Journal of the Korea
Statistical Society
Vol. 25, No. 1, 1996

A Model for a Continuous State System with (s, S) Repair Policy †

Won J. Park[1], Eui Yong Lee[2] and Honggie Kim[3]

Abstract

A model for a system whose state changes continuously with time is introduced. It is assumed that the system is modeled by a Brownian motion with negative drift and an absorbing barrier at the origin. A repairman arrives according to a Poisson process and repairs the system according to an (s, S) policy, i.e., he increases the state of the system up to S if and only if the state is below s. A partial differential equation is derived for the distribution function of $X(t)$, the state of the system at time t, and the Laplace-Stieltjes transform of the distribution function is obtained by solving the partial differential equation. For the stationary case the explicit expression is deduced for the distribution function of the stationary state of the system.

Key Words : Brownian motion; (s, S) Repair policy; Poisson process; Laplace-Stieltjes transform; Stationary distribution

† The Present Studies were Supported (in part) by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. BSRI-95-1403.

[1] Department of Mathematics and Statistics Wright State University, Dayton, Ohio 45435, USA

[2] Department of Mathematics, POSTECH, Pohang 790-784, Korea.

[3] Department of Statistics, Chungnam National University, Daejeon, 305-764, Korea

1. INTRODUCTION

In this paper, we introduce a diffusion model for a system the state of which deteriorates continuously with time. As a repair policy, we adopt a sort of (s, S) repair policy. The state of the system is initially $S > 0$ and thereafter follows Brownian motion with negative drift μ, variance σ^2 and an absorbing barrier at the origin. The state of the system is increased by a repairman who arrives according to a Poisson process of rate λ. If the state of the system when the repairman arrives exceeds a threshold $s(0 \leq s < S)$, he does nothing, otherwise the repairman instantaneously increases the state of the system up to S(or replaces the system with a new one).

Baxter and Lee (1987b) introduced a similar diffusion model for a continuous state system, where the amount of a repair is a random variable which is larger than s almost surely. The diffusion model with (s, S) policy has been also introduced as an inventory model with the demand process being a Brownian motion. See Bather (1966) and Harrison (1985). The most of the previous works, however, have been concentrated on the optimal policy concerning the barriers s and S which minimize the cost. We, in this paper, pay our attention to the distribution function of the state of the system for both finite time and limiting time.

The followings are two examples of the application of our model. For more examples, see Baxter and Lee (1987b).

Example 1 (Controlling a Production Process)
Consider a production process which may be modeled by a Brownian motion with negative drift (See Ross (1983, p.201)). Suppose that it is recommended to repair the process if the state falls below a threshold. However, due to the unpredictable demand for the product and/or to the availability of a capable repairman, it may not be possible to stop the process for repair either periodically or whenever the process reaches the threshold. In such a situation, it is better to check and repair (if necessary) the process whenever a possible moment for a repair arises. We assume that such moments comprise a Poisson process.

Example 2 (Fuelling a Power Station)
Suppose that fuel is supplied at a constant rate to a power station by a series of regular deliveries. The demand for electricity is continuously changing so that the stock of unused fuel may be modeled by Brownian motion. The fuel

is supplied at a rate slightly less than the average level of demand to prevent a large stock of fuel from accumulating. An additional supplier visits the power station according to a Poisson process and restocks the fuel if the stock, when he arrives, is below a threshold. Such a situation would arise if, for example, the regular deliveries were made by rail according to a specific timetable and the additional deliveries, perhaps at a reduced price, were made by a ship with an irregular schedule.

In section 2, we derive a partial differential equation for the distribution function of the state of the system at time t, and we solve the partial differential equation to obtain an expression for the Laplace-Stieltjes transform of the distribution function, where some terms are still needed to be evaluated. In section 3, we obtain an explicit formula for the distribution function for the state less than or equal to s by a purely probabilistic argument which does not make use of the partial differential equation, and we use this results to complete the expression for the Laplace-Stieltjes transform of the distribution function obtained in section 2. For the stationary case, the partial differential equation becomes an ordinary differential equation. In section 4, we solve the ordinary differential equation to obtain an explicit formula for the stationary distribution function of the state of the system.

2. THE Laplace-Stieltjes TRANSFORM

Let $X(t)$ denote the state of the system at time t and let $F(x,t)$ denote the distribution function of $X(t)$. If $\Delta(\delta t) = A(t+\delta t) - A(t)$, where $\{A(t), t \geq 0\}$ is Brownian motion with parameter $\mu < 0$ and σ^2, then we can have the following three mutually exclusive events during the small interval $(t, t+\delta t)$:
(a) The repairman does not come, then

$$X(t+\delta t) = \begin{cases} X(t) + \Delta(\delta t), & \text{almost surely if } X(t) + \Delta(\delta t) > 0, \\ 0, & \text{almost surely if } X(t) + \Delta(\delta t) \leq 0. \end{cases}$$

(b) The repairman comes but does nothing since $X(t) > s$, then $X(t+\delta t) = X(t) + \Delta(\delta t)$, almost surely.
(c) The repairman comes and repairs the system since $X(t) \leq s$, then $X(t+\delta t) = S + \Delta(\delta t')$, for some $\delta t' \leq \delta t$, almost surely.

Thus, for $x > 0$,

$$\begin{aligned} F(x, t+\delta t) &= (1-\lambda\delta t)P\{X(t)+\Delta(\delta t) \le x\} + \lambda\delta t P\{X(t)+\Delta(\delta t) \le x, \\ &\quad X(t) > s\} + \lambda\delta t P\{S+\Delta(\delta t') \le x, X(t) \le s\} + o(\delta t), \end{aligned}$$

and for $x = 0$,

$$F(0, t+\delta t) = (1-\lambda\delta t)P\{X(t)+\Delta(t) \le 0\} + o(\delta t).$$

Now, for $x \ge 0$,

$$\begin{aligned} & P\{X(t)+\Delta(\delta t) \le x\} \\ =\; & \int_{-\infty}^{\infty} F(x-y, t)\,dP\{\Delta(\delta t) \le y\} \\ =\; & F(x,t) - E[\Delta(\delta t)]\frac{\partial}{\partial x}F(x,t) + \frac{1}{2}E[\{\Delta(\delta t)\}^2]\frac{\partial^2}{\partial x^2}F(x,t) + o(\delta t) \end{aligned}$$

on performing a Taylor series expansion, assuming that $\frac{\partial}{\partial x}F(x,t)$ and $\frac{\partial^2}{\partial x^2}F(x,t)$ exist. Substituting this expression into the above equation, rearranging and letting $\delta t \to 0$, we have the following partial differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t) = \begin{cases} \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t) - \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(x,t), & \text{for } 0 \le x \le s, \\ \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t) - \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(s,t), & \text{for } s < x < S, \\ \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t) - \mu\frac{\partial}{\partial x}F(x,t), & \text{for } x \ge S. \end{cases} \tag{2.1}$$

Since the origin is an absorbing state, it follows from Cox and Miller (1965, p.219-220) that $f(0,t) = \frac{\partial}{\partial x}F(x,t)\mid_{x=0} = 0$, for all $t > 0$, which is a boundary condition. Taking the ordinary Laplace transform of equation(2.1) with respect to x yields

$$\begin{aligned} \frac{\partial}{\partial t}F^{\circ}(r,t) &= (\frac{1}{2}\sigma^2 r^2 - \mu r)F^{\circ}(r,t) - (\frac{1}{2}\sigma^2 r - \mu)F(0,t) - \lambda F^{s}(r,t) \\ &\quad + \frac{\lambda}{r}F(s,t)(e^{-rS} - e^{-rs}), \end{aligned}$$

where $F^{\circ}(r,t) = \int_0^{\infty} e^{-rx}F(x,t)dx$ and $F^{s}(r,t) = \int_0^{s} e^{-rx}F(x,t)dx$.
Solving the above equation with the boundary condition $F^{\circ}(r,0) = e^{-rS}/r$

gives

$$\begin{aligned} F^\circ(r,t) &= exp\{(\frac{1}{2}\sigma^2 r^2 - \mu r)t\}[\frac{e^{-rS}}{r} - \int_0^t exp\{(\mu r - \frac{1}{2}\sigma^2 r^2)u\} \\ &\times\{(\frac{1}{2}\sigma^2 r - \mu)F(0,u) + \lambda F^s(r,u) - \frac{\lambda}{r}F(s,u)(e^{-rS} - e^{-rs})\}du]. \end{aligned}$$

Since the Laplace-Stieltjes transform of $F(x,t)$ with respect to x, $F^*(r,t)$ say, satisfies $F^*(r,t) = rF^\circ(r,t)$, it follows that

$$\begin{aligned} F^*(r,t) &= exp\{(\frac{1}{2}\sigma^2 r^2 - \mu r)t\}[e^{-rS} - \int_0^t exp\{(\mu r - \frac{1}{2}\sigma^2 r^2)u\} \\ &\times\{(\frac{1}{2}\sigma^2 r^2 - \mu r)F(0,u) + \lambda r F^s(r,u) \\ &-\lambda F(s,u)(e^{-rS} - e^{-rs})\}du]. \end{aligned} \tag{2.2}$$

Equation(2.2) contains the terms $F(0,u), F^s(r,u)$ and $F(s,u)$ which are still needed to be evaluated, and this will be done in the next section.

3. A FORMULA FOR $F(x,t)$, $0 \leq x \leq s$

In this section, we obtain a formula for $F(x,t)$, for $0 \leq x \leq s$, by using an argument similar to that of Baxter and Lee (1987b). Consider the points at which the state of the system crosses the threshold s for the first time after a visit by the repairman. Since the exponential distribution is memoryless and Brownian motion has independent increments, the times between successive points are independent and identically distributed, and hence the sequence of these points forms an embedded renewal process. Let T^* be the generic random variable denoting the time between successive renewals. Then

$$T^* \stackrel{D}{=} T + Y_{U-s} \tag{3.1}$$

where $\stackrel{D}{=}$ denotes equality in distribution, T is an exponential random variable with parameter λ, U is the state of the system immediately after the first visit by the repairman following the previous renewal and Y_{a-b} denotes the first passage time from a to b $(a > b)$ in Brownian motion with parameters $\mu < 0$ and σ^2.

Observe that

$$U \stackrel{D}{=} \begin{cases} S, & \text{if } Z(T) \leq s, \\ Z(T), & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\{Z(t), t \geq 0\}$ denotes Brownian motion with initial state s and absorbing barrier at the origin, and hence the distribution function of U, V say, is given by

$$\begin{aligned} V(x) &= P\{S \leq x, Z(T) \leq s\} + P\{Z(T) \leq x, Z(T) > s\} \\ &= \begin{cases} \int_0^\infty \{B(x,t) - B(s,t)\}\lambda e^{-\lambda t} dt, & \text{for } s < x < S, \\ \int_0^\infty B(x,t)\lambda e^{-\lambda t} dt, & \text{for } x \geq S, \end{cases} \end{aligned}$$

on conditioning on T, where $B(x,t) = P\{Z(t) \leq x\}$. An argument similar to that of Cox and Miller (1965, p.220-221) shows that, for $x \geq 0$,

$$\begin{aligned} & 1 - B(x,t) \\ = & \int_x^\infty \frac{1}{\sigma(\sqrt{2\pi t})}[exp\{-\frac{(z-s-\mu t)^2}{2\sigma^2 t}\} - exp\{-\frac{2\mu s}{\sigma^2} - \frac{(z+s-\mu t)^2}{2\sigma^2 t}\}]dz. \end{aligned}$$

In summary, the distribution function of T^*, K say, is given by

$$K(t) = \int_0^t \int_s^\infty W_{u,s}(t-x)\lambda e^{-\lambda x} dV(u)dx,$$

where $W_{u,s}$ is the distribution function of Y_{u-s}. From Karlin and Taylor [6, p.363], it follows that

$$W_{u,s}(t) = \int_0^t \frac{u-s}{\sigma(\sqrt{2\pi x^3})} exp\{-\frac{(u-s+\mu x)^2}{2\sigma^2 x}\}dx.$$

Let H denote the renewal function of the embedded renewal process, then

$$H(t) = W_{S,s}(t) + \sum_{n=1}^{\infty} W_{S,s} * K^{(n)}(t),$$

where the asterisk denotes Stieltjes convolution and the superscript denotes n-fold recursive Stieltjes convolution.

We now deduce an expression for $F(x,t)$, for $0 \leq x \leq s$. Notice that after a renewal the state of the system follows $\{Z(t), t \geq 0\}$ until the repairman visits and after the repairman's visit the state of the system is over s until the next renewal. Hence, for $0 \leq x \leq s$,

$$F(x,t) = \int_0^t B(x, t-u)e^{-\lambda(t-u)}dH(u). \tag{3.3}$$

Equation (3.3) also enables us to evaluate the undetermined terms in equation(2.2), that is,

$$\begin{aligned} F(0,t) &= \int_0^t B(0, t-u)e^{-\lambda(t-u)}dH(u), \\ F(s,t) &= \int_0^t B(s, t-u)e^{-\lambda(t-u)}dH(u) \quad \text{and} \\ F^s(r,t) &= \int_0^s e^{-rx} \int_0^t B(x, t-u)e^{-\lambda(t-u)}dH(u)dx. \end{aligned}$$

4. THE STATIONARY DISTRIBUTION

We now derive an explicit formula for the stationary distribution of $X(t)$, for the case where $\frac{\partial}{\partial t}F(x,t) = 0$. Notice that this exists, since $\mu < 0$ and is the same as the limiting distribution $F(x) = lim_{t\to\infty}F(x,t)$(c.f. Baxter and Lee (1987a)).

Consider, firstly, the case when $0 \leq x \leq s$. Applying the key renewal theorem to equation (3.3) yields

$$F(x) = \frac{1}{E(T^*)}\int_0^\infty B(x,t)e^{-\lambda t}dt.$$

Baxter and Lee (1987b) have shown that

$$\int_0^\infty B(x,t)e^{-\lambda t}dt = \frac{\theta_2(\lambda)e^{\theta_1(\lambda)(x-s)} - \theta_1(\lambda)e^{\theta_2(\lambda)x-\theta_1(\lambda)s}}{\lambda[\theta_2(\lambda) - \theta_1(\lambda)]}, \tag{4.1}$$

where $\theta_1(\lambda) = (\mu + \sqrt{\mu^2 + 2\lambda\sigma^2})/\sigma^2$ and $\theta_2(\lambda) = (\mu - \sqrt{\mu^2 + 2\lambda\sigma^2})/\sigma^2$. From equation (3.1) and (3.2), it follows that

$$E(T^*) = \frac{1}{\lambda} + \frac{s}{\mu} - \frac{1}{\mu}[SP\{Z(T) \leq s\} + E\{Z(T) \mid Z(T) > s\}P\{Z(T) > s\}].$$

Again, from Baxter and Lee (1987b), it can be seen that

$$P\{Z(T) \leq s\} = \frac{\theta_2(\lambda)}{\theta_2(\lambda)-\theta_1(\lambda)} - \frac{\theta_1(\lambda)}{\theta_2(\lambda)-\theta_1(\lambda)} e^{s[\theta_2(\lambda)-\theta_1(\lambda)]} = q(s,\lambda), \text{say}, \tag{4.2}$$

and

$$E\{Z(t)\} = s + \mu t - \mu \int_0^t B(0,u)du. \tag{4.3}$$

Conditioning on T and making use of equations (4.1), (4.2) and (4.3), we can show that

$$\begin{aligned} &E\{Z(T) \mid Z(T) > s\}P\{Z(T) > s\} \\ &= s - sq(s,\lambda) + \frac{\mu}{\lambda} + \frac{(\theta_2(\lambda))^2 - (\theta_1(\lambda))^2 e^{[\theta_2(\lambda)-\theta_1(\lambda)]s}}{\theta_1(\lambda)\theta_2(\lambda)[\theta_2(\lambda)-\theta_1(\lambda)]}. \end{aligned}$$

In summary, $F(x)$, for $0 \leq x \leq s$, is given by

$$\begin{aligned} F(x) &= \frac{1}{E(T^*)} \int_0^\infty B(x,t)e^{-\lambda t}dt \\ &= \frac{\mu\theta_1(\lambda)\theta_2(\lambda)e^{-s\theta_1(\lambda)}\{\theta_1(\lambda)e^{\theta_2(\lambda)x} - \theta_2(\lambda)e^{\theta_1(\lambda)x}\}}{G(s,S,\lambda)}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} G(s,S,\lambda) &= \lambda(S-s)\theta_1(\lambda)\theta_2(\lambda)\{\theta_2(\lambda) - \theta_1(\lambda)e^{[\theta_2(\lambda)-\theta_1(\lambda)]s}\} \\ &\quad + \lambda(\theta_2(\lambda))^2 - \lambda(\theta_1(\lambda))^2 e^{[\theta_2(\lambda)-\theta_1(\lambda)]s}. \end{aligned}$$

We now consider the case when $x > s$. From equation (2.1), it follows that

$$\begin{cases} \lambda F(s) = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2}F(x) - \mu\frac{d}{dx}F(x), & \text{for } s < x < S, \quad (4.5) \\ 0 = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2}F(x) - \mu\frac{d}{dx}F(x), & \text{for } x \geq S, \quad (4.6) \end{cases}$$

The general solution to equation (4.6) is given by

$$F(x) = C_1 + C_2 e^{2\mu x/\sigma^2},$$

for $x \geq S$, and by using the method of variation constants, we can show that the general solution to equation (4.5) is given by

$$F(x) = C_3 + C_4 e^{2\mu x/\sigma^2} - \frac{\lambda}{\mu}F(s)(x-s) - \frac{\sigma^2\lambda}{2\mu^2}F(s)\{1 - e^{2\mu(x-s)/\sigma^2}\},$$

for $s < x < S$. To evaluate the constants C_1, C_2, C_3 and C_4, we consider the following four conditions:

(1) $F(x)$ is continuous at $x = s$, i.e., $lim_{x \to s^+} F(x) = F(s)$,
(2) $F(x)$ is continuous at $x = S$, i.e., $lim_{x \to S^-} F(x) = F(S)$,
(3) $F(x)$ is differentiable at $x = S$, i.e., $lim_{x \to S^+} F'(x) = lim_{x \to S^-} F'(x)$,
(4) $lim_{x \to \infty} F(x) = 1$.

An easy calculation shows that $F(x)$, for $x > s$, is given by

$$F(x) = \begin{cases} 1 + \frac{\lambda}{\mu}F(s)(S-x) - \frac{\sigma^2\lambda}{2\mu^2}F(s) + \{F(s) - 1 - \frac{\lambda}{\mu}F(s)(S-s) \\ +\frac{\sigma^2\lambda}{2\mu^2}F(s)\}e^{2\mu(x-s)/\sigma^2}, & \text{for } s < x < S, \\ 1 + \{F(s) - 1 - \frac{\lambda}{\mu}F(s)(S-s) + \frac{\sigma^2\lambda}{2\mu^2}F(s)\}e^{2\mu(x-s)/\sigma^2} \\ -\frac{\sigma^2\lambda}{2\mu^2}F(s)e^{2\mu(x-S)/\sigma^2}, & \text{for } x \geq S, \end{cases}$$

where, from equation (4.4),

$$F(s) = \frac{\mu\theta_1(\lambda)\theta_2(\lambda)\{\theta_1(\lambda)e^{[\theta_2(\lambda)-\theta_1(\lambda)]s} - \theta_2(\lambda)\}}{G(s, S, \lambda)}.$$

The stationary distribution obtained in this section is illustrated in the following figures for several cases.

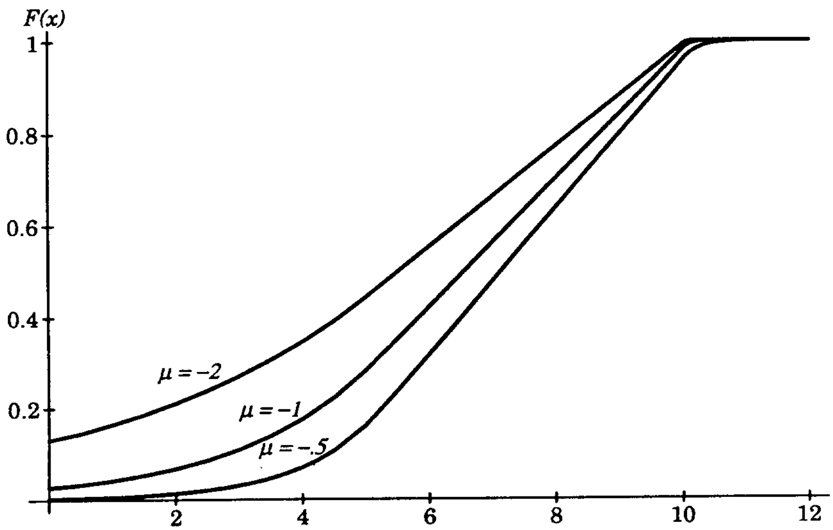

Figure 1. Stationary distribution when $s = 5$, $S = 10$, $\lambda = .5$, $\sigma^2 = .25$ and μ varies.

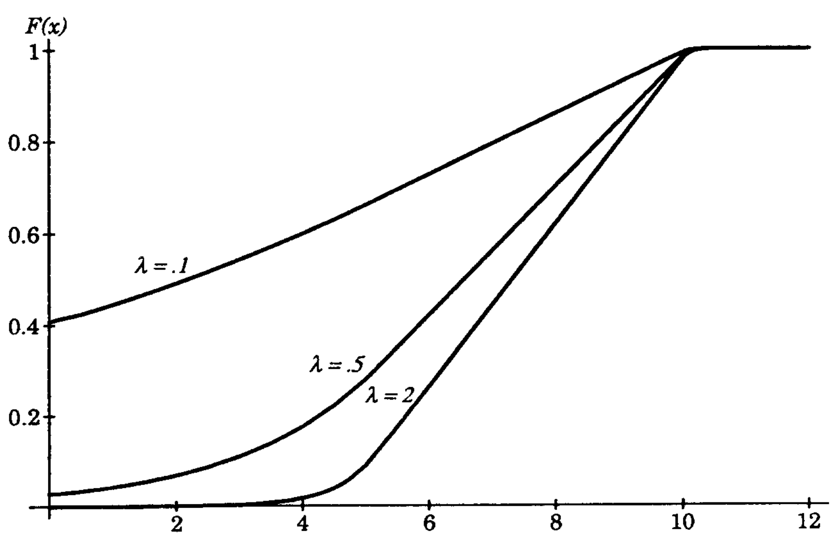

Figure 2. Stationary distribution when $s = 5$, $S = 10$, $\mu = -1$, $\sigma^2 = .25$ and λ varies.

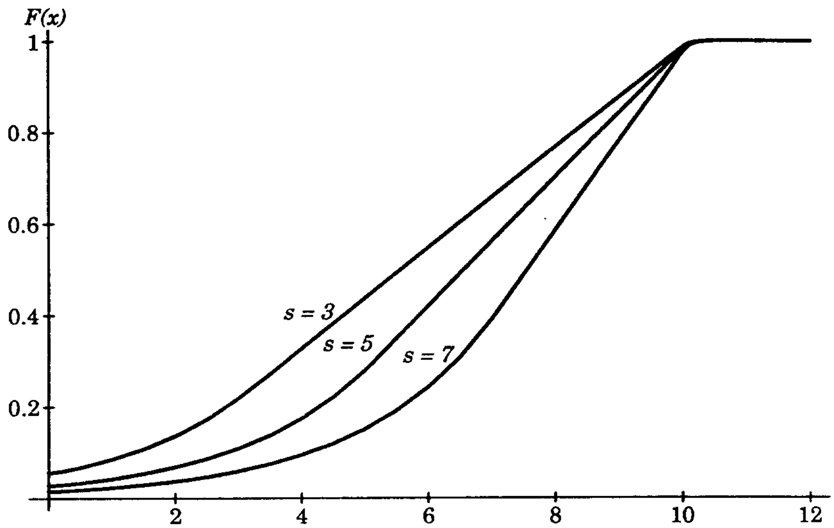

Figure 3. Stationary distribution when $S = 10$, $\lambda = .5$, $\mu = -1$, $\sigma^2 = .25$ and s varies.

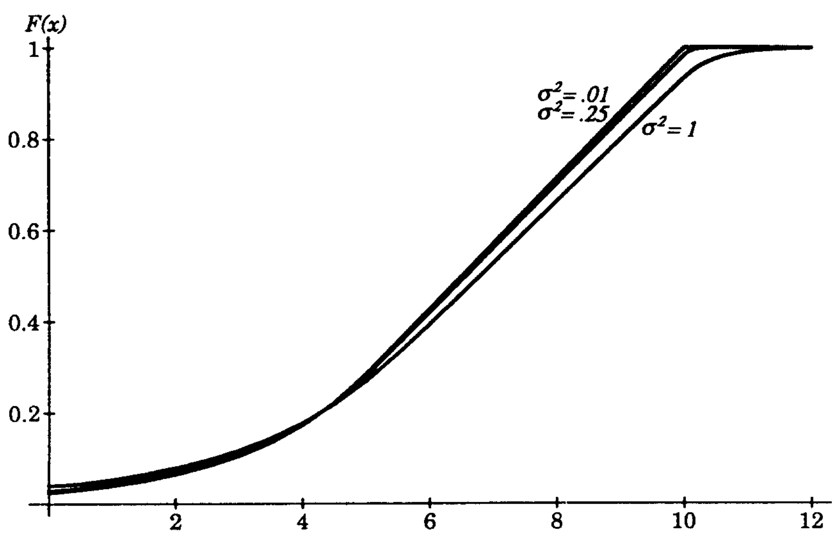

Figure 4. Stationary distribution when $s = 5$, $S = 10$, $\lambda = .5$, $\mu = -1$ and σ^2 varies.

ACKNOWLEDGEMENT

The authors would like to thank a referee for the various helpful comments.

REFERENCES

(1) Bather, J. A. (1966). A continuous time inventory model, *Journal of Applied Probability,* **3**, 538-549.

(2) Baxter, L. A. and Lee, E. Y. (1987). An inventory with constant demand and poisson restocking, *Probability in the Engineering and Informational Sciences,* **1**, 203-210.

(3) Baxter, L. A. and Lee, E. Y. (1987). A diffusion model for a system subject to continuous wear, *Probability in the Engineering and Informational Sciences*, **1**, 405-416.

(4) Cox, D. R. and Miller, H. D. (1965). *The Theory of Stochastic Processes*, Methun, London.

(5) Harrison, J. M. (1985). *Brownian Motion and Stochastic Flow Systems*, John Wiely, New York.

(6) Karlin, S. and Taylor, H. M. (1975). *A First Course in Stochastic Processes*, Academic Press, New York.

(7) Ross, S. M. (1983). *Stochastic Processes*, John Wiely (1983), New York.