A Study on the Stochastic Sensitivity in Structural Dynamics

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구조물의 동적 응답에 대한 확률 민감도 해석에 관한 연구

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要 約

구조물의 동적 응답 해석 문제에 대해서, 확률 유한요소법을 논의코자, 기존의 유한요소 해석법에 수반 변수법(adjoint variable approach)과 2차 섭동법(second order perturbation method)을 적용한다. 동적 민감도의 시간 응답을 고려하기 위해서 모든 시간에 대해서 갖는 구속 조건의 범함수 형태를 취하고, 시간 적분에 있어서 중첩법(fold superposition technique)에 근거를 둔 수치 해석이 훨씬 더 효과적임을 보인다. 본 논문의 확률 유한요소 해석법은 기존의 유한요소 코드에 맞추어 쉽게 적용할 수 있는 잇점이 있음을 보이며, 이의 검정을 위해서, 2차원과 3차원 프레임 구조물에 대한 수치 해석을 하고 그 결과를 검토해 보았다.

INTRODUCTION

The randomness are concerned with the uncertainties in the design parameters, which is the so-called inherent variability. Randomness may exist in the characteristics of the structure itself (e.g. material properties, member size) or in the environment to which the structure is exposed (e.g. load, support conditions). In some structure, the response is sensitive to both the material and geometric properties of the structure as well as to the applied loads even small uncertainties in these characteristics can adversely affect the structural performance. Moreover, since such uncertainties are usually spatially distributed over the region of the structure and must be modelled as random fields, and the structures themselves are frequently so complex as to exclude their analytical analysis even in the deterministic case, the need for an effective numerical tool to deal with a broad class of stochastic structural problems becomes evident.

The basic methodology being adopted in this paper to quantify structural response uncertainties is a Taylor series expansion to formulate linear relationships between some characteristics of the random response and the random structural parameters on the basis of the perturbation approach. The SFEA as the second-order version in this paper is exclusively used as it seems both theortically sound and computationally feasible to apply even for very large structual systems by using techniques typical of contemporary computational mechanics. In particular, we shall present a version of the finite element approach which accounts for uncertaints in both the geometry and/or meterial properties of the structure, as well as in the applied load. The purpose of the SDS analysis in the SFEA is to consider the dependence of a structural state functional on design variables, i.e. to evaluate the change in structural response with the variations of structural parameters. In the SFEA of structure, the probabilistic characteristics of the response are estimated in terms of the variations of the structural parameters due to their randomness. The generalized co - ordinates are normalized and the correlated random variables are transformed to uncorrelated variables, whereas the secularities are eliminated by the fast Fourier Transform of complex valued sequences[1)-4)].

The stochastic finite element models have been developed by Hisada and Nakagiri[5)], and Liu, Belytschko, Besterfield and Mani[1),6)-7)]. The theoretical foundation of sensitivity analysis has been formulated by Zienkiewicz and Campbell[8)], Haug and Arora[9)], Haug[10)], Rousselet[11)] and Mroz and Haftka[12)]. The computational aspects in problems of statics, free vibrations and dynamics have been discussed by Haug et al.[13)] and Hien and Kleiber[14)], for instance. On the other hand, to incorporate uncertainties in geometrical and material properties of structural members into the displacement and stress fields the concepts of discretized random fields has been introduced by Vanmarcke[15)].

This paper is concerned with this topic. Attention is restricted to linear structural mechanics, i.e. to systems whose governing equations of motion are linear in state variables once the random variables and design variables are fixed.

STOCHASTIC FINITE ELEMENT FORMULATION

Stochastic Finite Element Approach

In order to apply the finite element technique let us first assume that the region of interest Ω has been discretized by a finite element mesh. The basic idea of the mean - based, second - order, second - moment analysis in SFEA is to expand, via Taylor series, all the vector and matrix stochastic field variables typical of deterministic FEM about the mean values of random variables $b_r(x_k)$, to retain only up to second - order terms and to use in the analysis only the first two statistical moments. In this way equations for the expectations and cross - covariances(auto covariances) of the nodal displacements can be obtained in terms of the nodal displacement derivatives with respect to the random variables.

In the structure of the FEM concept the fields $b_r(x_k)$ have to be represented by a set of basic random variables. Thus, it is necessary to discretize $b_r(x_k)$ by expressing them in terms of some nodal values of the appropriate means and covariances. The following approximation is adopted :

$$b_r(x_k)=\phi_{\bar{\alpha}}(x_k)b_{r\bar{\alpha}}$$
$$r=1, 2, \cdots, R \; ; \; \bar{\alpha}=1, 2, \cdots, \bar{N} \qquad (1)$$

where, $\phi_{\bar{\alpha}}$ is the shape function for the $\bar{\alpha}$ - th nodal point, $\bar{N}$ is the number of nodal points in the mech and $b_{r\bar{\alpha}}$ is the matrix of random parameter nodal values ; for a fixed r the vector $b_{r\bar{\alpha}}$; $\bar{\alpha}=1, 2, \cdots, \bar{N}$, contains as its entries the successing nodal values of the random variables b_r.

By introducing now a vector of nodal random variables b_ρ, $\rho=1, 2, \cdots, \tilde{N}=R\times\bar{N}$, related to the matrix $b_{r\bar{\alpha}}$ by an approximate transformation

$$b_{r\bar{\alpha}}=A_{r\bar{\alpha}\rho}b_\rho \tag{2}$$

The random variables of equation (1), b_r becomes

$$b_r(x_k)=\phi_{\bar{\alpha}}(x_k)A_{r\bar{\alpha}\rho}b_\rho=\phi_{r\rho}(x_k)b_\rho \tag{3}$$

From equation (1)

$$\mathrm{E}[b_r(x_k)]=b_r^0(x_k)=\phi_{r\rho}(x_k)b_\rho^0 \tag{4-1}$$

$$\mathrm{Cov}(b_r(x_k), b_s(x_k))=S_b^{rs}=\phi_{r\rho}(x_k)\phi_{s\rho}(x_k)S_b^{\rho\sigma} \tag{4-2}$$

and $\quad \Delta b_r(x_k)=\phi_{r\rho}(x_k)\Delta b_\rho \quad (5)$

where $\quad \Delta b_\rho=b_\rho-b_\rho^0 \quad (6)$

and b_ρ^0 and $S_b^{\rho\sigma}$ stand for the mean values vector and the covariance matrix of the nodal random variables vector b_ρ, respectively. The remaining random field variables in the problem considered, i.e. elastic moduli $C_{ijkl}(x_k)$, mass density $\rho(x_k)$, body force $f_i(x_k, \tau)$, boundary traction $\hat{t}_i(x_k, \tau)$ and displacements $u_i(x_k, \tau)$ are expand the same shape functions as :

$$\begin{aligned}
C_{ijkl}[b_r(x_k)\,;\,x_k]&=C_{ijkl}[\phi_{r\rho}(x_k)b_\rho\,;\,x_k]\\
&=\phi_{\bar{\alpha}}(x_k)C_{ijkl\bar{\alpha}}(b_\rho)\\
\rho[b_r(x_k)\,;\,x_k]&=\rho[\phi_{r\rho}(x_k)b_\rho\,;\,x_k]=\phi_{\bar{\alpha}}(x_k)\rho_{\bar{\alpha}}(b_\rho)\\
f_i[b_r(x_k)\,;\,x_k,\,\tau]&=f_i[\phi_{r\rho}(x_k)b_\rho\,;\,x_k,\,\tau]\\
&=\phi_{\bar{\alpha}}(x_k)f_{i\bar{\alpha}}(b_\rho\,;\,\tau)\\
\hat{t}_i[b_r(x_k)\,;\,x_k,\,\tau]&=\hat{t}_i[\phi_{r\rho}(x_k)b_\rho\,;\,x_k,\,\tau]\\
&=\phi_{\bar{\alpha}}(x_k)\hat{t}_{i\bar{\alpha}}(b_\rho\,;\,\tau)\\
u_i[b_r(x_k)\,;\,x_k,\,\tau]&=u_i[\phi_{r\rho}(x_k)b_\rho\,;\,x_k,\,\tau]\\
&=\phi_{\bar{\alpha}}(x_k)u_{i\bar{\alpha}}(b_\rho\,;\,\tau)\\
&=\phi_{i\alpha}(x_k)q_\alpha(b_\rho\,;\,\tau)
\end{aligned}$$

$$\bar{\alpha}=1, 2, \cdots, \bar{N}\,;\ \alpha=1, 2, \cdots, N\,;\ \rho=1, 2, \cdots, \tilde{N}\,;\ i=1, 2, 3 \tag{7}$$

where, N is the total number of degree of freedom in the discretized model.

Substituting the finite elements approximation into the zeroth, first and second - order variational statements, employing the standard perturbation procedure and using the arbitariness of δq_α the following 'hierarchical' finite element equations of motion are obtained :

- zeroth - order ;

$$\begin{aligned}
&M_{\alpha\beta}^0(b_\rho^0)\ddot{q}_\beta^0(b_\rho^0\,;\,\tau)+C_{\alpha\beta}^0(b_\rho^0)\dot{q}_\beta^0(b_\rho^0\,;\,\tau)\\
&\quad+K_{\alpha\beta}^0(b_\rho^0)q_\beta^0(b_\rho^0\,;\,\tau)=Q_\alpha^0(b_\rho^0\,;\,\tau)
\end{aligned} \tag{8}$$

- lst - order ;

$$\begin{aligned}
&M_{\alpha\beta}^0(b_\rho^0)\ddot{q}_\beta^{,\rho}(b_\rho^0\,;\,\tau)+C_{\alpha\beta}^0(b_\rho^0)\dot{q}_\beta^{,\rho}(b_\rho^0\,;\,\tau)\\
&\quad+K_{\alpha\beta}^0(b_\rho^0)q_\beta^{,\rho}(b_\rho^0\,;\,\tau)=Q_2^{,\rho}(b_\rho^0\,;\,\tau)\\
&\quad-[M_{\alpha\beta}^{,\rho}(b_\rho^0)\ddot{q}_\beta^0(b_\rho^0\,;\,\tau)+C_{\alpha\beta}^{,\rho}(b_\rho^0)\dot{q}_\beta^0(b_\rho^0\,;\,\tau)\\
&\quad+K_{\alpha\beta}^{,\rho}(b_\rho^0)q_\beta^0(b_\rho^0\,;\,\tau)
\end{aligned} \tag{9}$$

- 2nd - order ;

$$\begin{aligned}
&M_{\alpha\beta}^0(b_\rho^0)\ddot{q}_\beta^{(2)}(b_\rho^0\,;\,\tau)+C_{\alpha\beta}^0(b_\rho^0)\dot{q}_\beta^{(2)}(b_\rho^0\,;\,\tau)\\
&\quad+K_{\alpha\beta}^0(b_\rho^0)q_\beta^{(2)}(b_\rho^0\,;\,\tau)=\{Q_\alpha^{,\rho\sigma}(b_\rho^0\,;\,\tau)\\
&\quad-2[M_{\alpha\beta}^{,\rho}(b_\rho^0)\ddot{q}_\beta^{,\sigma}(b_\rho^0\,;\,\tau)+C_{\alpha\beta}^{,\rho}(b_\rho^0)\dot{q}_\beta^{,\sigma}(b_\rho^0\,;\,\tau)\\
&\quad K_{\alpha\beta}^{,\rho}(b_\rho^0)\dot{q}_\beta^{,\sigma}(b_\rho^0\,;\,\tau)]-[M_{\alpha\beta}^{,\rho\sigma}(b_\rho^0)\ddot{q}_\beta^0(b_\rho^0\,;\,\tau)\\
&\quad+C_{\alpha\beta}^{,\rho\sigma}(b_\rho^0)\dot{q}_\beta^0(b_\rho^0\,;\,\tau)+K_{\alpha\beta}^{,\rho\sigma}(b_\rho^0)q_\beta^0(b_\rho^0\,;\,\tau)]\}S_b^{\rho\sigma}
\end{aligned} \tag{10}$$

where, $\quad q_\alpha^{(2)}(b_\rho^0\,;\,\tau)=q_\alpha^{,\rho\sigma}(b_\rho^0\,;\,\tau)S_b^{\rho\sigma} \quad (11)$

From equations (8) - (11), the zeroth - order mass, damping and stiffness matrices and load vector and their first and second mixed derivatives with respect to nodal random variables b_ρ are defined as follows.

- zeroth - order functions ;

$$M_{\alpha\beta}^0(b_\rho^0)=\int_\Omega \phi_{\bar{\alpha}}\,\rho_{\bar{\alpha}}^0\,\phi_\alpha\,\phi_\beta\,d\Omega \tag{12-1}$$

$$C_{\alpha\beta}^0(b_\rho^0)=\int_\Omega \phi_{\bar{\alpha}}\,\phi_{\bar{\beta}}(\alpha_{\bar{\alpha}}^0\,\rho_{\bar{\beta}}^0\,\phi_\alpha\,\phi_\beta+\beta_{\bar{\alpha}}^0\,C_{ijkl\bar{\beta}}^0\,B_{ij\alpha}\,B_{kl\beta})d\Omega \tag{12-2}$$

$$K_{\alpha\beta}^0(b_\rho^0)=\int_\Omega \phi_{\bar{\alpha}}\,C_{ijkl\bar{\alpha}}^0\,B_{ij\alpha}\,B_{kl\beta}\,d\Omega \tag{12-3}$$

$$Q_\alpha^0(b_\rho^0;\tau)=\int_\Omega \phi_{\bar{\alpha}}\,\phi_{\bar{\beta}}\,\rho_{\bar{\alpha}}^0\,f_{i\beta}^0\,\phi_\alpha\,d\Omega+\int_{\partial\Omega_\sigma}\phi_{\bar{\alpha}}\,\hat{t}_{i\alpha}^0\,\phi_\alpha\,d(\partial\Omega) \tag{12-4}$$

- first partial derivatives ;

$$M_{\alpha\beta}^{,\rho}(b_\rho^0)=\int_\Omega \phi_{\bar{\alpha}}\,\rho_{\bar{\alpha}}^{,\rho}\,\phi_\alpha\,\phi_\beta\,d\Omega \tag{13-1}$$

$$C_{\alpha\beta}^{,\rho}(b_\rho^0)=\int_\Omega \phi_{\bar{\alpha}}\,\phi_{\bar{\beta}}[(\alpha_{\bar{\alpha}}^{,\rho}\,\rho_{\bar{\beta}}^0+\alpha_{\bar{\alpha}}^0\,\rho_{\bar{\beta}}^{,\rho})\phi_\alpha\,\phi_\beta$$

$$+(\beta_{\bar{\alpha}}^{,\rho} C_{ijkl\bar{\beta}}^{0} + \beta_{\bar{\alpha}}^{0} C_{ijkl\bar{\beta}}^{,\rho}) B_{ij\alpha} B_{kl\beta}] d\Omega \tag{13-2}$$

$$K_{\alpha\beta}^{,\rho}(b_\rho^0) = \int_\Omega \phi_{\bar{\alpha}} C_{ijkl\bar{\alpha}}^{,\rho} B_{ij\alpha} B_{kl\beta}\, d\Omega \tag{13-3}$$

$$Q_\alpha^{,\rho}(b_\rho^0;\tau) = \int_\Omega \phi_{\bar{\alpha}} \phi_{\bar{\beta}} (\rho_{\bar{\alpha}}^{,\rho} f_{i\beta}^0 + \rho_{\bar{\alpha}}^0 f_{i\bar{\beta}}^{,\rho}) \phi_\alpha d\Omega + \int_{\partial\Omega_\sigma} \phi_{\bar{\alpha}} \hat{t}_{i\bar{\alpha}}^{,\rho} \phi_\alpha\, d(\partial\Omega) \tag{13-4}$$

- second partial derivatives ;

$$M_{\alpha\beta}^{,\rho\sigma}(b_\rho^0) = \int_\Omega \phi_{\bar{\alpha}} \rho_{\bar{\alpha}}^{,\rho\sigma} \phi_\alpha \phi_\beta\, d\Omega \tag{14-1}$$

$$C_{\alpha\beta}^{,\rho\sigma}(b_\rho^0) = \int_\Omega \phi_{\bar{\alpha}} \phi_{\bar{\beta}} [(\alpha_{\bar{\alpha}}^{,\rho\sigma} \rho_{\bar{\beta}}^0 + \alpha_{\bar{\alpha}}^{,\rho} \rho_{\bar{\beta}}^{,\sigma} + \alpha_{\bar{\alpha}}^{,\sigma} \rho_{\bar{\beta}}^{,\rho} + \alpha_{\bar{\alpha}}^0 \rho_{\bar{\beta}}^{,\rho\sigma}) \phi_\alpha \phi_\beta + (\beta_{\bar{\alpha}}^{,\rho\sigma} C_{ijkl\bar{\beta}}^0 + \beta_{\bar{\alpha}}^{,\rho} C_{ijkl\bar{\beta}}^{,\sigma} + \beta_{\bar{\alpha}}^0 C_{ijkl\bar{\beta}}^{,\rho\sigma}) B_{ij\alpha} B_{kl\beta}] d\Omega \tag{14-2}$$

$$K_{\alpha\beta}^{,\rho\sigma}(b_\rho^0) = \int_\Omega \phi_{\bar{\alpha}} C_{ijkl\bar{\alpha}}^{,\rho\sigma} B_{ij\alpha} B_{kl\beta}\, d\Omega \tag{14-3}$$

$$Q_\alpha^{,\rho\sigma}(b_\rho^0;\tau) = \int_\Omega \phi_{\bar{\alpha}} \phi_{\bar{\beta}} (\rho_{\bar{\alpha}}^{,\rho\sigma} f_{i\beta}^0 + \rho_{\bar{\alpha}}^{,\rho} f_{i\beta}^{,\sigma} + \rho_{\bar{\alpha}}^{,\sigma} f_{i\bar{\beta}}^{,\rho} + \rho_{\bar{\alpha}}^0 f_{i\bar{\beta}}^{,\rho\sigma}) \phi_\alpha d\Omega + \int_{\partial\Omega_\sigma} \phi_{\bar{\alpha}} \hat{t}_{i\bar{\alpha}}^{,\rho\sigma} \phi_\alpha\, d(\partial\Omega) \tag{14-4}$$

All the functions (12) - (14) are evaluated at the expectations b_ρ^0 of the nodal random variables b_ρ. Having solved equations (8) - (11) for $q_\alpha^0(b_\rho^0\,;\,\tau)$, $q_\alpha^{,\rho}(b_\rho^0\,;\,\tau)$, $q_\alpha^{(2)}(b_\rho^0\,;\,\tau)$ and their time derivatives, the probabilistic distributions for the nodal displacements, velocities and accelerations as well as for the element strains and stresses can be determined. By the assumption that the coefficient of variation in the problem considered are not too large (variances of random variables are small when compared with their expected values) all the solutions can be obtained fomally by setting $\varepsilon=1$ in the expansions.

Thus, the random displacement field can now expressed as,

$$q_\alpha(b_\rho\,;\,\tau) = q_\alpha^0(b_\rho^0\,;\,\tau) + q_\alpha^{,\rho}(b_\rho^0\,;\,\tau)\Delta b_\rho + \frac{1}{2} q_\alpha^{,\rho\sigma}(b_\rho^0\,;\,\tau)\Delta b_\rho \Delta b_\sigma \tag{15}$$

The dependence of random fields on $b_r(x_m)$ of b_ρ^0 will not be explicitly indicated unless confusion is likely to arise. By the definition of the nodal displacement expectations at any time instant τ=t and cross - covariances at $\xi_1=(x_m^{(1)}, t_1)$ and $\xi_2=(x_m^{(2)}, t_2)$ we have, repectively

$$E[q_\alpha(t)] = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{\tilde{N}\text{-fold}} q_\alpha(t) p_{\tilde{N}}(b_1, b_2, \cdots, b_{\tilde{N}})\, db_1 db_2 \cdots db_{\tilde{N}} \tag{16}$$

$$\mathrm{Cov}(q_\alpha(t_1), q_\beta(t_2)) = S_q^{\alpha\beta}(t_1, t_2) = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{\tilde{N}\text{-fold}} \{q_\alpha(t_1) - E[q_\alpha(t_1)]\} \{q_\beta(t_2) - E[q_\beta(t_2)]\} \times p_{\tilde{N}}(b_1, b_2, \cdots, b_{\tilde{N}})\, db_1 db_2 \cdots db_{\tilde{N}} \tag{17}$$

where, $p_{\tilde{N}}(b_1, b_2, \cdots, b_{\tilde{N}})$ is the tilde $\tilde{N}$- variate probability density function(PDF).

Substituting equation (15) into eqation (16), using equation (6) and observing that the terms involving the first variation of Δb_ρ vanish, yields the second - order accurate expectations for the nodal displacements at any $\tau=t$ as

$$E[q_\alpha(t)] = q_\alpha^0(t) + \frac{1}{2} q_\alpha^{,\rho\sigma}(t) S_b^{\rho\sigma} = q_\alpha^0(t) + \frac{1}{2} q_\alpha^{(2)}(t) \tag{18}$$

By using equation (15), the same as the nodal displacement, (15) - (17), random field responses of strain and stress, i.e. the strain and stress tensor (random field), the second - order accurate expectation of strain and stress components at any time instant $\tau=t$ and their first - order accurate cross - covariances at $\xi_1=(x_m^{(1)}, t_1)$ and $\xi_2=(x_m^{(2)}, t_2)$ of the strain and stress, may be obtained.

Stochastic Structural Sensitivity

Consider structural response of the linear - elastic system with N degrees of freedom described by an integral functional

$$\phi = \int_0^T G[q(h,b,t), h]\, dt \tag{19}$$

The system satisfies the equations of motion

$$M_{ij}(h,b)\ddot{q}_j(h,b,t)+D_{ij}(h,b)\dot{q}_j(h,b,t)+K_{ij}(h,b)$$
$$q_j(h,b,t)=f_i(h,d,t)$$
$$q_j(h,b,0)=0,$$
$$q_j(h,b,0)=0, i,j=1,\cdots,N \quad (20)$$

where symbols $h=\{h^e\}$, $e=1,\cdots,E$, $b=\{b^r\}$, $r=1,\cdots,R$, and t, $t\in[0, T]$ denote the vector of design variables, the vector of random variables and time variable, respectively. Clearly, some or all components in the vectors h and b ca n coincide. The random functions $q=q_i(h,b,t)$, $M_{ij}(h,b)$, $D_{ij}(h,b)$, $K_{ij}(h,b)$ and $f_i(h,b,t)$ represent the nodal displacement vector, mass, damping, stiffness matrices and load vector, respectively.

To incorporate into the formulation uncertainties of structural material, geometry and load the second order perturbation approach is used. For the random functions M_{ij}, D_{ij}, K_{ij}, f_i, $G_{,i}$, q_i, λ_i, M_{ij}^e, D_{ij}^e, K_{ij}^e, f_i^e, G^e the Taylor expansion is done about the spatial expectations of the random variables $b_0{}^r$, denoted by b^r, with a given small parameter θ. For instance, for the first derivative of the stiffness with respect to the design variable he we write

$$K_{ij}^{e}(h,b)=K_{ij}^{0,e}+\theta K_{ij}^{e,r}\Delta b^r+\frac{1}{2}\theta^2 K_{ij}^{e,rs}\Delta b^r\Delta b^s \quad (21)$$
$$i,j=1,\cdots,N,\ e=1,\cdots,E,\ r,s=1,\cdots,R$$

where $\theta\Delta b^r\equiv\delta b^r$ denotes the first order variation of b^r about $b_0{}^r$; $(.)^0$, $(.)^r$ and $(.)^{rs}$ represent the expectation, first and second(mixed) partial derivatives with respect to the random variables evaluated at their expectations, respectively. It is noted that functions with superscript '0' are deterministic, whereas functions with superscripts ',r' and ',rs' are random. These random functions can be expressed through the first two moments of random variables as

$$b_0^r=E(b^r)=\int_{-\infty}^{+\infty}b^r g(b^r)db^r \quad (22)$$

$$\mathrm{Cov}(b^r,b^s)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(b^r-b_0^r)(b^s-b_0^s)g(b^r,b^s)db^r db^s \quad (23-1)$$

or, equivalently

$$\mathrm{Cov}(b^r,b^s)=R(b^r,b^s)\sqrt{[\mathrm{Var}(b^r)\mathrm{Var}(b^s)]} \quad (23-2)$$

With

$$R(b^r,b^s)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}b^r b^s g(b^r,b^s)db^r db^s \quad (24)$$
$$\mathrm{Var}(b^r)=\alpha^2E^2(b^r) \quad (25)$$

where, $E(b^r)$, $\mathrm{Cov}(b^r,b^s)$, $R(b^r,b^s)$, $\mathrm{Var}(b^r)$, $g(b^r,b^s)$ and α denote expectation, covariance, correlation ; variance, joint probability density functions and coefficient of variation, respectively. Functions involving partial derivatives of the mass, damping, stiffness and loading with respect to random variables can be calculated exactly by partial differentiation by the finite difference technique or by the least square fit method[1]. The partial derivatives $G^{0,e}$, $G^{,e,r}$, $G^{,e,rs}$, $G_{,i}^0$, $G_{,i}^{,r}$, $G_{,i}^{,rs}$ can be calculated by the finite differences or the least square fits, since they are generally implicit function of random variables.

Substituting the expansion equations similar to equation(21) for M_{ij}, D_{ij}, K_{ij}, f_i, $G_{,i}$, q_i, λ_i collecting terms of order 1, θ and θ^2, the 0th-, 1st- and 2nd order equations of the structural and adjoint systems can be respectively obtained as :

one pair of the zeroth oeder equations

$$M_{ij}^0\ddot{q}_j^0(t)+D_{ij}^0\dot{q}_j^0(t)+K_{ij}^0q_j^0(t)=f_i^0(t)\ ; \quad (26-1)$$
$$M_{ij}^0\ddot{\lambda}_j^0(\tau)-D_{ij}^0\dot{\lambda}_j^0(\tau)+K_{ij}^0\lambda_j^0(\tau)=G_{,i}^0(t)\delta(\tau)\ ; \quad (26-2)$$

R pairs of the first order equations($r=1,\ldots,R$)

$$M_{ij}^0\ddot{q}_j^{,r}(t)+D_{ij}^0\dot{q}_j^{,r}(t)+K_{ij}^0q_j^{,r}(t)$$
$$=f_i^{,r}(t)-[M_{ij}^{,r}\ddot{q}_j^0(t)+D_{ij}^{,r}\dot{q}_j^0(t)+K_{ij}^{,r}q_j^0(t)] \quad (27-1)$$

$$M_{ij}^0\ddot{\lambda}_j^{,r}(t)+D_{ij}^0\dot{\lambda}_j^{,r}(t)+K_{ij}^0\lambda_j^{,r}(t)$$
$$=G_{,i}^{,r}(t)\delta(\tau)-[M_{ij}^{,r}\ddot{\lambda}_j^0(\tau)+D_{ij}^{,r}\dot{\lambda}_j^0(\tau)+K_{ij}^{,r}\lambda_j^0(t)] \quad (27-2)$$

one pair of the second order equations

$$M_{ij}^{0}\ddot{q}_j^{(2)}(t)+D_{ij}^{0}\dot{q}_j^{(2)}(t)+K_{ij}^{0}q_j^{(2)}(t)$$
$$=(\frac{1}{2}f_i^{,rs}(t)-[M_{ij}^{,r}\ddot{q}_j^{,s}(t)+D_{ij}^{,r}\dot{q}_j^{,s}(t)+K_{ij}^{,r}q_j^{,s}(t)]$$
$$-\frac{1}{2}[M_{ij}^{,rs}\ddot{q}_j^{0}(t)+D_{ij}^{,rs}\dot{q}_j^{0}(t)+K_{ij}^{,rs}q_j^{0}(t)])\mathrm{Cov}(b^r, b^s) \quad (28-1)$$

$$M_{ij}^{0}\ddot{q}_j^{(2)}(\tau)+D_{ij}^{0}\dot{q}_j^{(2)}(\tau)+K_{ij}^{0}q_j^{(2)}(\tau)$$
$$=(\frac{1}{2}G_i^{,rs}(t)\delta(\tau)-[M_{ij}^{,r}\ddot{\lambda}_j^{,s}(\tau)+D_{ij}^{,r}\dot{\lambda}_j^{,s}(\tau)+K_{ij}^{,r}\lambda_j^{,s}(\tau)]$$
$$-\frac{1}{2}[M_{ij}^{,rs}\ddot{\lambda}_j^{0}(\tau)+D_{ij}^{,rs}\dot{\lambda}_j^{0}(\tau)+K_{ij}^{,rs}\lambda_j^{0}(\tau)])\mathrm{Cov}(b^r, b^s) \quad (28-2)$$

with, $r, s=1,\cdots,R$, $i, j=1,\cdots,N$ and

$$q_j^{(2)}=\frac{1}{2}q_j^{,rs}\mathrm{Cov}(b^r, b^s) \quad (29)$$

$$\lambda_j^{(2)}=\frac{1}{2}\lambda_j^{,rs}\mathrm{Cov}(b^r, b^s) \quad (30)$$

Note that by the definition of the covariance function the 2nd - order equations are obtained by multiplying the joint probability density function by the second - order terms and integrating over the domain of the random field b^r.

In order to reduce the double summation in equations (28) to the single summation, the correlated random variables b^r are transformed to a set or uncorrelated random variables c_v through a standard eigenproblem[6]

$$\mathrm{Cov}(b^r, b^s)U=\mathrm{Var}(c^v)U \quad (31)$$

where the covariance matrix $\mathrm{Cov}(b^r,b^s)$, $r, s=1,\cdots,R$ is positive definite, U is an orthogonal $R\times R$ - dimensional fundamental matrix and $\mathrm{Var}(c_v)$ is an R - dimensional diagonal matrix. In contrast to the modal structural problem wherein the lowest modes are used, only the $V(V\ll R)$ highest values of $\mathrm{Var}(c_v)$are required to simulate the major characteristics of many probabilistic distributions[2]. If the random vector b^r is composed of three uncorrelated parts of random load geometry, and material, the highest modes for each of the three parts of $\mathrm{Cov}(b^r,b^s)$ are extracted to obtain $\mathrm{Var}(c_v)$. Thus, we get

$$c^v=U_{rv}b^r,\ E(c^v)=U_{rv}b_0,\ \partial(.)/\partial c^v=U_{rv}\partial(.)/\partial b^r \quad (32)$$

Substituting equations (31),(32) in equations (26) - (28) the mixed derivatives $(.)^{,rs}$ reduce to the second derivatives $(.)^{,rr}$ (no sum on r) ; and the double summation over r, s, r, $s=1,\ldots,R$, reduces to the single summation over v, $v=1,\ldots,V$. Thus, by employing uncorrelated random variables c_v, equations(26) - (28) read :

one pair of the transformed zeroth order equations

$$M_{ij}^{0}\ddot{q}_j^{0}(t)+D_{ij}^{0}\dot{q}_j^{0}(t)+K_{ij}^{0}q_j^{0}(t)=f_i^{0}(t) \quad (33-1)$$
$$M_{ij}^{0}\ddot{\lambda}_j^{0}(\tau)-D_{ij}^{0}\dot{\lambda}_j^{0}(\tau)+K_{ij}^{0}\lambda_j^{0}(\tau)=g_i^{0}(\tau) \quad (33-2)$$

V pairs of the first order equations($r=1,\ldots,V$)

$$M_{ij}^{0}\ddot{q}_j^{,v}(t)+D_{ij}^{0}\dot{q}_j^{,v}(t)+K_{ij}^{0}q_j^{,v}(t)=p_i^{,v}(t) \quad (34-1)$$
$$M_{ij}^{0}\ddot{\lambda}_j^{,v}(\tau)-D_{ij}^{0}\dot{\lambda}_j^{,v}(\tau)+K_{ij}^{0}\lambda_j^{,v}(\tau)=g_i^{,v}(\tau) \quad (34-2)$$

one pair of the transformed second order equations

$$M_{ij}^{0}\ddot{q}_j^{(2)}(t)+D_{ij}^{0}\dot{q}_j^{(2)}(t)+K_{ij}^{0}q_j^{(2)}(t)=p_i^{(2)}(t) \quad (35-1)$$
$$M_{ij}^{0}\ddot{\lambda}_j^{(2)}(\tau)-D_{ij}^{0}\dot{\lambda}_j^{(2)}(\tau)+K_{ij}^{0}\lambda_j^{(2)}(\tau)=g_i^{(2)}(\tau) \quad (35-2)$$

In the above equations $(.)^{,v}$ and $(.)^{,vv}$ denote the first and second derivatives with respect to c_v. Furthermore, the transfornratron of equation from the generalized co - ordinates $q_j(t)$ and $\lambda_j(\tau)$ to the normalized co - ordinates $\gamma_z(t)$ and $\vartheta_z(\tau)$ is used to decouple equations(33) - (35). We have then :

Z pairs of the uncoupled 0st - order equations

$$\ddot{\gamma}_z^{0}(t)+2\xi_z\,\omega_z\,\dot{\gamma}_z^{0}(t)+\omega_z^2\,\gamma_z^{0}(t)=p_z^{0}(t) \quad (36-1)$$
$$\ddot{\vartheta}_z^{0}(\tau)-2\xi_z\,\omega_z\,\dot{\vartheta}_z^{0}(\tau)+\omega_z^2\,\vartheta_z^{0}(\tau)=g_z^{0}(\tau) \quad (36-2)$$

$Z\times V$ pairs of the uncoupled 1st - order equations

$$\ddot{\gamma}_z^{,v}(t)+2\xi_z\,\omega_z\,\dot{\gamma}_z(t)+\omega_z^2\,\gamma_z^{,v}(t)=p_z^{,v}(t) \qquad (37-1)$$

$$\ddot{\vartheta}_z^{,v}(\tau)-2\xi_z\,\omega_z\,\dot{\vartheta}_z^{,v}(\tau)+\omega_z^2\,\vartheta_z^{,v}(\tau)=g_z^{,v}(\tau) \qquad (37-2)$$

Z pairs of the uncoupled 2nd - order equations

$$\ddot{\gamma}_z^{(2)}(t)+2\xi_z\,\omega_z\,\dot{\gamma}_z^{(2)}(t)+\omega_z^2\,\gamma_z^{(2)}(t)=p_z^{(2)}(t) \qquad (38-1)$$

$$\ddot{\vartheta}_z^{(2)}(\tau)-2\xi_z\,\omega_z\,\dot{\vartheta}_z^{(2)}(\tau)+\omega_z^2\,\vartheta_z^{(2)}(\tau)=g_z^{(2)}(\tau) \qquad (38-2)$$

The eigenproblem has to be solved only once and the same eigenpairs are used for either the structural or adjoint systems ; the 0th - . 1st - , 2nd - order structural and adjoint equations can be solved for $q_j^0(t)$, $\lambda_j^0(\tau)$; $q_j^{,v}(t)$, $\lambda_j^{,v}(\tau)$ and $q_j^{(2)}(t)$, $\lambda_j^{(2)}(\tau)$ in parallel and by the same algorithm for integrating equations of motion. In turn, the uncoupled equations (36) - (38) can be solved alternatively by the step - by - step direct integration techniques[16]. It is observed that, in contrast to the global mass, damping and stiffness matrices, most terms in the matrices of their derivatives with respect to design and random variables are equal to zero ; and almost all operations required compute the right hand sides can be carried out by vector multiplications and at the element level.

The second - order perturbation approach can give invalid solutions owing to the appearance of secular terms. Such an unbounded solution may occur even for conservative syste ms which are known to possess a bounded solution. To eliminate secularities the resonant parts from these force sequences have to be removed. For the case of multi - degree - of - freedom systems an efficient numerical procedure[7] has been proposed, in which the sine and cosine transform pairs are used.

Having solved equations (33) - (35) the probabilistic distributions for the sensitivities of structural response can be evaluated. The expectations and covariances of the sensitivity gradients can be presented as

$$E(\phi^e)=\int_{-\infty}^{+\infty}\phi^e g(b^r)db^r \qquad (39)$$

$$\mathrm{Cov}(\phi^e,\phi^f)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\phi^e-E(\phi^e)]\,[\phi^f-E(\phi^f)]\; g(b^r,b^s)db^r db^s \qquad (40)$$

Substituting the expansion equations for q_i, λ_i, M_{ij}^e, D_{ij}^e, K_{ij}^e, f_i^e, G^e(which are similar to equation (21)) into equations (39), (40), retaining variations of br up to the second order and observing that the terms involving the first variation vanish by the definition, yields the second - order - accurate expectations and the first - order accurate covariances of the sensitivity gradients evaluated at time $t, t=[0,T]$, expressed in terms of the transformed random variables as follows :

$$E(\phi^e(t))=G^{0,e}(t)+\frac{1}{2}\sum_{v=1}^{V}G^{,e,vv}\mathrm{Var}(c^v) + \int_0^t[A_i^e(\lambda_i^0+\lambda_i^{(2)})-F_i^{e2}\lambda_i^0 + \sum_{v=1}^{V}(B_i^{ev}\lambda_i^{,v}+C_i^{evv}\lambda_i^0)\mathrm{Var}(c^v)]\,d\tau \qquad (41)$$

$$\begin{aligned}\mathrm{Cov}(\phi^e(t),\phi^f(t))=\sum_{v=1}^{V}[&G^{,e,v}(t)G^{,f,v}(t)\\&+G^{,e,v}(t)\int_0^t[A_i^f(\tau)\lambda_i^{,v}(\tau)+B_i^{fv}(\tau)\lambda_i^0(\tau)]\,d\tau\\&+G^{,f,v}(t)\int_0^t[A_i^e(\tau)\lambda_i^{,v}(\tau)+B_i^{ev}(\tau)\lambda_i^0(\tau)]\,d\tau\\&+\int_0^t\int_0^t(A_i^e(\tau)A_j^f(v)\lambda_i^{,v}(\tau)\lambda_j^{,v}(v)\\&+B_i^{ev}(\tau)B_j^{fv}(v)\lambda_i^0(\tau)\lambda_j^0(v)\\&+[A_i^e(\tau)B_j^{fv}(v)+A_j^f(v)B_i^{ev}(\tau)]\\&\lambda_i^{,v}(\tau)\lambda_j^0(v))\,d\tau\,dv\}\mathrm{Var}(c^v)\end{aligned} \qquad (42)$$

where τ, v are dummy variables of integration, $\tau, v\in[0,t]$, $t\in[0,T]$.

In similar way, for the static case a system or the 0th - , 1st - , 2nd - order structral and adjoint equations can be obtained. Employing the transformed random variables the two moments of the sensitivity gradients read

$$E(\phi^e)=G^{0,e}+\frac{1}{2}\sum_{v=1}^{V}G^{,e,vv}\mathrm{Var}(c^v)+A_i^e(\lambda_i^0+\lambda_i^{(2)}) - k_{ij}^{0,e}q_j^{(2)}\lambda_i^0+\sum_{v=1}^{V}(B_i^{ev}\lambda_i^{,v}+C_i^{evv}\lambda_i^0)\mathrm{Var}(c^v) \qquad (41)'$$

$$\mathrm{Cov}(\phi^{e}, \phi^{f}) = \sum_{v=1}^{V} [G^{e,v}G^{f,v} + (G^{e,v}A_i^{f} + G^{f,v}A_i^{e})]\lambda_i^{v}$$
$$+ (G^{e,v}B_i^{fv} + G^{f,v}B_i^{ev})\lambda_i^{0} + A_i^{e}A_j^{f}\lambda_i^{v}\lambda_j^{v}$$
$$+ (A_i^{e}B_j^{fv} + A_j^{f}B_i^{ev})\lambda_i^{v}\lambda_j^{0} + B_i^{ev}B_j^{fv}\lambda_i^{0}\lambda_j^{0}]\,\mathrm{Var}(c^{v}) \quad (42)'$$

NUMERICAL EXAMPLES

Example 1

Authors considered the time response of the framework subjected to a concentrated time - varying load, Fig. 1. The response functional is assumed as

$$\phi(\tau) = \frac{q_y^2}{q_0^2} - 1 \le 0$$

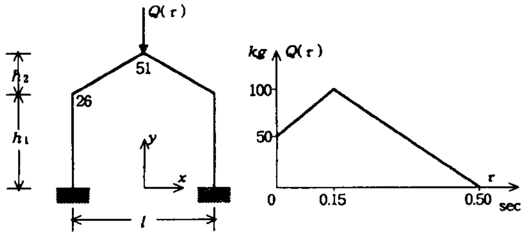

Fig. 1 2 - dimensional frame and load function

where, q_y is the vertical displacement at the nodal - point 51 of the frame, q_0 denotes an admissible value. The element cross sectional areas are assumed as random variables a^r, $r = 1, \cdots, 100$. The respective expectations, correlation function and coefficient of variation of the design are asseumed as follows :

$$E(A^r) = A^0 = 10.0$$
$$R(A^r, A^s) = \exp[-abs(x^r - x^s)/\lambda],\ \lambda = 0.5,$$
$$x^0 = 0., \cdots, x^{100} = 1.0$$
$$\alpha = 0.05$$

The deterministic data are assumed : length $l = 200$, $h = 100$, $h_1 = 50$, Young's modulus, $E = 2.0 \times 10^7$, Poisson's ratio $\nu = 0.2$, mass density $\lambda = 0.001$, damping factor $\xi = 0.05$, and allowable displacement $q_0 = 0.012$. To solve the initial - terminal problem the mode superposition technique is used with the 10 lowest eigensystems. The set of 100 correlated random variables is transformed to a set of uncorrelated vanables, out of which the 10 highest modes are used in the calculation. The equations are integrated

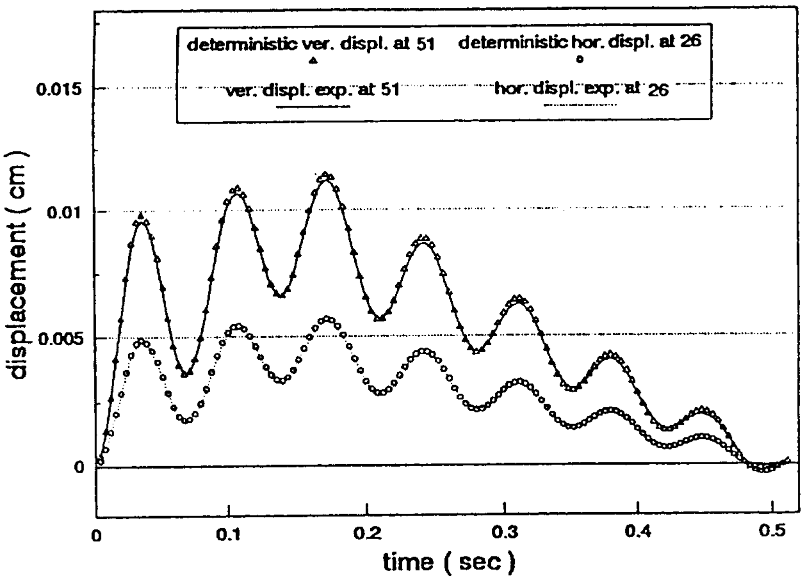

Fig. 2 Displacement response of deterministic and stochastic dynamics analysis at 26, 51 node of 2 - dim. frame.

with respect to time for 512 time steps ($\Delta t=0.001$). The secular terms are removed with the secularity elimination factor $r=0.15$ and with 1024 Fourier terms. Fig. 2, 3 shows the time response of expectations, variances and covariances of the vertical displacement at the nodal - point 51 and horizontal displacement at 26(compared against the deterministic solution). The time responses of expectations, variances and covariance of the sensitivity gradient

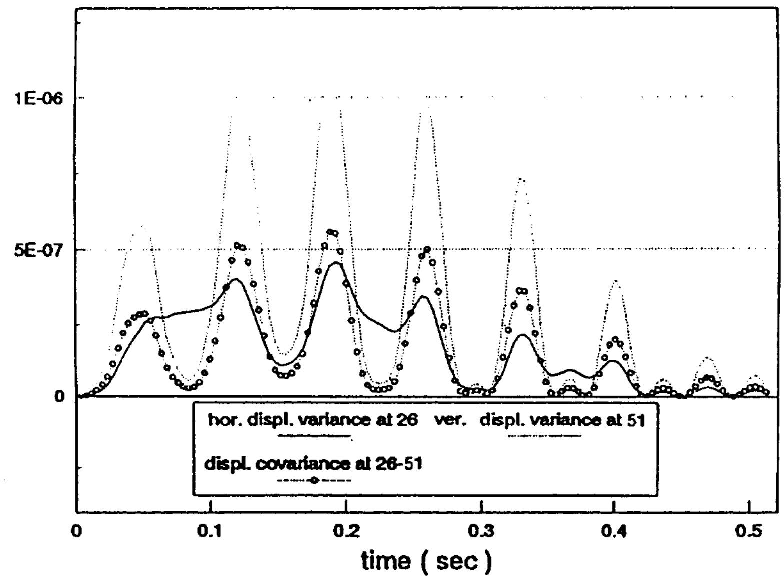

Fig. 3 Variance and covariance response of node displacement in stochastic sensitivity analysis at 26, 51 node of 2 - dim. frame.

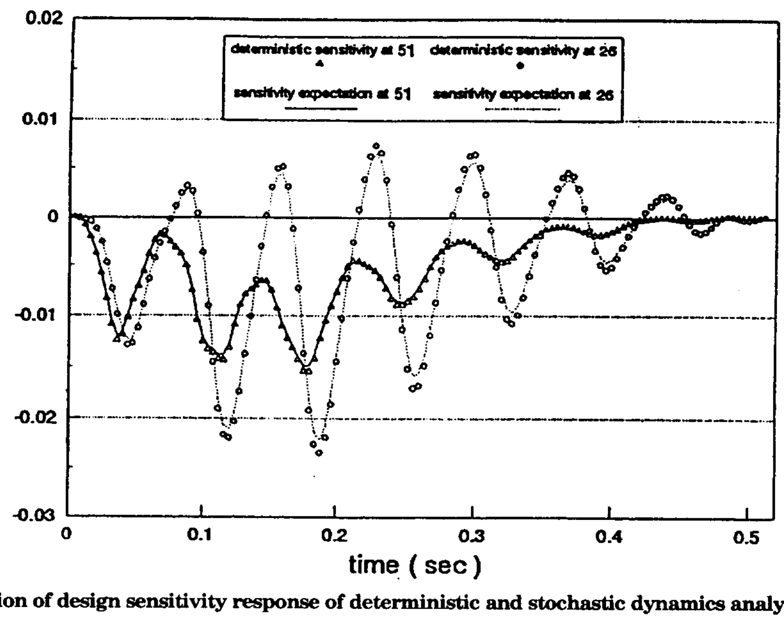

Fig. 4 Expectation of design sensitivity response of deterministic and stochastic dynamics analysis at 26, 51 node of 2 - dim. frame.

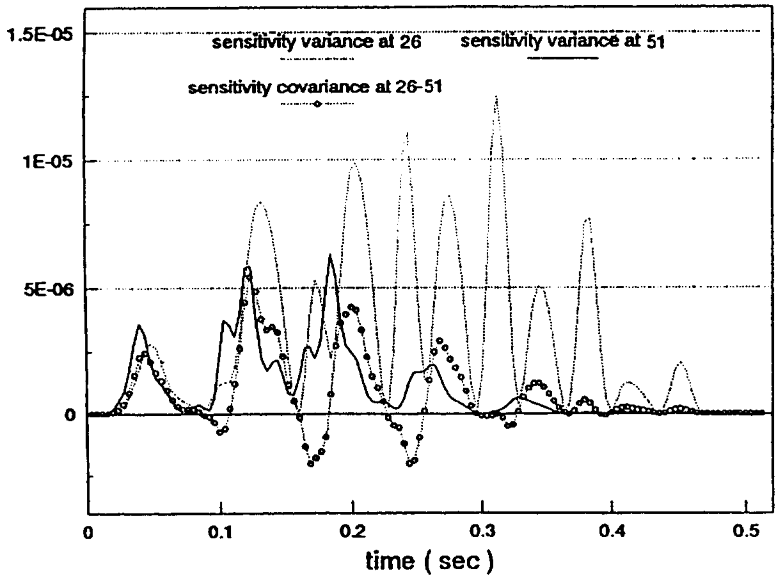

Fig. 5 Covariance of design sensitivity response in stochastic dynamics sensitivity analysis at 26, 51 node of 2 - dim. frame.

at the nodal - point 51 and at 26 are shown in Fig. 4, 5. The computation time was 1920 sec.

Example 2

Next, the authors estimated the time response of the sensitivity of a 3 - dimensional frame structure fixed at boundaries and subjected to a concentrated time - varying load at nodal - point 6, Fig. 6. The finite element mesh includes 40 elements(40 random design variables).

The response functional is assumed as

$$\phi(\tau)=\frac{q_z^2}{q_0^2}-1\leq 0$$

where q_z is the z - displacement at load point

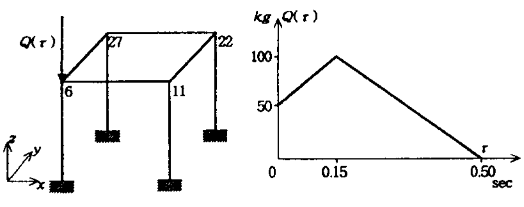

Fig. 6 3D frame and load function

No. 6, q_0 denotes an admissible value. The expectation, correlation function and coefficient of variation of the designs are given, respectively, as follows :

$$E[A^r]=A^0(1.0+\theta x^r/l)$$
$$E[A^{100-r+1}]=E[A^r],\ r=1,\cdots,40$$
$$R[A^r,A^s]=\exp[-abs(x^r-x^s)/\lambda],\ r,s=1,\cdots,40$$
$$\alpha=0.07,\ \lambda=0.1,\ A_0=10.0,\ \theta=0.3$$

where, $x_1=0.0$, $x_2=0.01$, $x_{100}=1.0$.

and, the deterministic data are assumed, i.e. length $l=100$, Young's modulus $E=2.0\times10^7$, Poisson's ratio $\nu=0.3$, mass density $\gamma=0.001$, damping factor $\xi=0.002$. To solve the initial - terminal problem the mode superposition technique is used with the 10 lowest eigensystems. The set of 60 correlated random variables is transformed to a set of uncorrelated variables, out of which the 10 highest modes are used in the calculation. The equations are integrated with respect to time for 512 time steps ($\Delta t=0.001$). The secula terms are removed with the

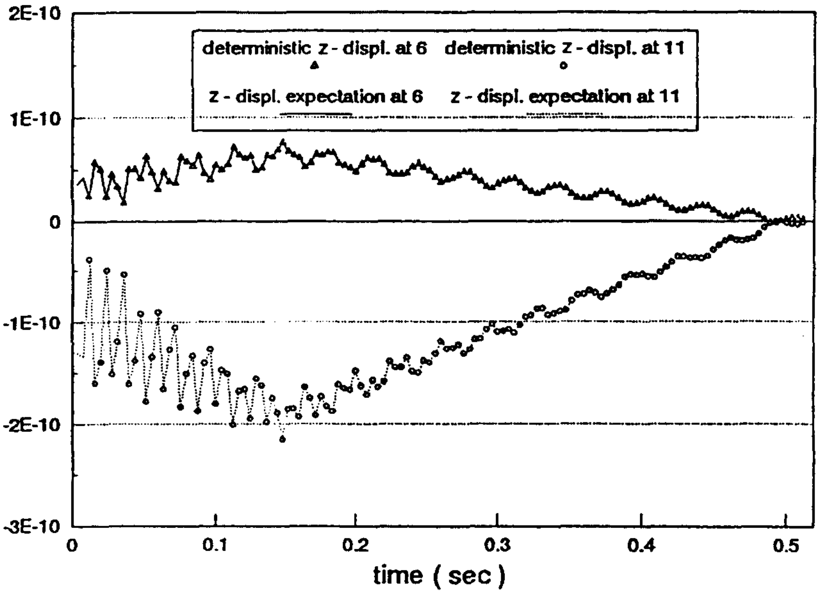

Fig. 7 Displacement response of deterministic and stochastic dynamics sensitivity analysis at 6, 11 node of 3 - dim. frame.

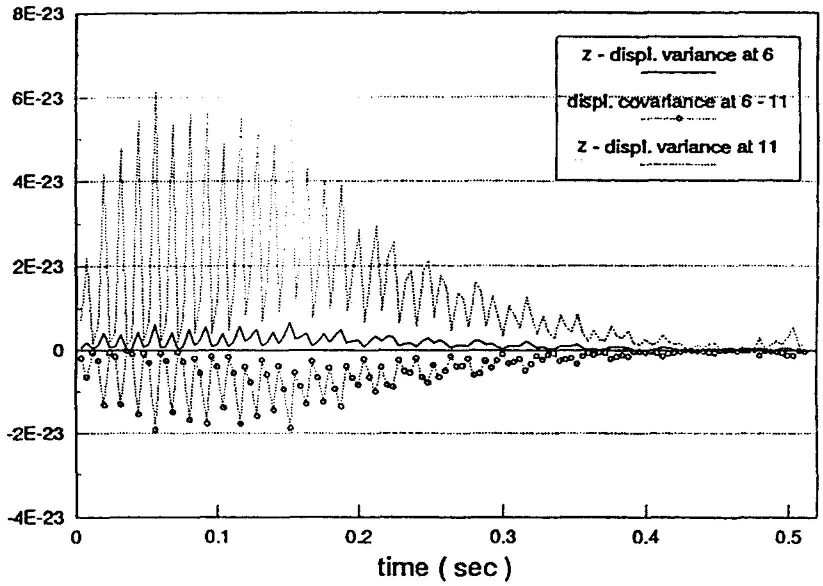

Fig. 8 Variance and covariance response of node displacement in stochastic sensitivity analysis at 6, 11 node of 3 - dim. frame.

secularity elimination factor $r=0.15$ and with 1024 Fourier terms. The time response of expectations and standard deviation of the z - direction at 6 - nodal point are displayed in Fig. 7, 8. The time distribution of expectations, variances and covariance of the sensitivity gradient at 6, 11 - nodal point and in Fig. 9, 10. The computation time was 1050 sec.

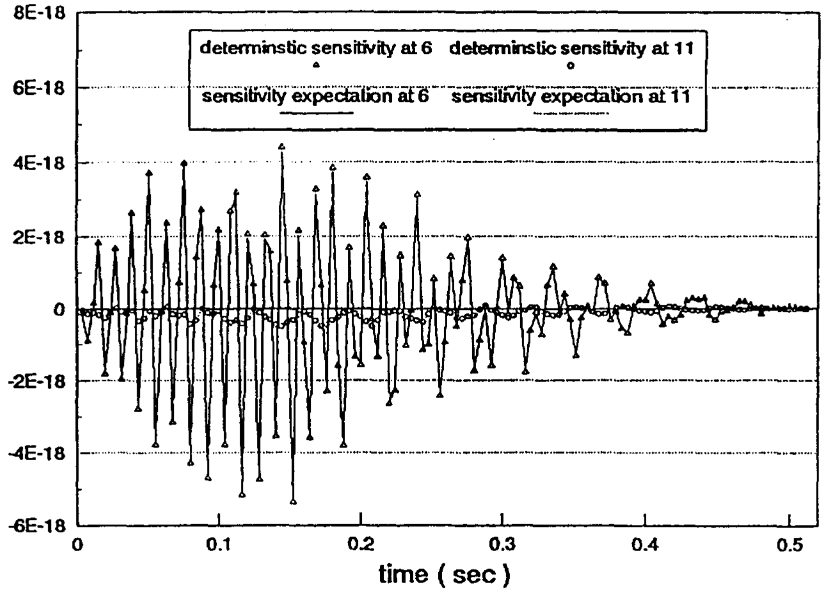

Fig. 9 Expectation of sensitivity of stochastic dynamics sensitivity analysis and sensitivity of deterministic analysis at 6, 11 node of 3 - dim. frame.

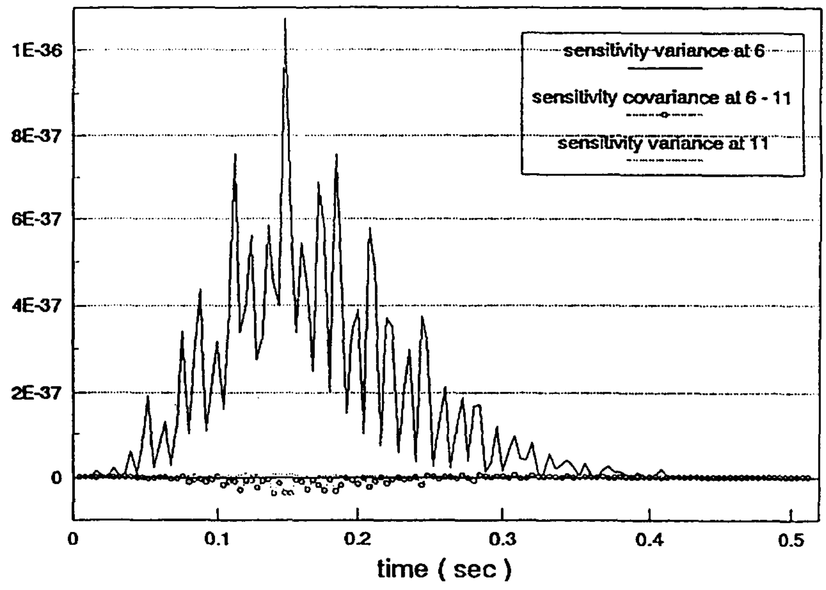

Fig. 10 Variance and covariance of design sensitivity response in stochastic dynamics sensitivity analysis at 6, 11 node of 3 - dim. frame.

CONCLUSION

The structural response of the stochastic finite element approach(SFEA) to stochastically described variations in geometrical, material and loading parameters is an important subject which can be studied using current finte element approach. Sinse the structural design sensitivity(SDS) and SFEA are closely related in problem statement, finite element modelling and computer implementation, the stochastic sensitivity analysis using the SFEA can be formulated in a natural and effective way. In fact, both the formulations for the SDS and SFEA are based on the perturbation approach. If the first-order perturbation is assumed, the models of SFEA and SDS are almostly identical. Another common feature is that involving the discretization of the parameter spaces(of a random field for SFEA and a design field for SDS). This enables one to approximate either the random field or the design field by the same interpolation functions. Therefore, if a random variable and a design variable are defined by the same quantity, i.e. $h \equiv b$, the derivatives of the element stiffness and loads must be calculated only once and can be used simultaneously as the derivative with respect to the design variable and the derivative with respect to the random variable. And the stochastic sensitivity analysis requires a much finer finite element mech than a typical structural problem. It is seen that the probablilistic characteristics in SFEA(for both the homogeneous and inhomogeneous random fields) and the sensitivity characteristics in SDS are translted entirely to the right hand sides of the equations. Thus, the global sriffness matrix has to be assembled and factorized only once and both the equilibrium equations and the adjoint equations can be solved simultaneously. Also, the zeroth-, first- and second-order equation pairs of the structural and adjoint systems can be solved in parallel. The algorithm developed have proved to be accurate and efficient(low computation cost, as shown from the computation times given for each numerical example) in the analysis of small- and medium-size systems and can be immediately adapted to fit into existing finite element programs, in which differentiation with respect to random and design variables can be carried out explicitly. However, the computational approach with implicitly generated finite elements does not seem to have been fully investigated in the literature and it requires future work to increase its applicability and effectiveness.

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