On the Error Bound of the Approximate Solution of a Nonclassically Damped Linear System under Periodic Excitations

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Abstract

One common procedure in the approximate solution of a nonclassically damped linear system is to neglect the off-diagonal elements of the normalized damping matrix. A tight error bound, which can be computed with relative ease, is given for this method of solution. The role that modal coupling plays in the control of error is clarified. If the normalized damping matrix is strongly diagonally dominant, it is shown that adequate frequency separation is not necessary to ensure small errors.

Ⅰ. Introduction

A linear dynamic system is said to have classical normal modes if the system possesses a complete set of real, orthonormal eigenvectors. In general, an undamped dynamic system always possesses c' :sical normal modes. When dissipative forces are present, the system may or may not possess classical normal modes. If it does, the system is said to be classically damped. Caughey and O'Kelly(1965) have established a necessary and sufficient condition for the existence of classical normal modes in a damped linear system. If the criterion of Caughey and O'Kellly is not met, then nonclassical damping is said to exist. In reality, nonclassical damping comes from drastic variations of energy absorption rates of the materials in different parts of a structure. Typical examples of nonclassically damped systems are a nuclear reactor containment vessel founded on soft soil subjected to earthquake motion(Clough and Mojtahedi, 1976), and a base-isolated structure in the same environment(Tsai and Kelly.1988).

When dissipative forces are nonclassical, it is generally difficult to analyze the system dynamics, owing to the complex nature of eigensolutions. Foss(1958) and Vigneron(1986) proposed a state space approach which takes into account the orthogonality relations between the complex eigenvectors of a nonclassically damped system. The key to the utility of the eigensolutions is, of course, orthogonality, which allows decoupling of the governing equations. One disadvantage of such exact methods is that they require significant numerical effort to determine the eigensolutions. The effort required is evidently intensified by the fact that the eigensolutions of a nonclassically damped system are complex. From the analysts' viewpoint, another disadvantage is the lack of physical insight afforded by methods which are intrinsically numerical in nature. Several authors have studied nonclassically damped linear systems by approximate techniques. For instance, Cronin(1976) obtained an approximate solution for a nonclassically damped system under harmonic excitation by perturbation techniques. Chung and Lee(1986) applied perturbation techniques to obtain the eigensolutions of damped systems with weakly nonclassical damping. Prater and Singh(1986), and Nair and Singh(1986) developed several indices to determine quantitatively the extent of nonclassical damping in discrete vibratory systems. Nicholson(1987) gave upper bounds for the response of nonclassically damped systems under impulsive loads and step loads.

In analyzing a nonclassically damped system, one common approximation is to neglect those damping terms which are nonclassical, and retain the classical ones. This approach is termed the method of approximate decoupling. For large-scale systems, the computational effort at adopting approximate decoupling is at least an order of magnitude smaller than the method of complex modes. The solution of the decoupled equations would be close to the exact solution of the coupled equations if the nonclassical damping terms are sufficiently small. A discussion on this topic is given, for example, by Meirovitch (1967), Thomson et al.(1974), and Cronin(1976). Approximate decoupling of a damped linear system is often convenient and practical. An attempt to evaluate the extent

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of approximation, introduced by neglecting the nonclassical damping term, has recently been reported by Shahruz and Ma(1988). It is claimed that the error bounds obtained in their paper are the tightest in a certain functional form, if the external excitation is arbitrary. However, computational effort to find the error bounds is quite involved.

The purpose of this paper is to derive a new error bound which can be obtained with less computational effort, as well as to highlight the modal coupling due to the nonclassical damping terms. It has been found that the new error bound is generally sharper than the previous ones given by Shahruz and Ma(1988), but the derivation of the new error bound requires an additional assumption that the external excitation be periodic. The organization of the paper is as follows. In section 2, an expression for the error bound that involves less computational effort is derived. The error bound due to approximate decoupling has an intimate relationship with modal coupling because of the nonclassical damping terms. Modal coupling between any two adjacent modes is highlighted in section 3. A criterion which ensures that approximate decoupling does not cause excessive errors in the response is also presented in this section. An example in section 4 illustrates the theoretical developments pursued in this paper. In section 5, a summary of findings is provided.

Ⅱ. Error bound for approximate decoupling

Consider the equation of motion of a discrete or discretized linear system under external excitation

$$M\ddot{x} + M\dot{x} + Kx = f(t),\ x(0) = x_0,\ \dot{x}(0) = \dot{x}_0,\ t \geq 0, \tag{2.1}$$

where the mass matrix M, the damping matrix C, and the stiffness matrix K are of order $n \times n$; the displacement vector $x(t)$ and the external excitation $f(t)$ are n-dimensional vectors. For passive systems, the matrices M, C, and K are symmetric and positive definite. These assumptions are not arbitrary, but in fact have solid footing in the theory of Lagrangian dynamics. Symmetry of M results naturally from the transformation from Cartesian to generalized coordinates for a scleronomic system, and the positive definiteness requirement is a property of kinetic energy. Symmetry of K results from linearization of the potential energy function about an equilibrium point, and the form of Rayleigh dissipation function ensures symmetry of C(Rosenberg, 1977; Greenwood, 1977).

Let U denote the $n \times n$ modal matrix corresponding to the system (2.1). The modal matrix is a nonsingular matrix whose columns are the eigenvectors of the generalized symmetric eigenvalue problem

$$Ku^{(i)} = \omega_i^2 Mu^{(i)} \tag{2.2}$$

where $\omega_i^2 > 0$ and $u^{(i)}$, $i = 1, \cdots, n$, are the eigenvalues and the corresponding eigenvectors, respectively. The modal matrix is usually orthonormalized according to $U^T MU = I$, where U^T denotes the transpose of U, and I represents the $n \times n$ identity matrix. Hence, $U^T KU = diag(\omega_1^2, \cdots, \omega_n^2) = \Omega^2$. It is well-known that by the linear transformation $x(t) = Uq(t)$, equation (2.1) can be written in the normalized form

$$\ddot{q} + \tilde{C}\dot{q} + \Omega^2 q = g(t),\ q(0) = U^T Mx_0,\ \dot{q}(0) = U^T M\dot{x}_0,\ t \geq 0, \tag{2.3}$$

where $\tilde{C} = U^T CU$, $g(t) = U^T f(t)$, $q(t)$ is the n-dimensional vector of normalized coordinates. The symmetric matrix $\tilde{C}$ is called the normalized damping matrix.

The normalized damping matrix $\tilde{C}$ is not diagonal in general. When $\tilde{C}$ is not diagonal, the system (2.1) is said to be nonclassically damped. If the damping matrix C is a linear combination of the mass and the stiffness matrices, then $\tilde{C}$ is diagonal; this is a sufficient condition under which $\tilde{C}$ is diagonal, and was originally given by Lord Rayleigh(1945). The necessary and sufficient condition under which $\tilde{C}$ is diagonal has been given by Caughey and O'kelly(1965). When $\tilde{C}$ is diagonal, system (2.3) is a set of n decoupled second-order differential equations, which can be solved for q(t) conveniently. Then, the solution of (2.1) is obtained from $x(t) = Uq(t)$, for all $t \geq 0$.

In the following, we shall use the L_∞ norm of a vector, defined by $\|x(p)\| = \max_{1 \leq i \leq n} |x_i(p)|$ for any vector $x(p) = [x_1(p), \cdots, x_n(p)]^T$, where p is a parameter. Rewrite system (2.3) as

$$\ddot{q} + (\tilde{C}_d + \tilde{C}_r)\dot{q} + \Omega^2 q = g(t),\ q(0) = U^T Mx_0,\ \dot{q}(0) = U^T M\dot{x}_0,\quad t \geq 0, \tag{2.4}$$

where $\tilde{C}_d + \tilde{C}_r = \tilde{C}$, $\tilde{C}_d = \mathrm{diag}(2\xi_1\omega_1, \cdots, 2\xi_n\omega_n)$ is a matrix composed of the diagonal elements of $\tilde{C}$, and $\tilde{C}_r = [\tilde{c}_{ij}]$ is a symmetric $n \times n$ matrix with zero diagonal elements. Notice that by the positive definiteness of $\tilde{C}$, $\xi_i > 0$ for all $i = 1, \cdots, n$. We now decouple system (2.4) by neglecting $\tilde{C}_r$, and denote the solution of the resulting equation by

$q_d(t)$. Thus, we have

$$\ddot{q}_a+\tilde{C}_d\dot{q}_a+\Omega^2 q_a=g(t),\quad t\geq 0, \tag{2.5}$$

with $q_a(0)=q(0)$, and $\dot{q}_a(0)=\dot{q}(0)$. Assume at this time that the excitation is harmonic. Introduce a diagonal transformation matrix D prescribed by

$$D=diag\left[\frac{1}{\sqrt{\omega_1^2-\omega^2+j2\xi_1\omega_1\omega}},\ \cdots,\ \frac{1}{\sqrt{\omega_n^2-\omega^2+j2\xi_n\omega_n\omega}}\right], \tag{2.6}$$

where ω is the excitation frequency and $j=\sqrt{-1}$. Premultiplying D to equations (2.4) and (2.5), and using the identity $I=DD^{-1}$, we have

$$DIDD^{-1}\ddot{q}+D(\tilde{C}_d+\tilde{C}_r)DD^{-1}\dot{q}+D\Omega^2DD^{-1}q=Dg(t), \tag{2.7}$$

$$DIDD^{-1}\ddot{q}_a+D\tilde{C}_dDD^{-1}\dot{q}_a+D\Omega^2DD^{-1}q_a=Dg(t). \tag{2.8}$$

Subtracting equation (2.8) from (2.7), and denoting the n-dimensional vector of error by $e=q-q_a$, we obtain,

$$DIDD^{-1}\ddot{e}+D\tilde{C}_dDD^{-1}\dot{e}+D\Omega^2DD^{-1}e=-D\tilde{C}_r\dot{q}, \tag{2.9}$$

where $e(0)=\dot{e}(0)=0$. If we examine the frequency response of equation (2.9), we have

$$[D\Omega^2D-D\omega^2D+Dj\omega\tilde{C}_dD]D^{-1}e(j\omega)=-Dj\omega\tilde{C}_rq(j\omega), \tag{2.10}$$

Since $D\Omega^2D-D\omega^2D+Dj\omega\tilde{C}_dD=I$, equation (2.10) can be cast in the form

$$e(j\omega)=Hq(j\omega), \tag{2.11}$$

where H is a matrix given by

$$H=-D^2j\omega\tilde{C}_r. \tag{2.12}$$

To obtain an upper bound for $\|e(j\omega)\|$ from equation (2.11), we have

$$\|e(j\omega)\|\leq\|H\|\|q(j\omega)\|, \tag{2.13}$$

where $\|H\|$ is the L_∞ induced norm of H defined by

$$\|H\|=\max_i\left[\sum_{k=1}^{n}|h_{ik}|\right]. \tag{2.14}$$

Here, h_{ik} is the ik-th element of the matrix H. From equation (2.12), h_{ik} can be expressed as

$$h_{ik}=\frac{-j\omega\tilde{c}_{ik}}{\sqrt{\omega_i^2-\omega^2+j2\xi_i\omega_i\omega}\sqrt{\omega_i^2-\omega^2+j2\xi_i\omega_i\omega}}. \tag{2.15}$$

The absolute value of h_{ik} can be written as

$$|h_{ik}|=|f_i(\omega)|^2|\tilde{c}_{ik}|, \tag{2.16}$$

where

$$f_i(\omega)=\frac{\omega^{1/2}}{\left\{(\omega_i^2-\omega^2)^2+(2\xi_i\omega_i\omega)^2\right\}^{1/4}}. \tag{2.17}$$

To evaluate the supremum of $|h_{ik}|$ over the set of excitation frequencies, we first write

$$\sup_\omega|h_{ik}|=\sup_\omega|f_i(\omega)|^2|\tilde{c}_{ik}|. \tag{2.18}$$

The supremum of $|f_i(\omega)|$ occurs at $\omega=\omega_i$, and

$$\sup_\omega|f_i(\omega)|=\frac{1}{\sqrt{2\xi_i\omega_i}}. \tag{2.19}$$

Substituting the formula (2.19) in expression (2.18), we obtain

$$\sup_\omega|h_{ik}|=\frac{|\tilde{c}_{ik}|}{2\xi_i\omega_i}, \tag{2.20}$$

which is valid for any $\xi_i>0$, $i=1,\ \ldots,\ n$. It now follows from equation (2.14) that

$$\|H\|\leq\max_i\left[\sum_{\substack{k=1\\k\neq i}}^{n}\frac{|\tilde{c}_{ik}|}{2\xi_i\omega_i}\right]. \tag{2.21}$$

If we define the row sum of the absolute values of the off-diagonal elements of the normalized damping matrix $\tilde{C}$ by

$$\sigma_i=\sum_{\substack{k=1\\k\neq i}}^{n}|\tilde{c}_{ik}|, \tag{2.22}$$

then expression (2.21) can be rewritten as

$$\|H\|\leq\max_i\left[\frac{\sigma_i}{2\xi_i\omega_i}\right]. \tag{2.23}$$

Therefore, in the frequency domain, equation (2.13) implies that

$$\|e(j\omega)\|\leq\max_i\left[\frac{\sigma_i}{2\xi_i\omega_i}\right]\|q(j\omega)\|. \tag{2.24}$$

Since $\sigma_i/2\xi_i\omega_i$ is constant, without loss of generality, expression (2.24) can be rewritten in the form

$$\|e(t)\| \le \max_i \left[\frac{\sigma_i}{2\xi_i\omega_i} \right] \|q(t)\|. \tag{2.25}$$

This result states that if the row sum of the absolute values of the off-diagonal elements, σ_i, of $\tilde{C}$ is small compared to the diagonal term $2\xi_i\omega_i$ of $\tilde{C}$, then the error committed in approximate decoupling is small. This is consistent with an earlier result given by Shahruz and Ma(1988), which shows that the error is small if and if only $\sigma_i/2\xi_i\omega_i \ll 1$ for all $i=1, \ldots, n$. By extensive numerical calculations, it has been found that the new error bound (2.25) is generally tighter than the previous error bounds given by Shahruz and Ma(1988).

To continue with the further analysis of approximation error, assume that σ_i is sufficiently small for all $i=1, \ldots, n$. If σ_i is small, q_a should be close to the actual solution. It would be of interest to obtain a neighborhood of q_a in which the actual solution q lies. Since $q=e+q_a$, by the triangle inequality, $\|q\| \le \|e\| + \|q_a\|$. From equation (2.13), we obtain

$$\|e(t)\| \le \frac{\|H\|}{1-\|H\|} \|q_a(t)\|. \tag{2.26}$$

That is, q lies in the $\frac{\|H\|}{1-\|H\|} \|q_a(t)\|$ neighborhood of q_a. It has been assumed that $1-\|H\|>0$, which is always the case if $\sigma_i/2\xi_i\omega_i$ is small for all $i=1, \ldots, n$. Define

$$r = \max_i \left[\frac{\sigma_i}{2\xi_i\omega_i} \right]. \tag{2.27}$$

Then, from equation (2.23), $\|H\| \le r$. Since $f(x)=x/(1-x)$ is an increasing function for $0 \le x < 1$, it follows from equation (2.26) that

$$\|e(t)\| \le \frac{r}{1-r} \|q_a(t)\|. \tag{2.28}$$

Notice that the above error bounds have been derived by assuming harmonic excitation, but the results can be extended to any other type of periodic excitation by the method of Fourier series. The utility of the error bound expressed by equations (2.27) and (2.28) lies in the relative ease with which they can be computed. The error bound can be evaluated by merely examining the normalized damping matrix $\tilde{C}$, before the process of approximate decoupling is performed. Moreover, the error bound prescribed by expressions (2.27) and (2.28) is generally tighter than previous tight error bound given by Shahruz and Ma(1988). This point will be illustrated by an example.

Ⅲ. Modal coupling

In this section, modal coupling between any two adjacent modes is investigated. The aim is to derive a criterion to ensure that approximate decoupling does not cause excessive errors in the response. Rewrite equation (2.7) as

$$DID\ddot{q}_D + D\tilde{C}_d D\dot{q}_D + D\Omega^2 D q_D + D\tilde{C}_r D\dot{q}_D = g_D(t), \tag{3.1}$$

where $q_D = D^{-1}q$ and $g_D(t) = Dg(t)$. Since $D\Omega^2 D - D\omega^2 D + Dj\omega\tilde{C}_d D = I$, the frequency response matrix of equation (3.1) can be cast in the form

$$[I - Z(j\omega)]\, q_D(j\omega) = g_D(j\omega), \tag{3.2}$$

where

$$Z(j\omega) = -Dj\omega\tilde{C}_r D. \tag{3.3}$$

It is easy to see that the diagonal elements of $Z(j\omega)$ are zero. The magnitude of each off-diagonal element z_{ik} of $Z(j\omega)$ provides a direct measure of the degree of coupling between the i-th and k-th modes. Coupling between the two modes may be neglected whenever the absolute value of z_{ik} is sufficiently small. This condition may be stated as

$$|z_{ik}| \ll 1. \tag{3.4}$$

It is straightforward to show that

$$z_{ik} = \frac{-j\omega\tilde{c}_{ik}}{\sqrt{\omega_i^2 - \omega^2 + j2\xi_i\omega_i\omega}\,\sqrt{\omega_k^2 - \omega^2 + j2\xi_k\omega_k\omega}}. \tag{3.5}$$

The absolute value of z_{ik} is given by

$$|z_{ik}| = |f_i(\omega)||f_k(\omega)||\tilde{c}_{ik}|, \tag{3.6}$$

where

$$f_k(\omega) = \frac{\omega^{1/2}}{\{(\omega_k^2-\omega^2)^2 + (2\xi_k\omega_k\omega)^2\}^{1/4}}, \tag{3.7}$$

and $f_i(\omega)$ is as defined in the previous section. As shown in equations (3.6) and (3.7), the value of $|z_{ik}|$ depends on ω_i, ω_k, ξ_i, ξ_k, $\tilde{c}_{ik}$, and ω. Notice that $|z_{ik}|$ attains the maximum neither at $\omega=\omega_i$ nor at $\omega=\omega_k$. The maximum of $|z_{ik}|$ always occurs somewhere between ω_i and ω_k. However, the functions $|f_i(\omega)|$, $|f_k(\omega)|$ attain the maximum at, $\omega=\omega_i$, ω_k respectively. The frequency separation $|\omega_i - \omega_k|$ and the ratio of the off-diagonal to diagonal terms of the normalized damping matrix $\tilde{C}$ would have strong influence on $|z_{ik}|$. Coupling between the i-th and k-th modes is the strongest when $|z_{ik}|=c$, where

$$c=\sup_{\omega}|z_{ik}|. \quad (3.8)$$

It is clear that

$$\sup_{\omega}|z_{ik}| \le \sup_{\omega}|f_i(\omega)|\sup_{\omega}|f_k(\omega)|\,|\tilde{c}_{ik}|. \quad (3.9)$$

Since the supremum of $|f_i(\omega)|$ and $|f_k(\omega)|$ can be written as

$$\sup_{\omega}|f_i(\omega)|=\frac{1}{\sqrt{2\xi_i\omega_i}}, \quad (3.10)$$

and

$$\sup_{\omega}|f_k(\omega)|=\frac{1}{\sqrt{2\xi_k\omega_k}}, \quad (3.11)$$

it follows by taking positive square roots that

$$c\le\left[\frac{|\tilde{c}_{ik}|}{2\xi_i\omega_i}\right]^{\frac{1}{2}}\left[\frac{|\tilde{c}_{ik}|}{2\xi_k\omega_k}\right]^{\frac{1}{2}}, \quad (3.12)$$

which is valid for any $\xi_i>0$, $i=1, \ldots, n$. Therefore, the method of approximate decoupling would not cause excessive errors if $\tilde{C}_r$ is sufficiently small. In addition, expression (3.12) provides a convenient means of estimating the degree of modal coupling.

It is important to discuss our results in the light of observations made by Hasselman(1976) and Warburton and Soni(1977). Since Hasselman's result is equivalent to the result provided by Warburton and Soni, we shall only refer to the result of Hasselman for convenience. Two observations are worth reporting. First, the method of approximate decoupling would not cause serious errors if the normalized damping matrix $\tilde{C}$ is strongly diagonally dominant. For a multi-degree of freedom system, the ratio of the off-diagonal to diagonal terms of $\tilde{C}$ satisfies the relation

$$\frac{|\tilde{c}_{ik}|}{2\xi_i\omega_i}\le\frac{\sigma_i}{2\xi_i\omega_i}, \quad (3.13)$$

where σ_i has been defined by equation (2.22). If the normalized damping matrix is strongly diagonally dominant, i.e.

$$\frac{\sigma_i}{2\xi_i\omega_i}\ll 1, \quad (3.14)$$

for all i, then equation (3.12) implies that $c\ll 1$, and the errors introduced by approximate decoupling must be small. Hasselman has stated that the degree of coupling between two classical normal modes depends not only on the ratio of the off-diagonal to diagonal terms of the normalized damping matrix $\tilde{C}$, but also on the percentage of critical damping in the two modes and their frequency separation. This conclusion has been arrived with the assumption that $|z_{ik}|$ attains the maximum either at $\omega=\omega_i$ or ω_k. The assumption is generally not correct. If the modal damping matrix $\tilde{C}$ is strongly diagonally dominant, it is not necessary to stipulate an additional requirement, such as adequate frequency separation between the modes, in order to neglect the coupling effect of modes in the solution of the system. In other words, strong diagonal dominance of the matrix $\tilde{C}$ is already a sufficient condition for approximate decoupling. Although the proof of the above statement requires $\tilde{C}$ to be strongly diagonally dominant, it has been found, by numerous examples, that the statement may also be valid if $\tilde{C}$ is strictly diagonally dominant, i.e.

$$\frac{\sigma_i}{2\xi_i\omega_i}<1, \quad (3.15)$$

for all i. That means adequate frequency separation is not necessary to ensure small errors, if the normalized damping matrix is strictly diagonally dominant. Secondly, if the modal damping matrix $\tilde{C}$ is not diagonally dominant and all damping coefficients ξ_i are equivalent in magnitude, then an additional condition that adequate frequency separation exists between two modes is needed in order to neglect the modal coupling. This could be explained simply by using graphical method. In Fig. 1, two dashed

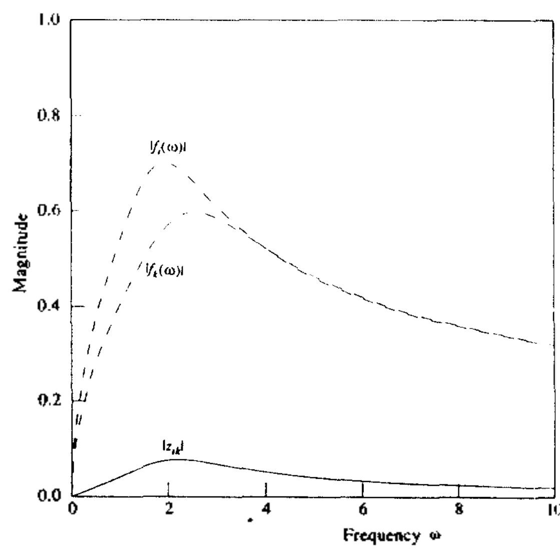

Figure 1. The effect of frequency separation on $|z_{ij}|$.

lines representing $|f_i(\omega)|$ and $|f_k(\omega)|$ are shown, and the solid line represents $|z_{ik}|$. The values of parameters used in Fig. 1 correspond to those of the example of section 4. The i-th and k-th modes are the first and third modes respectively. The qualitative features of Fig. 1, however, are typical of any multi-degree of freedom system. Notice that the curves of $|f_i(\omega)|$ and $|f_k(\omega)|$ have a bell-shape, $|f_i(0)|=|f_k(0)|=0$, and both $|f_i(\omega)|$ and $|f_k(\omega)|\to 0$ and $\omega\to\infty$. The larger the frequency separation between the two modes, the smaller the supremum of $|z_{ik}|$ becomes. Therefore, for a coupled system with many degrees of freedom, we can neglect, in every case, almost all off-diagonal terms of $\tilde{C}$ except for a few off-diagonal terms whose associated natural frequencies are adjacent to those of the corresponding diagonal terms.

Ⅳ. An example

In this section, an example is given to illustrate the possible application of the results obtained so far. A low order system is used for convenience. However, the major results of this paper are particularly useful for large-scale systems. Consider a system whose normalized equation is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\\ddot{q}_2\\\ddot{q}_3\end{bmatrix}+\begin{bmatrix}2.04&-0.2&-0.19\\-0.2&2.2&-0.3\\-0.19&-0.3&2.8\end{bmatrix}\begin{bmatrix}\dot{q}_1\\\dot{q}_2\\\dot{q}_3\end{bmatrix}+$$

$$\begin{bmatrix}3.8&0.0&0.0\\0.0&4.0&0.0\\0.0&0.0&6.5\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\sin 2t, \tag{4.1}$$

with zero initial conditions. We have $\omega_1=1.9493$, $\omega_2=2$, $\omega_3=2.55$, $\xi_1=0.523$, $\xi_2=\xi_3=0.55$. An approximate solution of system (4.1) is obtained by solving the following decoupled equations

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\ddot{q}_{1a}\\\ddot{q}_{2a}\\\ddot{q}_{3a}\end{bmatrix}+\begin{bmatrix}2.04&0&0\\0&2.2&0\\0&0&2.8\end{bmatrix}\begin{bmatrix}\dot{q}_{1a}\\\dot{q}_{2a}\\\dot{q}_{3a}\end{bmatrix}+$$

$$\begin{bmatrix}3.8&0.0&0.0\\0.0&4.0&0.0\\0.0&0.0&6.5\end{bmatrix}\begin{bmatrix}q_{1a}\\q_{2a}\\q_{3a}\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\sin 2t. \tag{4.2}$$

The exact solution $q_2(t)$ and the corresponding approximate solution $q_{2a}(t)$ are plotted in Fig. 2. It can be seen that the steady state is reached after a short transient behavior. The steady-state solution of system (4.2) is

$$q_s(t)=\begin{bmatrix}(0.245)\sin(2t+1.521)\\(0.227)\sin(2t-1.571)\\(0.163)\sin(2t-1.151)\end{bmatrix} \tag{4.3}$$

For the system (4.2), $\|q_a\|=\|q_s\|=0.245$. From equation (2.27), $r=0.227$. Using equation (2.28), we obtain $\|e\|<(0.294)\|q_a\|=0.072$. The error bound obtained by expressions (2.27) and (2.28) is quite close to the exact error and is plotted as solid lines in Fig. 2. To demonstrate the tightness of the error bound developed in this paper, comparison is made with an error bound obtained previously by Shahruz and Ma(1988). Their error bound, which is the tightest in a certain way, has been derived without any assumption on the external excitation, while the error bound (2.28) requires the external excitation to be harmonic or periodic. Substantially greater effort is needed to compute the error bound of Shahruz and Ma(1988). Their error bound, shown by dashed lines in Fig. 2, is equivalent to $\|e\|<0.425\|q_a\|=0.1042$, which is much larger than the error bound (2.28). As demonstrated by this example, the new error bound of this paper is superior, in terms of accuracy and computational effort. The new error bound can be obtained very readily by examining only the normalized damping matrix. Extensive numerical calculations have been performed by the authors, and all calculations have yielded the same qualitative conclusion on the advantages of the use of the new error bound.

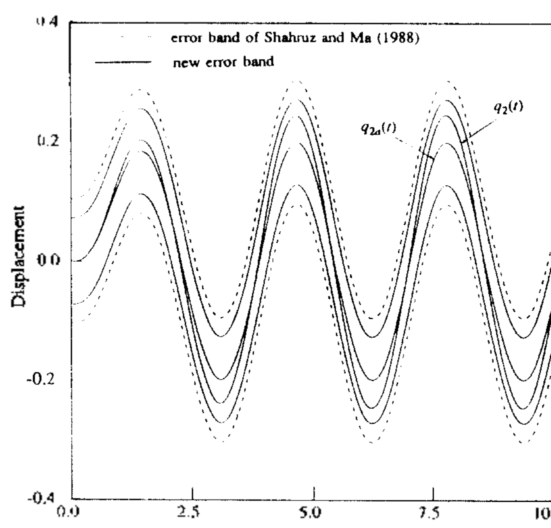

Figure 2. Comparison of error bounds.

Modal coupling between any two modes in this example is exhibited in Table 1. Modal coupling between the second and the third mode is the most significant, and the coupling between the first and the third is the weakest.

The numbers in the table have been normalized as equation (3.2) implies: 0 and 1 denote zero coupling and full coupling respectively. As shown in the table, estimates of modal coupling by equation (3.12) are quite accurate. The estimate of coupling between the first and second mode is particularly remarkable since the frequency of external excitation is very close to the natural frequencies of the first and second modes. The normalized damping matrix of system (4.1) is diagonally dominant. Even though the modal frequencies of the first and second mode are very close, or $(\omega_2-\omega_1)/(\omega_1)=2.5\%$, modal coupling between the two modes is only 0.094. This agrees qualitatively with our earlier discussion.

Table 1. Modal coupling between any two modes.

		1st mode	2nd mode	3rd mode
1st mode	Eq.(3.12)	1	0.0944	0.0795
	Exact sol.	1	0.0943	0.0731
2st mode	Eq.(3.12)	0.0944	1	0.1208
	Exact sol.	0.0943	1	0.1154
3rd mode	Eq.(3.12)	0.0795	0.1208	1
	Exact sol.	0.0731	0.1154	1

V. Conclusions

The normalized coordinates of a nonclassically damped system are coupled by nonzero off-diagonal elements in the normalized damping matrix. A common procedure in the solution of a damped linear system with small off-diagonal damping elements is to neglect the off-diagonal elements of the normalized damping matrix. In this paper, the extent of approximation introduced by this method of decoupling the system is evaluated. The major results, summarized in the following, are applicable to any linear system with nonclassical damping elements.

(1) A new error bound (2.28) has been derived for the case of harmonic or periodic excitation. In terms of accuracy and computational effort, the new error bound is superior to earlier ones. And unlike the error bounds of Shahruz and Ma(1988), the new error bound is valid for all $\xi_i>0$. Because of relative ease with which the new error bound can be computed, it is straightforward to first examine the error bound before the method of approximate decoupling is applied.

(2) It has been shown in expression (3.12) how modal coupling between any two modes can be estimated by the corresponding ratios of the off-diagonal to diagonal terms of the normalized damping matrix. If the normalized damping matrix is strongly diagonally dominant, modal coupling can be neglected without causing serious errors in the approximate solution. In the case, it is not necessary to stipulate an additional requirement, such as adequate frequency separation between any two modes, to apply the method of approximate decoupling.

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