studied as a function of the cost of sampling.

**Key Words** : Bayes sequential procedure; Optimal fixed sample size Bayes procedure; Minimax sequential risk.

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## 1. INTRODUCTION

A sequential problem is considered in which independent observations are taken on a random variable $X$ which is distributed as $Uniform(0, \theta)$, where the parameter $\theta > 0$ is an unknown constant. Suppose that we want to test

$$H_0 : \theta = \theta_0 \quad against \quad H_1 : \theta = \theta_1 \tag{1.1}$$

where $\theta_1 = 1$ and $\theta_0 > 1$. Let $\pi_0$ denote the prior probability for $H_0$ and let $c$ be the constant cost of sampling. It is assumed that the decision loss is $0 - l_i$ loss, i.e., $L(\theta_0, a_0) = L(\theta_1, a_1) = 0, L(\theta_0, a_1) = l_1$ and $L(\theta_1, a_0) = l_0$. Let $n$ denote the number of observations ultimately taken. It is assumed that the overall loss is

$$L(\theta_i, a_j, n) = L(\theta_i, a_j) + nc, \quad i = 0, 1, \quad j = 0, 1.$$

The Bayes stopping time is, in general, of the form: Stop either at time 0 or at the first time $n$ such that

$$\rho_0(\pi^n) \leq \rho^*(\pi^n)$$

where $\rho_0(\pi^n)$ is the posterior Bayes decision risk in the fixed sample size problem with a sample of size $n$ and data $\mathbf{X}^n$ and $\rho^*(\pi^n)$ is the minimum Bayes risk that can be attained if at least $n+1$ observations are taken. (The "decision" risk does not include the cost of sampling $c$.) Notice that $\rho_0(\pi^n)$ does not involve the cost $c$ while $\rho^*(\pi^n)$ does. It turns out that the Bayes procedure is just the immediate Bayes decision with no observation or a SPRT(sequential probability ratio test) which is of the following form:(Berger(1985))

At stage $n (n \geq 1)$,

if $L_n \leq A$, stop sampling and decide $a_0$;

if $L_n \geq B$, stop sampling and decide $a_1$;

if $A < L_n < B$, take another observation;

here, $A < 1$ and $B > 1$ are appropriate stopping boundaries and $L_n$ is the likelihood ratio of $\theta_1$ to $\theta_0$ at stage $n$,

$$L_n = \frac{\Pi_{i=1}^n f(x_i|\theta_1)}{\Pi_{i=1}^n f(x_i|\theta_0)}.$$

(For the SPRT, as a nature of Bayes test, see Wald(1947), Wald(1950), Wald and Wolfowitz (1948), Ferguson(1967) and Berger(1985).) Observe that

$$\begin{aligned} L_n &= \frac{I_{(x_i \leq 1, \forall i=1,\ldots,n)}(\theta)}{I_{(x_i \leq \theta_0, \forall i=1,\ldots,n)}(\theta)} \\ &= \begin{cases} 0 & \text{if } 1 \leq x_{(n)} < \theta_0 \\ \theta_0^n & \text{if } x_{(n)} < 1 \end{cases} \end{aligned} \tag{1.2}$$

where $x_{(n)}$ is the $nth$ order statistic. So only two things can happen:
a either we get $x_n > 1$ at some stage; then, clearly, we should stop and accept $H_0$ or
b we keep getting $x_i's \leq 1$ and therefore since $\theta_0 > 1$, by (1.2) the ratio sooner or later goes above the fixed bound $B$ and we will reject $H_0$.

Thus Bayes rules must be one of the following rules:
$d_0$ : stop and take the optimal action without taking any observations; $d_J(J \geq 1)$ : necessarily stop before $n \leq J$ ; if $\exists n < J \ni x_n \geq 1$, then stop at stage $n$ and accept $H_0$, while if $\forall n < J, x_n < 1$, then continue sampling untill the $Jth$ obsevation and accept $H_0$ if $x_J \geq 1$ and decide $H_1$ otherwise.

In section 2, efficiency of the Bayes sequential procedure with respect to the optimal fixed sample size Bayes procedure is considered. Let $r_{\pi_0}(c)$ be Bayes sequential risk and let $r_{\pi_0}^F(c)$ be the optimal fixed sample size Bayes risk. Then it is proved that

$$\lim_{c \to 0} \frac{r_{\pi_0}^F(c)}{r_{\pi_0}(c)} = \frac{1}{1-\pi_0}.$$

On the other hand, for a fixed value of $\pi_0$, for the Bayes sequential procedure and the optimal fixed sample size Bayes procedure to have the same Bayes risk, i.e., $r_{\pi_0}(c) = r_{\pi_0}^F(c^F)$, we prove that the ratio of the sampling costs satisfies

$$\lim_{c \to 0} \frac{c^F}{c} = 1 - \pi_0.$$

Notice that the two definitions of efficiency are not equivalent. In section 3, the minimax sequential procedure is considered. The minimax sequential rule is determined among the set of Bayes rules, $d_J, J \geq 0$. This is justified, as established in Brown, Cohen and Strawderman(1980). It will be shown that the minimax sequential risk increases as the sampling cost increases while for minimax sequential strategy $d_{J^m}$, $J^m$ decreases as $c$ increases.

The results of this paper hold also for the general nonregular case in which independent observations have a common density of the form $b(\eta)h(x)I_{(x<\eta)}$ and (1.1) is replaced by

$$H_0 : \eta = \eta_0 \quad vs \quad H_1 : \eta = \eta_1$$

for some $\eta_0 > \eta_1$. This is easily seen on making a transformation of the form $Y = 1/(\eta_1 b(X))$. For asymptotic efficiency of sequential procedure of more general type of distributions (for example, a distribution of exponential (Koopman-Darnois) type), see Kiffer and Sacks (1963). The efficiency of the optimal sequential procedure has been studied from a classical perspective before; see Ghosh(1970). But such calculations were lacking altogether for Bayes sequential procedures. The problem here is particularly amenable to those calculations.

## 2. ASYMPTOTIC EFFICIENCY OF BAYES SEQUENTIAL PROCEDURE

Let $r(\pi_0, d_J)$ denote the Bayes risk for the procedure $d_J$ is defined in section 1. Since a Bayes sequential rule is $d_J$, for some $J \geq 0$, for a given $\pi_0$, the Bayes sequential procedure is determined by minimizing $r(\pi_0, d_J)$ over $J \geq 0$.
For the procedure $d_J, J \geq 1$, (Berger(1985))

$$\begin{aligned}
\alpha_0 &= P_{\theta=\theta_0}(reject \;\; H_0) \\
&= P_{\theta=\theta_0}(x_i < 1, \forall i = 1, \ldots, J) \\
&= (\frac{1}{\theta_0})^J, \\
\alpha_1 &= P_{\theta=1}(accept \;\; H_0) \\
&= 1 - P_{\theta=1}(reject \;\; H_0) \\
&= 0.
\end{aligned}$$

Let $N_J$ be a stopping time for the procedure $d_J, J \geq 1$. Then

$$E(N_J|H_0) = \sum_{n=0}^{J-1} P_{H_0}(N > n)$$

$$\begin{aligned} &= \sum_{n=0}^{J-1} P_{\theta=\theta_0}(x_i < 1, \forall i = 1, \ldots, n) \\ &= \sum_{n=0}^{J-1} \theta_0^{-n} \\ &= \frac{1-\theta_0^{-J}}{1-\theta_0^{-1}}, \\ E(N_J|H_1) &= J. \end{aligned}$$

Thus, the Bayes sequntial risk for the procedure $d_J, J \geq 1$, is given by

$$r(\pi_0, d_J) = \pi_0(\theta_0^{-J} l_1 + c\frac{1-\theta_0^{-J}}{1-\theta_0^{-1}}) + (1-\pi_0)cJ.$$

Also, $r(\pi_0, d_0) = \min\{\pi_0 l_1, (1-\pi_0)l_0\}$. Let $f(J) = r(\pi_0, d_J), J \geq 1$. Pretending that $J$ is a continuous variable and differentiating with respect to $J$ gives

$$f'(J) = \pi_0(\frac{\theta_0}{\theta_0 - 1}c - l_1)\theta_0^{-J} \log\theta_0 + (1-\pi_0)c.$$

Thus $f'(J) > 0$ if $l_1 < \theta_0/(\theta_0 - 1)c$. So $J = 0$ is the optimal value for $l_1 < \theta_0/(\theta_0 - 1)c$. If $l_1 \geq \theta_0/(\theta_0 - 1)c$, the second derivative of $f(J)$ is positive, so $f(J)$ is strictly convex function in $J$. Setting $f'(J) = 0$ and solving gives the approximate optimal value of $J$ which is

$$\frac{\log(\pi_0(l_1 - \frac{c}{1-1/\theta_0})\log\theta_0) - \log((1-\pi_0)c)}{\log\theta_0}.$$

Let $r_{\pi_0}(c)$ denote the Bayes sequential risk. Thus if $l_1 \geq \frac{\theta_0}{\theta_0-1}c$, since $J^*$ is not an integer value, $r_{\pi_0}(c) = f(J^*)$ approximately, where

$$f(J^*) = \pi_0(\theta_0^{-J^*} l_1 + c\frac{1-\theta_0^{-J^*}}{1-\frac{1}{\theta_0}}) + (1-\pi_0)cJ^*, \tag{2.1}$$

and

$$J^* = \max\{0, \frac{\log(\pi_0(l_1 - \frac{c}{1-1/\theta_0})\log\theta_0) - \log((1-\pi_0)c)}{\log\theta_0}\}.$$

And if $l_1 < \frac{\theta_0}{\theta_0 - 1}c$,

$$r_{\pi_0}(c) = \min\{\pi_0 l_1, (1-\pi_0)l_0\}.$$

Now, let us consider the optimal fixed sample size procedure.
If $\mathbf{X}^n = (X_1, \ldots, X_n)$ is observed, the Bayes decision rule is to select $a_0$ if

$$\begin{aligned} & E^{\pi} E_{\theta}^{x} L(\theta, a_0) \leq E^{\pi} E_{\theta}^{x} L(\theta, a_1) \\ \Leftrightarrow \quad & (1-\pi_0) l_0 I_{(x_{(n)}<1)} \leq \pi_0 l_1 \left(\frac{1}{\theta_0}\right)^n I_{(x_{(n)}<\theta_0)}. \end{aligned}$$

Thus the Bayes decision rule is

$$\delta_{\pi}^{n} = \begin{cases} a_0 & \text{if } x_{(n)} > 1 \text{ or } (1-\pi_0)l_0 \leq \pi_0 l_1 (1/\theta_0)^n \\ a_1 & \text{otherwise.} \end{cases}$$

Let $r^n(\pi)$ denote the Bayes decision risk for $\delta_{\pi}^{n}$. Then

$$\begin{aligned} r^n(\pi) &= E^{\pi} E_{\theta}^{x} L(\theta, \delta_{\pi}^{n}) \\ &= \pi_0 l_1 P_{\theta_0=\theta_0}(x_{(n)} < 1 \ \ and \ \ (1-\pi_0)l_0 \leq \pi_0 l_1 (1/\theta_0)^n) \\ &\quad + (1-\pi_0) l_0 (P_{\theta_1=1}(x_{(n)} \geq 1) \\ &\quad + P_{\theta_1=1}(x_{(n)} < 1 \ \ and \ \ (1-\pi_0)l_0 > \pi_0 l_1 (1/\theta_0)^n) \\ &= \begin{cases} (\frac{1}{\theta_0})^n \pi_0 l_1 & \text{if } n > \frac{\log(\pi_0 l_1) - \log((1-\pi_0)l_0)}{\log \theta_0} \\ (1-\pi_0) l_0 & \text{otherwise.} \end{cases} \end{aligned}$$

If we let $r_{\pi_0}^{F}(c)$ denote the optimal fixed sample size Bayes risk, then

$$r_{\pi_0}^{F}(c) = \min_{n \geq 0} (r^n(\pi) + nc).$$

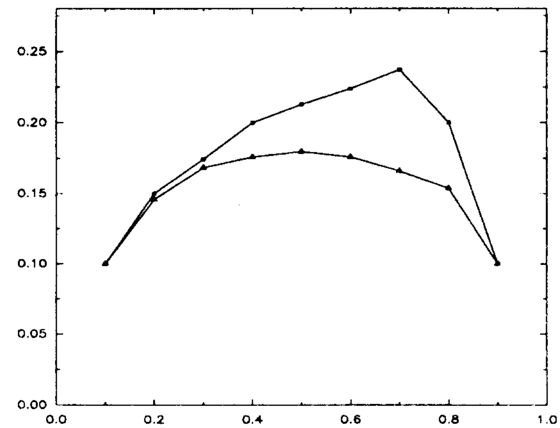
($n = 0$ corresponds to making a decision without taking observations so that $r^0(\pi) = \min\{\pi_0 l_1, (1-\pi_0)l_0\}$.) Let $g(n) = (1/\theta_0)^n \pi_0 l_1 + nc$. Pretending that $n$ is a continuous variable and differentiating with repect to $n$ gives

$$g'(n) = -\theta_0^{-n} \pi_0 l_1 \log \theta_0 + c.$$

Setting equal to zero and solving gives

$$n^* = \frac{\log(\pi_0 l_1 \log \theta_0) - \log c}{\log \theta_0}.$$

Since the second derivative of $g(n)$ is positive, $g(n)$ is strictly convex in $n$.



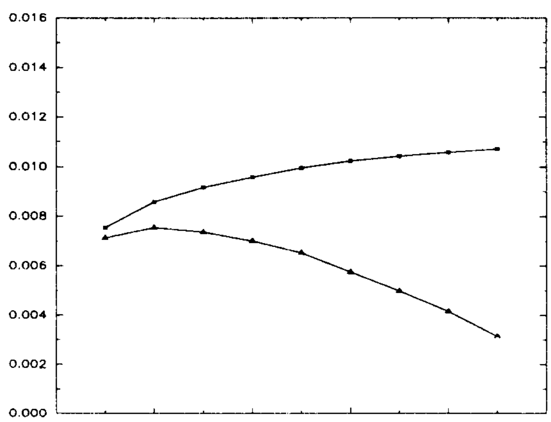




**Figure 1.** Plots of sequential Bayes risk (-$\triangle$-) and the optimal sample size Bayes risk (-$\square$-) for cost $c = 0.05$ (above) and $c = 0.001$(below)

Define $q = \frac{\log(\pi_0 l_1) - \log((1-\pi_0)l_0)}{\log\theta_0}$ for notational convenience.
(i) Suppose that $q \le 0 (\Leftrightarrow \pi_0 l_1 \le (1-\pi_0)l_0)$.
Then $r^n(\pi) = (1/\theta_0)^n \pi_0 l_1$, so

$$r^F_{\pi_0}(c) \approx (1/\theta_0)^{n^*} \pi_0 l_1 + n^* c \tag{2.2}$$

unless $n^* < 0$, in which case $r^F_{\pi_0}(c) = \pi_0 l_1$.
(ii) Suppose that $q > 0 (\Leftrightarrow \pi_0 l_1 > (1-\pi_0)l_0)$.
If $n^* > q$,

$$r^F_{\pi_0}(c) \approx \min\{(1/\theta_0)^{n^*} \pi_0 l_1 + n^* c, (1-\pi_0)l_0\}. \tag{2.3}$$

And if $n^* \le q$,

$$r^F_{\pi_0}(c) = (1-\pi_0)l_0.$$

Figure 1 shows the sequntial risk and the optimal sample size Bayes risk for $0-1$ decision loss.

The next theorem is for the asymptotic efficiency of the Bayes sequential procedure with respect to the optimal fixed sample size Bayes procedure in terms of their risks.

**Theorem 1.** For a fixed prior probability $\pi_0$, the asymptotic ratio of the Bayes sequential risk to the optimal fixed sample size Bayes risk has the property

$$\lim_{c\to 0} \frac{r^F_{\pi_0}(c)}{r_{\pi_0}(c)} = \frac{1}{1-\pi_0}.$$

**Proof.** Let $\pi_0$ be fixed and let $c > 0$ be sufficiently small.
(a) For the Bayes sequential procedure, note that $l_1 \ge \frac{\theta_0}{\theta_0 - 1} c$ for $c$ near 0. Then $r_{\pi_0}(c) = f(J^*)$ approximately, where $f(J^*)$ is defined as (2.1). Also, since $J^* \to \infty$ as $c \to 0$, $r_{\pi_0}(c) = f(J^*)$ asymptotically for $c$ near 0. Now, we get

$$\begin{aligned} f(J^*) &= \pi_0\left[\frac{(1-\pi_0)l_1 c}{\pi_0(l_1 - \frac{c}{1-\theta_0^{-1}})\log\theta_0} + \frac{c}{1-\theta_0^{-1}}\left(1 - \frac{(1-\pi_0)c}{\pi_0(l_1 - \frac{c}{1-\theta_0^{-1}})}\right)\right] \\ &\quad + (1-\pi_0)c\frac{\log(\pi_0(l_1 - \frac{c}{1-\theta_0^{-1}})\log\theta_0) - \log((1-\pi_0)c)}{\log\theta_0} \end{aligned}$$

$$\begin{aligned} &= O\{\frac{c}{\log\theta_0}[1-\pi_0+\frac{1}{1-\theta_0^{-1}}(\pi_0\log\theta_0-\frac{(1-\pi_0)c}{l_1}) \\ &\quad +(1-\pi_0)(\log(\pi_0 l_1\log\theta_0)\log(1-\pi_0)-\log c)]\} \\ &= O(c-\frac{1-\pi_0}{\log\theta_0}c\log c) \\ &= O(-\frac{1-\pi_0}{\log\theta_0}c\log c). \end{aligned}$$

(b) For the optimal fixed sample size Bayes procedure, since

$$n^* \le 0 \Leftrightarrow \pi_0 l_1 \log\theta_0 \le c \ \ \textit{and}$$
$$n^* \le q \Leftrightarrow (1-\pi_0)l_0 \le c,$$

$n^* > 0$ & $q \le 0$ or $n^* > q$ & $q > 0$ for small $c$. Also, $\lim_{c\to 0} n^* c = \lim_{c\to 0}(\frac{-c\log c}{\log\theta_0}) = 0$. Thus by (2.2) and (2.3),

$$r_{\pi_0}^F(c) = (1/\theta_0)^{n^*}\pi_0 l_1 + n^* c, \tag{2.4}$$

where

$$n^* = \frac{\log(\pi_0 l_1\log\theta_0)-\log c}{\log\theta_0}.$$

Now, the expression on the right in (2.4) is

$$\begin{aligned} &\frac{c}{\pi_0 l_1\log\theta_0}\pi_0 l_1 + \frac{\log(\pi_0 l_1\log\theta_0)-\log c}{\log\theta_0}c \\ &= O\left(\frac{c}{\log\theta_0}+c\frac{\log(\pi_0 l_1\log\theta_0)-\log c}{\log\theta_0}\right) \\ &= O(c-\frac{c\log c}{\log\theta_0}) \\ &= O(\frac{-c\log c}{\log\theta_0}) \end{aligned}$$

This proves the stated assertion.

Next, an efficiency will be considered in terms of sampling cost of each procedure.

**Theorem 2.** For given $c > 0$, let $c^F$ be such that $r_{\pi_0}(c) = r_{\pi_0}^F(c^F)$ for a given $\pi_0$. Then

$$\lim_{c\to 0}\frac{c^F}{c} = 1-\pi_0.$$

**Proof.** Assume that $c$ is sufficiently small. From the above theorem

$$r_{\pi_0}(c) = O(-\frac{1-\pi_0}{\log\theta_0} c \log c). \tag{2.5}$$

Again from the above theorem, since

$$\frac{r^F_{\pi_0}(c)}{r_{\pi_0}(c)} = \frac{1}{1-\pi_0} > 1,$$

$r_{\pi_0}(c) < r^F_{\pi_0}(c)$ for small $c$. Thus to have them equal, $c^F$ must be less than $c$, implying $\lim_{c\to 0} c^F = 0$. Now, for $c^F$ small,

$$r^F_{\pi_0}(c^F) = O(-\frac{c^F \log c^F}{\log\theta_0}). \tag{2.6}$$

The Theorem now follows from (2.5) and (2.6).

## 3. MINIMAX STRATEGY AND THE MINIMAX RISK

We now consider the minimax sequential procedure for the problem. A minimax sequential procedure is a procedure which minimizes $\sup_\theta R(\theta, \mathrm{d})$ among all proper sequential procedures. We begin with definitions of the needed concepts.

**Definition** A sequential rule $\delta_1$ is R-better than a sequential rule $\delta_2$ if $R(\theta,\delta_1) \leq R(\theta,\delta_2)$ for all $\theta \in \Theta$, with strict inequality for some $\theta$.

**Definition** A class $C$ of sequential rules is said to be complete if, for any sequential rule $\delta$ not in $C$, there is a sequential rule $\delta' \in C$ which is R-better than $\delta$.

It is shown in Brown, Cohen, and Strawderman(1980) that for simple versus simple testing problems the Bayes sequential rules form a complete class. Since a Bayes rule is represented by $d_J$ for some $J \geq 0$, the minimax sequential procedure can be considered only among the procedures $d_J$, $J \geq 0$. Let $r_m(c)$ denote the minimax risk. Then

$$r_m(c) = \inf_{\mathrm{d}} \sup_\theta R(\theta, \mathrm{d}) = \inf_{d_J} \sup_\theta R(\theta, d_J).$$

Let us recall from the previous section that the Bayes risk for the procedure $d_J, J \geq 1$, is

$$r(\pi_0, d_J) = \pi_0(\theta_0^{-J} l_1 + c\frac{1-\theta_0^{-J}}{1-\theta_0^{-1}}) + (1-\pi_0)cJ.$$

Since for $J \geq 1$

$$\begin{aligned}\sup_\theta R(\theta, d_J) &= \max\{\theta_0^{-J} l_1 + c\frac{1-\theta_0^{-J}}{1-\theta_0^{-1}}, cJ\} \\ &= \max_{\pi_0} r(\pi_0)\end{aligned}$$

and

$$\sup_\theta R(\theta, d_0) = max\{l_0, l_1\},$$

the minimax sequential risk is

$$r_m(c) = \min\{\min_{J\geq 1}(\max\{\theta_0^{-J} l_1 + c\frac{1-\theta_0^{-J}}{1-\theta_0^{-1}}, cJ\}), \max\{l_0, l_1\}\}.$$

The following lemma gives that for all small c we will need more than 1 observation under the alternative as the minimax sequential strategy.

**Lemma 1.** Let $d_{J^m(c)}$ denote the minimax sequential strategy. Then $J^m(c)$ is at least 1 for $0 < c \leq l_1(\theta_0 - 1)/\theta_0$.

**Proof.** Let $g_1(x) = \theta_0^{-x} l_1 + c\frac{1-\theta_0^{-x}}{1-\theta_0^{-1}}$ and let $g_2(x) = cx$ for $x \geq 0$. Since the first derivative of $g_1(x)$ is positive if $c \leq l_1(\theta_0 - 1)/\theta_0$, $g_1$ is decreasing function with $g_1(0) = l_1$. Thus $g_1(x)$ meets with $g_2(x)$ exactly once and the crossing point will minimize $\max\{g_1(x), g_2(x)\}$. Let $x^m(c)$ be the crossing point for a given $c$. Solving the equation $g_1(x) = g_2(x)$ gives that

$$c = \frac{l_1(\theta_0 - 1)}{\theta_0 - \theta_0^{x^m(c)}(\theta_0 + x^m(c)(1-\theta_0))}. \tag{3.1}$$

Let

$$h_1(x) = \theta_0 - \theta_0^x(\theta_0 + x(1-\theta_0)). \tag{3.2}$$

Then

$$h_1'(x) > 0 \Leftrightarrow x > 1 + \frac{1}{\theta_0 - 1} - \frac{1}{\log \theta_0}. \tag{3.3}$$

Since

$$0 < 1 + \frac{1}{\theta_0 - 1} - \frac{1}{\log \theta_0} < 1$$

and $h_1(0) = h_1 1 = 0$, $h_1(x) \leq 0$ for $0 \leq x \leq 1$. It follows that this contradicts (3.1), since $c$ is necessarially positive. Hence $x^m(c) > 1$ and therefore $J^m(c) \geq 1$ for $0 < c \leq l_1(\theta_0 - 1)/\theta_0$.

**Theorem 3.** For the minimax sequential strategy $d_{J^m(c)}$, $J^m(c)$ is monotonically decreasing in $c$.

**Proof.** Let $0 < c_1 < c_2 < l_1(\theta_0 - 1)/\theta_0$ and let $x_1 = x^m(c_1)$ and $x_2 = x^m(c_2)$. $c_1 - c_2 < 0$ implies by virtue of (3.1),

$$\frac{(\theta_0 - 1)(\theta_0^{x_1}(\theta_0 + x_1(1 - \theta_0)) - \theta_0^{x_2}(\theta_0 + x_2(1 - \theta_0)))}{(\theta_0 - \theta_0^{x_1}(\theta_0 + x_1(1 - \theta_0)))(\theta_0 - \theta_0^{x_2}(\theta_0 + x_2(1 - \theta_0)))} < 0. \tag{3.4}$$

Again, let $h_1(x) = \theta_0 - \theta_0^x(\theta_0 + x(1 - \theta_0))$. Then, from (3.3), $h_1(x)$ is strictly increasing for $x \geq 1$. But $h_1 1 = 0$. Thus $h_1(x) > 0$ for all $x > 1$. Hence

$$(3.4) \Leftrightarrow$$

$$\theta_0 x_1(\theta_0 + x_1(1 - \theta_0)) < \theta_0 x_2(\theta_0 + x_2(1 - \theta_0)). \tag{3.5}$$

Let $h_2(x) = \theta_0^x(\theta_0 + x(1 - \theta_0))$. Then

$$\begin{aligned} h_2'(x) &= \theta_0^x((1 - \theta_0) + (\theta_0 + x(1 - \theta_0)) \log \theta_0) \\ &< \theta_0^x(1 - \theta_0 + \log \theta_0) < 0. \end{aligned}$$

i.e., $h_2(x)$ is decreasing in $x$. Thus

$$(3.5) \Rightarrow x_1 > x_2.$$

Also, by Lemma 1, $x_2 > 1$.
Suppose $n < x_m(c_2) < x_m(c_1) < n + 1$ for some integer $n \geq 1$. Then

$$\begin{aligned} J^m(c_1) &= \begin{cases} n & \text{if } g_1(n) < g_2(n+1) \\ n+1 & \text{otherwise} \end{cases} \\ &= \begin{cases} n & \text{if } \frac{1}{c_1} < \frac{1}{l_1}(\frac{\theta_0}{\theta_0 - 1} - (\frac{1}{\theta_0 - 1} - n)\theta_0^n) \\ n+1 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$J^m(c_2) = \begin{cases} n & \text{if } g_(n) < g_2(n+1) \\ n+1 & \text{otherwise} \end{cases}$$
$$= \begin{cases} n & \text{if } \frac{1}{c_2} < \frac{1}{l_1}(\frac{\theta_0}{\theta_0-1} - (\frac{1}{\theta_0-1} - n)\theta_0^n) \\ n+1 & \text{otherwise.} \end{cases}$$

Since

$$1/c_2 < 1/c_1 < \frac{1}{l_1}(\frac{\theta_0}{\theta_0 - 1} - (\frac{1}{\theta_0 - 1} - n)\theta_0^n),$$

$$J^m(c_1) = n \Longrightarrow J^m(c_2) = n.$$

Thus $J^m(c_1) \geq J^m(c_2)$. Suppose that $n < x_1$ and $m < x_2$ where $n > m > 0$. Then obviously $J^m(c_1) > J^m(c_2)$. Hence $J^m(c)$ is monotone decreasing of $c$ if $0 < c \leq l_1(\theta_0 - 1)/\theta_0$.
Now, if $c \geq l_1(\theta_0 - 1)/\theta_0$,

$$g_1(x) = \theta_0^{-x} l_1 + c\frac{1 - \theta_0^{-x}}{1 - \theta_0^{-1}}$$

is increasing in $x$. Thus $\max\{g_1(x), cx\}$ is minimized at $x = 0$. Hence $J^m(c) = 0$ if $c \geq l_1(\theta_0 - 1)/\theta_0$. This proves $J^m(c)$ is monotone decreasing for all $c$. Next, it will be shown that the minimax sequential risk $r_m(c)$ is monotonely decreasing in $c$.

**Theorem 4.** The minimax risk

$$r_m(c) = \min_J \max_{\pi_0} r(\pi_0, d_J)$$

is monotone increasing in $c$.

**Proof.** (a) Assume that $0 < c < l_1(\theta_0 - 1)/\theta_0$. Let $0 < c_1 < c_2 < l_1(\theta_0 - 1)/\theta_0$ be given. Let $J_1 = J^m(c_1), J_2 = J^m(c_2)$. Then by the above theorem, $J_1 \geq J_2 > 1$. And

$$r_m(c_i) = \max\{\theta_0^{-J_i} l_1 + c_i\frac{1 - \theta_0^{-J_i}}{1 - 1/\theta_0}, c_i J_i\}, \quad i = 1, 2.$$

(i) Suppose that $J_1 = J_2$.; Then obviously $r_m(c_1) \le r_m(c_2)$.
(ii) Suppose that $J_1 = J_2 + n$ for some $n \ge 1$. ; If $r_m(c_1) = c_1 J_1$,

$$\begin{aligned} c_1 J_1 &< l_1\theta_0^{J_1-1} + c\frac{1-\theta_0^{J_1-1}}{1-1/\theta_0} (by\ the\ definition\ of\ J_1) \\ &\le l_1\theta_0^{J_2} + c_1\frac{1-\theta_0^{J_2}}{1-1/\theta_0} \quad (since\ g_1(x)\ is\ decreasing\ in\ x) \\ &< l_1\theta_0^{J_2} + c_2\frac{1-\theta_0^{J_2}}{1-1/\theta_0} \quad (c_1 < c_2) \\ &\le r_m(c_2). \end{aligned}$$

If $r_m(c_1) = l_1\theta_0^{-J_1} + c_1\frac{1-\theta_0^{-J_1}}{1-1/\theta_0}$, then since $g_1(x)$ is decresing of $x$ and $c_1 < c_2$,

$$\begin{aligned} r_m(c_1) &= l_1\theta_0^{-J_1} + c_1\frac{1-\theta_0^{-J_1}}{1-1/\theta_0} \\ &< l_1\theta_0^{-J_2} + c_2\frac{1-\theta_0^{-J_2}}{1-1/\theta_0} \\ &\le r_m(c_2). \end{aligned}$$

(b) Assume that $c > l_1(\theta_0 - 1)/\theta_0$. Then $J^m(c) = 0$ from the above theorem. Thus

$$r_m(c) = \max\{l_0, l_1\}.$$

Since $r_m(c) < l_1$ for $0 < c < l_1(\theta_0 - 1)/\theta_0$, $r_m(c)$ is monotone increasing in $c$.

Figure 2 shows the minimax risk with respect to the sampling cost when $\theta_0 = 2$ for $0 - 1$ decision loss.

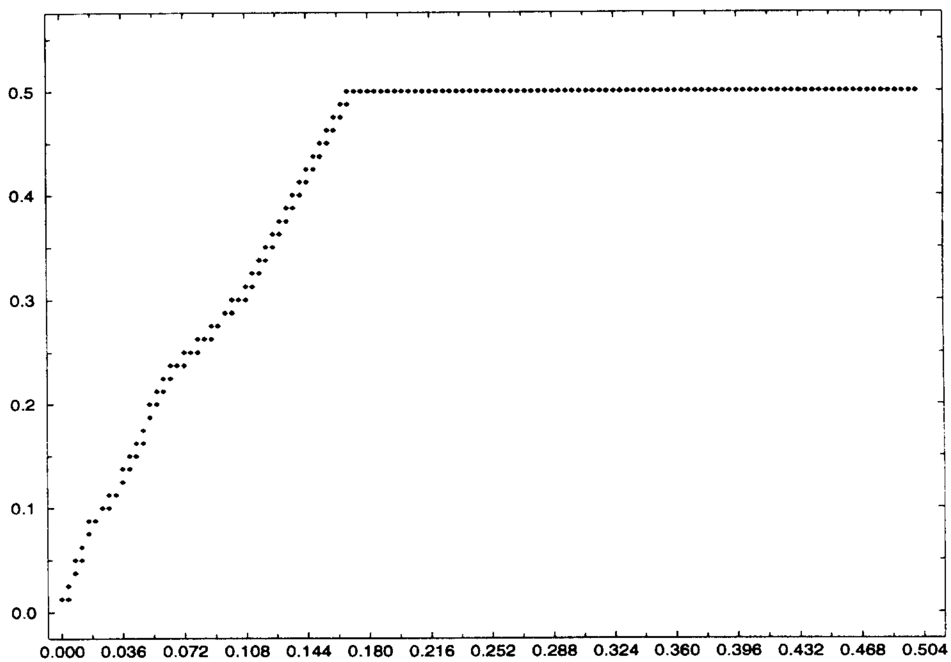



**Figure 2.** Plot of Minimax risk and the sampling cost when $\theta_0 = 2$

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