A Generalized Fourier Transform Based on a Periodic Window

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Abstract

An extension of the well-known Fourier transform is developed in this paper. It is denoted as the generalized Fourier transform(GFT), since it encompasses the Fourier transform as its special case. The first idea of this extension can be found on [1]. In the definition of the N-point discrete GFT, it first construct a passband in time which functions as a window in the time domain. An appropriate interpretation of each variables are introduced during the definition of the GFT, followed by the formal derivation of the inverse GFT. This transform pair is similar to the windowing in the frequency domain such as the subband coding technique (or filter bank approach) and could be extended to the wavelet transform.

Ⅰ. Introduction

We are familiar with various windowing schemes in the time domain such as Hamming, Hanning, or rectangular window. The window isolates a section of input signal so as to analyze its local property. A series of overlapping windows are then applied to the whole input signal, which leads to the well-known short-time Fourier Transform (STFT). This approach has been quite successful for the analysis of quasi-stationary signal. However, there would be an situation where it is necessary to assume a periodical isolation instead of the local isolation of the input signal. This idea has been a motivation of this study, which employs a sequence of windows whose location and width are determined by the designer.

An extension of the well-known Fourier transform is developed in this paper. It is denoted as the generalized Fourier transform(GFT), since it encompasses the Fourier transform as a special case. The first idea of this extension starts on [1]. In the definition of the GFT, which covers the next section, it first constructs a passband in time which functions as a periodic window in the time domain. A definition of the GDFT, followed by the formal derivation of the inverse GFT, is presented. This transform pair is similar to the windowing in the frequency domain such as the subband coding technique (or filter bank approach). The basic difference is that the former is in the time domain and the later is in the frequency domain. Section III studies the structure of the coefficient matrix in a search for an efficient way calculating its forward and backward conversion.

Ⅱ. Generalized Fourier transform

Start from the original Fourier transform of a function $v(t)$ such that

$$V(f)=\int_{-\infty}^{+\infty} v(t)e^{-j2\pi ft}\,dt \tag{1}$$

it is straight-forward to show that, for all square integrable signals [1]

$$V(f)=\int_{0}^{T} \sum_{n=-\infty}^{\infty} v(t+nT)e^{-j2\pi ft}\,dt \tag{2}$$

where $T=1/f$. For example, if $v(t)=e^{-t}$, $t\geq 0$, it is easy to verify Eqns. (1) and (2) produce the same result. In Eq. (2), a signal is cumulated within the interval $[0, T]$ for each frequency f, then multiplied by the exponential term, $\exp\{-j2\pi ft\}$, which completes one period in each interval of length T.

1. The generalized Fourier transform

Suppose that two interesting events occur periodically in the signal with different frequencies and duration which is short compared to the entire time interval of interest. Two pulse trains with close frequency and different width would be a good example. Also suppose that we want to enhance the detectability of these pulses. An algorithm with good resolutions on both time and frequency domain would be required for this purpose. As one approach, the GFT first modifies the basis function set of the usual Fourier transform, that is, from the complex exponential $\{\exp(-j2\pi ft)\}$ to $\{\exp(-j2\pi\beta ft)\}$. Then the generaliza-

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tion is achieved by constructing a sequence of windows in time which is periodic. Since it is different from the normal window, we shall denote it as a passband in time. If one of the pulse is restricted within only one of the passband, then the GFT would do a better job of detecting this occurrence than the Fourier transform. These passbands can be made as small as desired, thus increasing the detectability of the prescribed events. Also, we can increase the effective length of the detection interval as a percentage of the data length. The reduction in passband width is limited only by the length of the pulse to be detected. When this passband is centered at t_P where $0 \le t_P \le T$, and of width T/β, we could use a variable, $\phi = 2\pi t_P/T$ and define the generalized Fourier transform by

$$V(f,\phi,\beta)=\int_{\frac{T}{2\pi}\left(\phi-\frac{\pi}{\beta}\right)}^{\frac{T}{2\pi}\left(\phi+\frac{\pi}{\beta}\right)}\sum_{n=-\infty}^{\infty} v\left(t+\frac{n}{f}\right)e^{-j2\pi\beta ft}\,dt \qquad (3)$$

where $\beta > 0$ and $0 \le \phi \le 2\pi$. Note that the Fourier transform is a special case of the GFT for $\phi=\pi$ and $\beta=1$. Figure 1 shows graphical description of the transform, where one can see that the variable ϕ determines the location of passband and β specifies its width. In this way, we can get extra two-degrees of freedom. As an example, consider the exponential signal, $v(t)=e^{-t}$, $t \ge 0$. From Eq.(2),

$$V(f,\phi,\beta)=\int_{\frac{T}{2\pi}\left(\phi-\frac{\pi}{\beta}\right)}^{\frac{T}{2\pi}\left(\phi+\frac{\pi}{\beta}\right)}\sum_{n=0}^{\infty} e^{-(t+n/f)}e^{-j2\pi\beta ft}\,dt$$

$$=\frac{1}{1-e^{-1/f}}\int_{\frac{T}{2\pi}\left(\phi-\frac{\pi}{\beta}\right)}^{\frac{T}{2\pi}\left(\phi+\frac{\pi}{\beta}\right)} e^{-(1+j2\pi\beta f)t}\,dt$$

$$=\frac{\exp\left\{-(1+j2\pi\beta f)\frac{T}{2\pi}\left(\phi-\frac{\pi}{\beta}\right)\right\}-\exp\left\{-(1+j2\pi\beta f)\frac{T}{2\pi}\left(\phi-\frac{\pi}{\beta}\right)\right\}}{(1-\exp(-1/f))(1+j2\pi\beta f)}$$

The third equation follows from the geometric series Notice that for $\phi=\pi$ and $\beta=1$, this expression reduces to the Fourier transform for the exponential. To see this, note that the numerator reduces to $1-\exp\{-1/f\}$ thus yielding

$$V(f,\pi,1)=\frac{1}{1+j2\pi f}$$

By introducing β, one could adjust the frequency of the rotating phasor so that one cycle is completed within a band of length T/β. However, β should be restricted to be an integer, unless we lost the orthogonality and hence the completeness of the basis function set $\{\exp(-j2\pi\beta ft)\}$. For an integer value of β, Eq. (3) becomes by interchanging summation and integration

$$V(f,\phi,\beta)=\int_{-\infty}^{\infty}\bar{v}(t)e^{-j2\pi\beta ft}\,dt \qquad (4a)$$

where

$$\bar{v}(t)=v(t), \quad \text{for } |\phi-2\pi ft\cdot \mathrm{mod}(2\pi)| < \frac{\pi}{\beta}$$

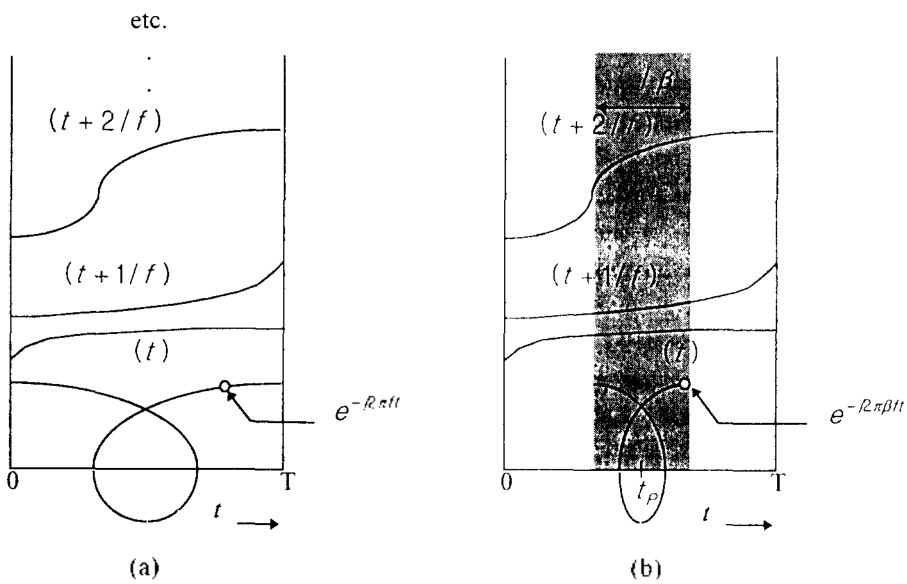

Figure 1. An illustration for (a) Fourier Transform and (b) Passband (for $\beta=3$)

$$= 0, \qquad otherwise \tag{4b}$$

If we discretize the value of ϕ in such a way that the first passband of width T/β starts from $t = 0 \cdot \mathrm{mod}(T)$, then, the GFT can be represented as

$$V_{i,\beta}(f) = \int_{i\_th} \sum_{k=-\infty}^{\infty} v(t+kT) e^{-j2\pi\beta f t} dt \tag{5}$$

$$= V(f, \phi, \beta)|_{\phi=(1+2i)\pi/\beta} \tag{5}$$

where an index for the i^{th} passband is used instead of ϕ for $i = 0, \cdots, \beta$.

Since the basis function set $\{\exp(-j2\pi\beta f t)\}$ is complete, the original time signal one could be reconstructed, if $V_{i,\beta}(f)$ is known for all f, and i. One can show that the inverse can be calculated by

$$v(t) = \frac{\beta}{\pi} \int_{-\infty}^{\infty} \sum_{i=0}^{\beta-1} V_{i,\beta}(f) e^{j2\pi\beta f t} df \tag{6}$$

Also, since

$$V(\beta f) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi\beta f t} dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{0}^{T} v(t+kT) e^{-j2\pi\beta f t} dt \tag{7}$$

$$= \sum_{i=0}^{\beta-1} \int_{iT/\beta}^{(i+1)T/\beta} \sum_{k=-\infty}^{\infty} v(t+kT) e^{-j2\pi\beta f t} dt$$

it is clear that the Fourier transform of $v(t)$ at frequency βf is equal to the sum of passbands for the GFT at frequency f, that is,

$$V(\beta f) = \sum_{i=0}^{\beta-1} V_{i,\beta}(f) \tag{8}$$

Thus, to evaluate the inverse transform for fixed integer β using Eq. (6), the procedure is as follows: add the terms $V_{i,\beta}(f)$ from each passband, multiply by $\exp\{j2\pi\beta f t\}$, and integrate it over f. Then multiply the result by $\beta/2\pi$.

2. The generalized discrete Fourier transform

As we used the Fourier transform to derive the GFT, we could make use of the N-point discrete Fourier transform (DFT) in deriving the generalized discrete Fourier transform (GDFT). Since the N-point DFT is defined by

$$V(k) = \sum_{n=0}^{N-1} v(n) W^{nk} \tag{9}$$

where $W = \exp(-j2\pi/N)$, we can get through a simple operation

$$V(k) = \sum_{\ell=0}^{\ell<N/k} \sum_{n=0}^{k-1} v(\ell + \frac{nN}{k}) W^{\ell k} \tag{10}$$

This corresponds to Eq. (2) for the Fourier transform. The N-point GDFT is then defined by selecting a portion of the time interval of length $N/k\beta$, multiplying time function by the proper exponential, and then summing over this interval. For integer β, it is defined to be

$$V(k, \phi, \beta) = \sum_{n=0}^{N-1} \bar{v}(n) W^{\beta nk} \tag{11a}$$

where

$$\bar{v}(n) = v(n), \quad \text{for } \left| \phi - \frac{2\pi nk}{N} \cdot \mathrm{mod}(2\pi) \right| < \frac{\pi}{\beta}$$

$$= 0, \qquad otherwise \tag{11b}$$

and $n, k = 1, \cdots, N-1$, and $i = 0, \cdots, \beta-1$. For a given frequency k, we divide the interval, N/k, into β equidistant intervals so that Eq. (11) becomes for the i^{th} passband

$$V_{i,\beta}(k) = V(k, \phi, \beta)|_{\phi=(1+2i)\pi/\beta} \tag{12}$$

When discrete frequency is used in the calculation of GDFT, the effect of the complex exponential term on the discrete time samples is the same on both frequency k in the GDFT and frequency $\beta k \le N$, in the DFT. For example, when $k = 3$ in DFT, $v(1)$, $v(5)$ and $v(9)$ are multiplied by the same exponential term, W^3, since $W^{3\times1} = W^{3\times5} = W^{3\times9}$ with $N = 12$. Similarly, when $\beta = 3$ and $k = 1$ in GDFT, they are multiplied by the same term, W^3, since $W^{\beta kn} = W^{3\times1} = W^{3\times5} = W^{3\times9}$ before summation. Moreover, for each frequency k, one time sample belongs to only one passband. Therefore, we have

$$\sum_{i=0}^{\beta-1} V_{i,\beta}(k) = \sum_{n=1}^{N} v(n) W^{\beta nk} = V(\beta k) \tag{13}$$

This gives the values of $V_{i,\beta}(k)$ at frequencies βk with $\beta > 1$, skipping values at intermediate frequencies. It causes a problem of missing frequencies terms. Even though we have all the information on $V_{i,\beta}(k)$ for all k and i, it may not be sufficient to invert $V(\beta k)$ in Eq. (13) using the usual inverse DFT equation. When, for example, $N = 12$ and, only $v(0)$, $v(3)$, $v(6)$, and $v(9)$ are available.

In GDFT, one period of the time segment for each discrete frequency is divided into β parts, each of which is characterized by one passband. The maximum number of discrete time samples that could be sampled in each passband is $\lceil N/\beta \rceil$, where $\lceil x \rceil$ represents the smallest integer greater than or equal to x. Therefore, it is natural to yield only $\lfloor N/\beta \rfloor$ meaningful spectral components. In order to avoid this situation, the sampling rate needs to be increased by a factor of β and thus provide enough discrete-time samples. If a fraction of the continuous waveform which belongs to a particular passband be sampled enough, then it becomes possible to reconstruct the original waveform based on the Eq. (13). Otherwise, there would be some missing frequency terms during the inverse GDFT computation as explained.

When the sampling rate is increased by a factor of β in order to sample N data within the given interval of a passband, the discrete frequency values need to be changed accordingly. It is possibly done with a series of interpolation and decimation process. Other possible approach is using fractional values of frequency, $k = 1/\beta, \cdots, N/\beta$, so that each passband for each discrete frequency have up to N data samples. Then, the inverse GDFT becomes

$$v(n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{\beta-1} V_{i,\beta}(\frac{k}{\beta}) W^{-nk} \tag{14}$$

Ⅲ. Structure of GDFT

From Eq. (11), the computation of $V_{i,\beta}(k)$ could be expressd by a matrix equation

$$\mathrm{A}_{i,\beta}\,\mathrm{q} = \mathrm{v}_{i,\beta} \tag{15}$$

where the coefficient matrix by $\mathrm{A}_{i,\beta}$, $i = 0, \cdots, \beta - 1$ is defined for each passband and its components consists of the sum of time samples. The vectors $\mathrm{v}_{i,\beta}$ and **q** are GF-transformed column vector and phasor column vector such that

$$\mathrm{v}_{i,\beta} = [v_{i,\beta}(1/\beta)\ v_{i,\beta}(2/\beta) \cdots v_{i,\beta}(N/\beta)]^T$$
$$\mathrm{q} = [W\ W^2 \cdots W^N]^T \tag{16}$$

Note that the column vector, **q**, does not depends on β, but is determined only by the exponential terms.

Since the GDFT is an extension of the DFT, it still has some properties of the DFT which leads to the implementation of the FFT algorithm. They could be analyzed through the investigation of the structure of the GDFT coefficient matrix. The coefficient matrix $\mathrm{A}_{i,\beta}$ is an $(N \times N)$ square matrix whose row and column correspond to the frequency and sampling points within the passband, respectively. Therefore, the element $\mathrm{A}_{i,\beta}(k, \ell)$ is the sum of discrete-time samples which share the multiplication factor W^k for a discrete frequency k/β and the first step to the formation of the coefficient matrix would be detecting which samples belong to a given passband for each frequency. One could see that the formation of $\mathrm{A}_{i,\beta}$ is controlled by a modulo function mapping samples into the frequency domain. The mapping function depends on both integer frequency and sampling point, that is, a discrete-time sample $v(n)$ is summed to $\mathrm{A}_{i,\beta}(k, \ell)$ if and only if it satisfies the following two conditions

$$(i-1)\frac{N}{\beta} < nk \le i\frac{N}{\beta} \tag{17a}$$

$$\beta nk \cdot \mathrm{mod}(N) = \ell \tag{17b}$$

The formation of $\mathrm{A}_{i,\beta}$ shows some interesting properties which would be used for the fast calculation of GDFT. Most of its property are the same as those used in the development of the FFT, which are [2]

(P.1) $W^{k(N-n)} = W^{-kn}$

(P.2) $W^{kn} = W^{k(n+N)} = W^{(k+N)n}$

These two properties allow us to decompose the computation of the N-point DFT into successively smaller DFTs. The cyclic property (P.2) of the complex sequence, W^{kn} comes from the assumption of the finite duration. In GDFT, only (P.2) can be used, since GDFT involves with different exponential term, $W^{\beta kn}$. Therefore, an efficient computation of GDFT should be pursued through investigating the behavior of time samples, $v(n)$, depicted in $\mathrm{A}_{i,\beta}$, which results in the following properties.

(P.3) For all integer $d > 1$, if $(N, k) = d$ then there exists at least one column ℓ with more than one n satisfying Eq. (17), where (a, b) represents the greatest common divisor (GCD) between integers a and b.

(P.4) For a given column number ℓ, the same occurrence in mapping time samples iterates with a certain period within each set of frequencies given by $\{x : (x, N) = d \text{ and } x \in (0, N]\}$ and the period is

$$P(k, \ell) = \frac{(N, k)}{(N, \ell)} N \tag{18}$$

Proof for (P.3) is straightforward using the basic number theory. (P.4) is based on the fact that, if $(N, m) = a$, $(N, k) = b$ and $L = a/b > 1$, the modulo equation $kn \cdot \mathrm{mod}(N) = \ell$ has at least one solution for n, if and only if L is a divisor of m.

Ⅳ. Conclusion

We have presented a method for effectively control the detection interval as a percentage of the data length. This procedure is called the generalized Fourier transform since the special case of $\phi=\pi$ and $\beta=1$ gives the usual formula for the Fourier transform. This transform pair is derived in this paper, followed by appropriate interpretation of each variable. In a way to perform GDFT operation, the structure of the coefficient matrix is analyzed. However, the application of GFT remains uncertain at this point. Search for the proper application is under investigation.

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