구조체의 위상학적 최적화를 위한 비선형 프로그래밍

A Nonlinear Programming Formulation for the Topological Structural Optimization

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요 약

구조물에 있어서 위상학적 최적화 문제는 최적화를 구하는 과정에서 구조체가 변화함으로 인한 어려움 때문에 최적화 분야에서 가장 어려운 문제로 간주되어 왔다. 종래의 방법으로는 일반적으로 구조 요소 사이즈가 영으로 접근할 때 강성 매트릭스의 singularity를 발생시킴으로써 최적의 해를 얻지 못하고 도중에 계산이 종료되어 버린다. 본 연구에 있어서는 이러한 문제점들을 해결하기 위한 비선형 프로그래밍 formulation을 제안하는 것을 목적으로한다. 이 formulation의 주된 특성은 요소 사이즈가 영이 되는 것을 허용한다. 평형 방정식을 등제약조건으로 간주함으로써 강성 매트릭스의 singularity를 피할 수 있다. 이 formulation을 하중을 받는 구조물에 있어서 응력과 변위의 제약조건하에서 중량을 최소화할때의 유한 요소의 두께를 구하는 디자인 문제에 적용하여, 이 formulation이 위상학적 최적화에 있어서의 효과를 입증하였다.

Abstract

The focus of this study is on the problem of the design of structure of undetermined topology. This problem has been regarded as being the most challenging of structural optimization problems, because of the difficulty of allowing topology to change. Conventional approaches break down when element sizes approach to zero, due to stiffness matrix singularity. In this study, a novel nonlinear programming formulation of the topology problem is presented. Its main feature is the ability to account for topology variation through zero element sizes. Stiffness matrix singularity is avoided by embedding the equilibrium equations as equality constraints in the optimization problem. Although the formulation is general, two dimensional plane elasticity examples are presented. The design problem is to find minimum weight of a plane structure of fixed geometry but variable topology, subject to constraints on stress and displacement. Variables are thicknesses of finite elements, and are permitted to assume zero sizes. The examples demonstrate that the formulation is effective for finding at least a locally minimal weight.

Keywords : topology, optimization, nonlinear programming formulation (NLP), hierarchical method, simultaneous method, stiffness matrix singularity, SQP

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1. Introduction

The problem of determining the optimal topology of structures modeled by finite elements is addressed. The problem is defined as follows: given a structure with fixed nodal locations and a list of possible element incidences (the ground structure), and given upper and lower bounds on displacements and stresses arising from loading condition, find the subset of elements, and corresponding sizes, which minimize some function of the design variables. The design problem then includes configurational as well as sizing decisions. Examples of design variables include bar cross-sectional area, plate thickness, and beam moment of inertia.

In principle, the topological nature of the problem can be molded by introducing binary variables which represent the absence or presence of an element. The resulting problem then belongs to the class of mixed-integer nonlinear programming problems (MINLP). This class of problems can be solved in principle by Generalized Benders Decomposition[7] or the Outer Approximation Method of Duran and Grossmann.[5] In general neither of the two methods can guarantee a global optimum for problems in which the nonlinear programming problems (NLP) created by relaxing the integrality constraints is nonconvex. Since this is the case in general for structural optimization problems (e.g.[14]), there is no guarantee that such methods will produce a global optimum. Motivated by the complexity of such approaches, we present a nonlinear programming formulation of the problem which, while not guaranteeing global optimality, does avoid the use of integer variables, thereby greatly reducing the complexity of the problem. The formulation is based on simultaneous analysis and design, in which behavioral constraints are embedded as equality constraints in the optimization model, in contrast to the conventional hierarchical formulation, in which state variables are eliminated from the constraints. In section 2, we shall see that this allows deletion of elements as required by topological optimization.

Our development addresses weight minimization of (possibly) inhomogeneous plate structure subject to stress, displacement; however, the topological formulation for other structures discretized by finite elements and other constraints types is possible and follows a similar development. We assume the optimization problem is solved by projected Lagrangian techniques,[6] which require at least zero- (values of objective and constraints) and first- (objective gradient and constraint Jacobian) order information to construct a linearly-constrained subproblem, the solution of which determines a search direction. For example, the popular sequential quadratic programming (SQP) algorithm uses a quadratic programming subproblem to determine the search direction.

The obvious approach to solving topological optimization problems of allowing zero lower bounds on the size of elements breaks down with a conventional hierarchical formulation, i.e., a formulation which eliminates state variables (e.g. displacements, stress) from the model by solving the equilibrium equations at each optimization iteration. In this formulation, an analysis is performed to provide zero-order information for constraints, and sensitivity information is computed based on the analysis, yielding first-order information. The results are then used to construct a linearly-constrained subproblem, the solution of which is used to find a new search direction. Consequently, the number of optimization variables is equal to the number of design variables, and the constraints are limited to those dictating design. The resulting Jacobian and Hessian matrices are small and dense. If critical element sizes assume zero value (as desired in topological optimization), stiffness matrix singularity can ensue, and the algorithm terminates at a suboptimal solution. The simple fix of altering the structural model as an element size reaches a small value ε is not satisfactory: if ε is too large, the decision to alter the model by dropping an element may be premature (which is important since the element can not be recovered, since it is not contained within the model); if ε is too small, the resulting stiffness matrix may be ill-conditioned, leading to poor calculated displacements and stresses (which can mislead the optimization).

On the other hand, the simultaneous formulation includes the equilibrium equations as equality constraints, and requires only their evaluation and not their solution at each iteration. Its use results in a larger number of constraints and variables, which now include state variables as well as design variables as unknowns. Even though the Jacobian and Hessian matrices are larger, they are sparse, and the total number of nonzeroes is typically much smaller than in the hierarchical formulation. With proper exploitation of sparsity, and especially if the behavior is nonlinear, greater efficiency can be achieved. The optimization process now moves towards a set of variables which simultaneously satisfy equilibrium and minimize the objective. In contrast to the hierarchical formulation, invertibility of stiffness matrix is not required, and substructures can be created by deleting elements (which might cause singularity of the stiffness matrix of the original structure). This is a consequence of the fact that only the residual of the equilibrium equations is required for zero-order information, and only the pseudo-force vectors associated with sensitivity analysis and an evaluation of the stiffness matrix are required for first-order information. The linearly-constrained subproblem is well-posed, the Jacobian matrix has full row rank, and a numerical solution to the subproblem can be readily obtained.

2. NLP Formulation for the Optimal Topology Problem

2.1 General Nonlinearly Constrained Optimization

The general constrained optimization problem may be expressed as:

minimize $F(\mathbf{x})$ objective function

subject to: (1)

$g_i(\mathbf{x})=0 \quad i=1,2,\cdots,m_e$ equality constraints

$g_i(\mathbf{x})\pm 0 \quad i=m_{e+1},\cdots,m$ inequality constraints

where

m_e number of equality constraints

number of total constraints

$\mathbf{x}$ a vector containing optimization design variables

The objective function or any of the constraints imposed on the variables do not always involve only linear functions. Most often the case in design optimization involves nonlinear functions. Then, the problem is said to be one of the class of nonlinear programming problems (NLP).

2.2 Hierarchical Method and Simultaneous Method

2.2.1 Hierarchical Method

Figure 1 shows the general optimum design

process.[1] Thus conventional optimization formulation for structural design (the hierarchical method) does not include equilibrium equations (Ku=P) in its constraints. It eliminates displacement variables in constraints by solving equilibrium equations at each optimization iteration. Hence, the hierarchical formulation is expressed as follows (assume the constraints are related to displacements and stresses
:

$$g_1(x) = C_1[K(x)^{-1}P(x)] - u_b \geq 0 \qquad (2)$$

$$g_2(x) = C_2[K(x)^{-1}P(x)] - \sigma_b \geq 0 \qquad (3)$$

where

$g_1(x)$ displacement constraints

$g_2(x)$ stress constraints

C_1, C_2 matrix of constant coefficients

$K(x)$ stiffness matrix

$P(x)$ a vector of applied loads

u_b displacement limits

σ_b stress limits

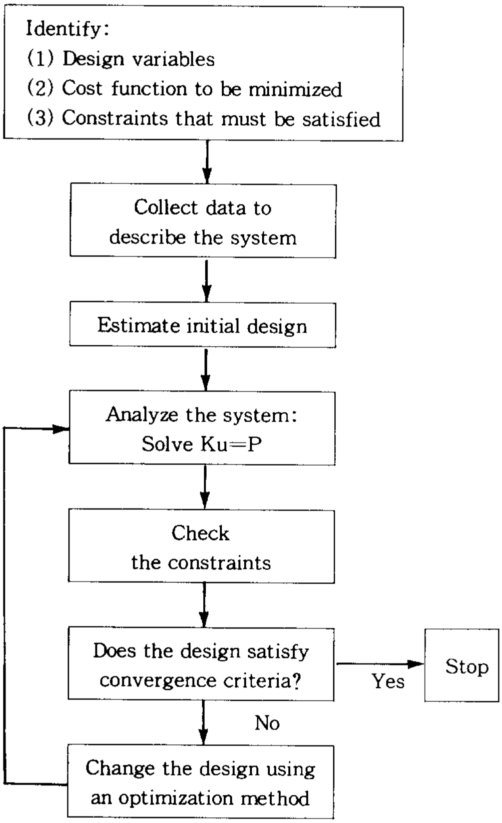


Fig. 1 Optimum design process

Clearly, these constraints are not meaningful when element sizes assume zero values, since the stiffness matrix becomes singular and its inverse no longer exists.

2.2.2 Simultaneous Method

The simultaneous formulation directly includes the equilibrium equations as equality constraints. The simultaneous formulation is expressed as follows:

$$g_1(x) = C_1u(x) - u_b \geq 0 \qquad (4)$$

$$g_1(x) = C_2u(x) - \sigma_b \geq 0 \qquad (5)$$

$$g_e(x) = K(x)u(x) - P(x) = 0 \qquad (6)$$

The problem size becomes larger than that of the hierarchical method because of the larger number of variables in the constraints. But by including the equilibrium equations as equality constraints, one can avoid its singularity. It does not require stiffness matrix inversion. It requires only their evaluations, not their solution, at each optimization iteration.

2.3 NLP Formulation for the Topological Structural Optimization

As stated above, our development addresses the minimum weight of structures. It incorporates zero sizes; hence, the simultaneous method is used to insure that matrix singularity is avoided.

Formulation

The NLP for the optimal topology is formulated and stated as follows:

objective function:

$$\text{minimize } F = \text{total weight} = \sum_{i=1}^{k} A_i \rho_i t_i \quad (7)$$

constraints:

subject to:

Equilibrium equation:

$$\mathbf{K}\mathbf{u} - \mathbf{P} = 0 \quad (8)$$

Stress constraints:

$$\sigma_i^L \le \sigma_i \le \sigma_i^U \qquad i = 1, 2, \cdots, k \quad (9)$$

Displacement constraints:

$$\mathbf{u}^L \le \mathbf{u} \le \mathbf{u}^U \quad (10)$$

Thickness constraints:

$$t_i^L \le t_i \le t_i^U \qquad i = 1, 2, \cdots, k \quad (11)$$

Parameters are defined as:

- k number of total elements
- n number of degree of freedom after applying boundary condition
- $\mathbf{K}$ $n \times n$-stiffness matrix
- $\mathbf{P}$ n-vector of applied nodal loads
- σ_i^L, σ_i^U stress lower (upper) bounds of element i
- $\mathbf{u}^L$, $\mathbf{u}^U$ n-vector of nodal displacement lower (upper) bounds
- t_i^L, t_i^U thickness lower (upper) bounds of element i
- A_i area of element i
- ρ_i density of element i

and the variables are defined as:

- t_i thickness of element i
- $\mathbf{u}$ n-vector of nodal displacement

Remarks

- All functions (7)-(11) are assumed to be continuously differentiable.
- The nonlinearity in this formulation is found in the equilibrium equations (8) and stress constraints (9), which include bilinear product of displacement and thickness. The objective function (7) and all other constraints are linear.
- If none of the t_i^L is zero, then the NLP (7)-(11) is no longer a topological design problem and topology is fixed by the thickness lower bounds.
- There is no guarantee that a unique minimum exist, or that a local minimizer coincides with a global minimizer.
- A single stress constraints (9) or displacement constraints (10) can be chosen, if needed.

2.4 Sequential Quadratic Programming Algorithm

The sequential quadratic programming (SQP) method is generally regarded as the best technique solving the NLP (1),[9] and will be the method of choice in this study. SQP can be derived as a Newton method for solving the first-order constrained stationary conditions.[6] It is based on the iterative formulation and solution of quadratic programming subproblems. These subproblems are defined by an objective function consisting of a quadratic approximation of the Lagrangian function, the minimization of which is subject to linear approximations of the original constraints. That is:

$$\text{minimize } \frac{1}{2} \mathbf{p}_k^T \mathbf{B}(\mathbf{x}_k, \lambda_k) \mathbf{p}_k + F(\mathbf{x}_k)^T \mathbf{p}_k$$

subject to:

$$\nabla g_i(\mathbf{x}_k)^T \mathbf{p}_k + g_i(\mathbf{x}_k) = 0 \qquad i = 1, 2, \cdots, m_e$$

$$\nabla g_i(\mathbf{x}_k)^T \mathbf{p}_k + g_i(\mathbf{x}_k) \le 0 \qquad i = m_{e+1}, \cdots, m$$

$$\mathbf{x}^L - \mathbf{x}_k \le \mathbf{p}_k \le \mathbf{x}^U - \mathbf{x}_k$$

where $\mathbf{B}_k$ is a positive definite approximation of the Hessian of the Lagrangian function. $\mathbf{x}_k$ represents the current iterate points. Let p_k be the solution of the subproblem. A line search is used to find a new point x_{k+1}, where

$$x_{k+1} = x_k + \alpha p_k \qquad \alpha \in (0, 1]$$

such that a merit function will have a lower function value at the new point. The augmented Lagrange function is used here as the merit function. When optimality is not achieved, B_k is updated according to the BFGS formula.

Remark

- SQP applied to this problem requires at least the gradient of objective function and Jacobian matrix of the constraint set with respect to the optimization variables. Second derivative informations can be approximated from differences of first derivatives. These techniques are known as quasi-Newton method.

3. Examples of NLP for Topology Optimization

In this section, several examples are tested to verify and examine the NLP formulation for optimal topology. The NLP is solved using IMSL implementation of the SQP algorithm, which does exploit sparsity of Jacobian and Hessian matrix.

Initial guesses for the displacements are computed from the equilibrium equations for an initial design to initiate the SQP method.

Common data for problems

- Aluminum (Al 6061-T6) is the material, i.e. $E = 70GPa \quad \sigma_y = 240MPa \quad \tau_y = 140MPa$
  $\rho = 0.002710 kg/cm^3 \quad \nu = 0.34615$
- Triangular finite elements are used.
- The structure is in plane stress.
- For stress constraints, 2 principle stress (σ_1, σ_2) and maximum shear stress (τ_{max}) are calculated for each element, and these stresses should be less than (or equal to) the maximum tension (compression, shear) stresses. That is,

$$\sigma_1 \le \sigma_y$$
$$\sigma_2 \ge -\sigma_y$$
$$\tau_{max} \le \tau_y$$

- For thickness constraints, following is used:

$$0 \le \mathbf{t} \le 10^7 cm$$

- Density and areas of all elements are equal in each example, hence, the objective function is set to $F = \sum_{i=1}^{k} t_i$ with the exception of the Example 1. The minimum volume is multiplied by density and area to obtain weight.
  - If thickness of any element reaches zero, stress in that element is defined as zero.
  - SQP terminates when the optimality condition is less than 10^{-7}

3.1 Example 1

At first, the NLP formulation of the topology problem is tested with just a three element model depicted in Figure 2, using different initial thicknesses, density and lower bound. Table 1 shows the values of each case and optimal thicknesses

With elements of the same density, the optimal topology consists of element 1, as expected. However, as element 2 and element 3 decrease in density relative to element 1, the optimal topology should consist of element 2 and element 3, but the resulting topology from NLP is element 1. Since the design space changes abruptly, an initial guess arbitrarily close to the global solution (elements 2 and 3) may not converge to it. For case 1.4, the initial thickness of element 1 taken as zero. Then NLP seeks to reduce the large stress in element 1 by increasing t1, and as a result, an optimal topology consists of element 1.

Convergence depends on initial weight, that is, the larger initial weight of element 1 is, the faster the convergence. Hence, convergence was fastest in case 1.3(it took just 5 iterations). Convergence is fast because the optimal topology consists of element 1, and the initial weight of element 1 is larger than those of elements 2 and 3. Convergence of case 1.1 is slowest (it took 130 iterations). Interestingly,

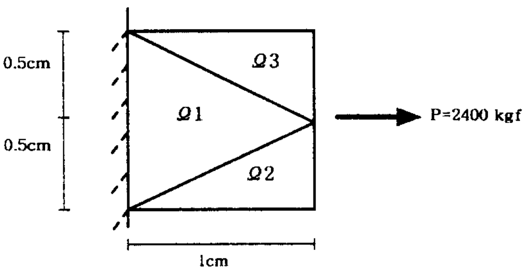


Fig. 2 Model for Example 1

Table 1 Optimal thickness and minimum weight of example 1

case number	element number	initial thick. (cm)	density (kg/cm³)	Lower bound(cm)	optimal thick.(cm)	min. weight (kg)
case 1.1	Ω1	1.50	2.71	0.00	2.00	2.71
	Ω2	1.50	2.71	0.00	0.00	
	Ω3	1.50	2.71	0.00	0.00	
case 1.2	Ω1	1.50	2.71	0.00	2.00	2.71
	Ω2	1.50	0.00271	0.00	0.00	
	Ω3	1.50	0.00271	0.00	0.00	
case 1.3	Ω1	1.50	2.7	0.00	2.00	2.71
	Ω2	0.50	0.00271	0.00	0.00	
	Ω3	0.50	0.00271	0.00	0.00	
case 1.4	Ω1	0.00	2.71	0.00	2.00	2.71
	Ω2	5.50	0.00271	0.00	0.00	
	Ω3	5.50	0.00271	0.00	0.00	
case 1.5	Ω1	1.50	2.71	0.50	1.72	3.0081
	Ω2	1.50	2.71	0.50	0.50	
	Ω3	1.50	2.71	0.50	0.50	

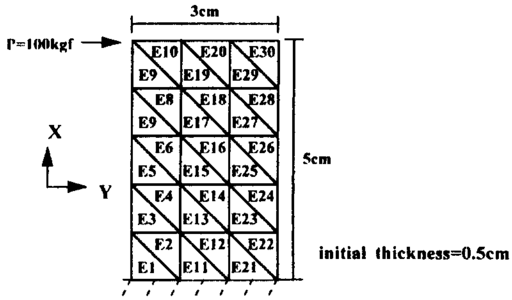


Fig. 3 Model for Example 2

in case 1.5, lower bound of thickness (t_i^L) was set to 0.5 cm, and the other conditions were the same as case 1.1. The resulting optimal thicknesses were different from case 1.1, however the resulting displacements were identical.

3.2 Example 2

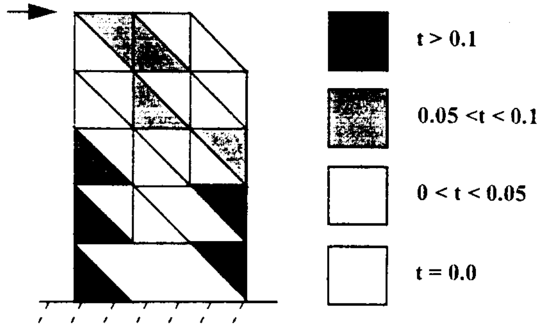
In this section, the NLP formulation is tested, using each constraint with the test model of Figure 3. Constraints are chosen as follows:

Case 2.1: stress constraints only

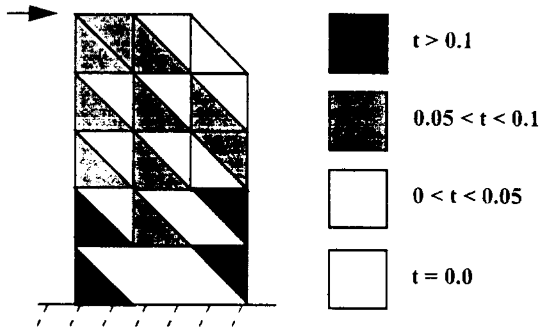
Case 2.2: displacement constraints only($-0.1cm \leq u \leq 0.1cm$)

Case 2.3: displacement and stress constraints

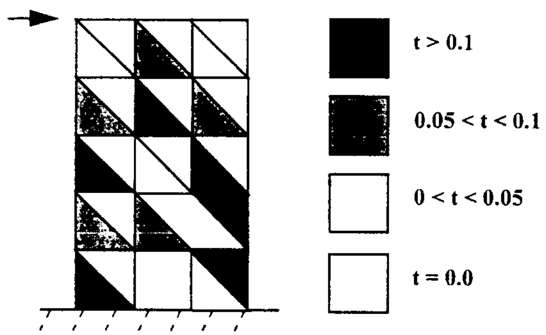
Figure 4 shows the optimal topology. The topologies of case 2.1 and case 2.2 are similar, but the element thicknesses are different, and the stresses of case 2.2 are very high(some values are twice of σ_y). In this example, we found that the optimal topology usually depends on only one constraint even though two constraints (stress and displacement) are applied. That is, the topology with two constraints is same as the topology with just stress constraint or displacement constraint. Case 2.3 was chosen so that both stress and displacement constraints are active.



(a) stress constraints



(b) displacement constraints



(c) stress+displacement constraints

Fig. 4 Optimal topologies of example 2

Table 2 Minimum weight and maximum displacement of example 2

case number	minimum weight (kg)	maximum displacement (cm)
case 2.1	0.0014635	0.043382
case 2.2	0.00063429	0.1
case 2.3	0.00306549	0.027285

4. Conclusion

We have presented an NLP formulation for the optimal topology problem of structure. This problem has been regarded as posing the greatest difficulty to successful optimal design. The formulation guarantees at least a local minimum. Potential singularity of the stiffness matrix is avoided by embedding the behavioral equations as equality constraints in the optimization problem. Arbitrary objective functions, stress and displacement constraints, and upper and lower bounds on and linking of the design variables can be easily handled. The formulation is demonstrated on a number of examples of topology optimization of plate structures loaded in plane, and shown to be robust under a variety of constraints.

In this study, the formulation was tested using coarse meshes and applied under a single loading condition. However, it would be desirable to apply the formulation to multiple loading conditions with finer meshes. Thus, we will test the formulation under multiple loading conditions in the not-too-distance future.

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