Implementation of the Discrete Rotational Fourier Transform

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Abstract

In this paper we implement the Discrete Rotational Fourier Transform(DRFT) which is a discrete version of the Angular Fourier Transform and its inverse transform. We simplify the computation algorithm in [4], and calculate the complexity of the proposed implementation of the DRFT and the inverse DRFT, in comparison with the complexity of a DFT(Discrete Fourier Transform).

I. Introduction

In general, the N-point Discrete Fourier Transform (DFT) is defined by the following pair

$$X(k) = W\, x(n) \tag{1}$$

$$x[n] = W^{-1} X(k) \tag{2}$$

where the DFT operator W is defined as an $N \times N$ matrix with entries

$$W_{nk} = \frac{1}{\sqrt{N}} \exp(-j\frac{2\pi nk}{N}) \tag{3}$$

for $n, k = 0, 1, 2, \cdots, N-1$. [1, 2, 4]

The Discrete Rotational Fourier Transform (DRFT), i. e., an angular generalization of the DFT, is defined by the following pair

$$X_\alpha(k) = A_\alpha\, x[n] \tag{4}$$

$$x[n] = A_\alpha^{-1} X(k) \tag{5}$$

and the DRFT operator is given by

$$A_\alpha = W^{\frac{2\alpha}{\pi}} \tag{6}$$

where α represents the amount of the angle to be rotated [1, 2, 4]. The DRFT operator A_α becomes the DFT operator W when the rotation angle $\alpha = \frac{\pi}{2}$. Since we know that W is a unitary operator with a set of N eigenvectors and four distinct eigenvalues [1, -1, j, $-$j] [3], the DRFT operator can be represented, using the Taylor series expansion of the matrix operator A_α followed by application of the Cayley-Hamilton theorem [6], as follows;

$$A_\alpha = a_0(\alpha) I + a_1(\alpha) W + a_2(\alpha) W^2 + a_3(\alpha) W^3 \tag{7}$$

where the coefficients $a_i(\alpha)$ for i = 0, 1, 2, 3 is given by

$$a_0(\alpha) = \frac{1}{2}(1 + e^{j\alpha})\cos\alpha \tag{8}$$

$$a_1(\alpha) = \frac{1}{2}(1 + je^{j\alpha})\sin\alpha \tag{9}$$

$$a_2(\alpha) = \frac{1}{2}(-1 + e^{j\alpha})\cos\alpha \tag{10}$$

$$a_3(\alpha) = \frac{1}{2}(-1 - je^{j\alpha})\sin\alpha \tag{11}$$

Similarly, the inverse DRFT operator is given, using the unitary nature of the operator [6], as follows;

$$A_\alpha^{-1} = a_0^*(\alpha) I + a_3^*(\alpha) W + a_2^*(\alpha) W^2 + a_1^*(\alpha) W^3 \tag{12}$$

Thus, the DRFT operator $A\alpha$ can be interpreted as a projection onto the basis matrices I, W, W^2 and W^3 with the angular coefficients $a_i(\alpha)$ for i = 0, 1, 2, 3.

In this paper, the DRFT which is an angular generalization of the DFT and its inverse transform, are implemented. Namely, the computation algorithm that avoids eigenvalues decomposition is efficiently simplified and the complexity of the implementation is reduced, in compari-

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son with the DFT.

Ⅱ. Computing Algorithm of the DRFT

The computation of the DRFT is described as follow [5]:

Step 1. Generate the basis matrices I, W, W^2 and W^3 for a given dimension N.

Step 2. Evaluate A_α using (7) with the angular coefficients given by (8)-(11).

Step 3. Compute the DRFT using the equation (4).

Then, the DRFT of a discrete signal $x[n]$ becomes

$$X_\alpha(k)=a_0(\alpha)\sum_{n=0}^{N-1}x[n]\,\delta[n-k]$$
$$+\frac{a_1(\alpha)}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]\exp(-j\frac{2\pi nk}{N})$$
$$+a_2(\alpha)\sum_{n=0}^{N-1}x[n]\,\delta[(n+k)_N]$$
$$+\frac{a_3(\alpha)}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]\exp(j\frac{2\pi nk}{N}) \qquad (13)$$

where $\delta[(n)_N]$ means a finite sequence $\delta(n)$, whose duration is N.

The inverse DRFT is computed in a similar fashion:

$$x[n]=a_0^*(\alpha)\sum_{n=0}^{N-1}X_\alpha(k)\,\delta[n-k]$$
$$+\frac{a_3^*(\alpha)}{\sqrt{N}}\sum_{n=0}^{N-1}X_\alpha(k)\exp(-j\frac{2\pi nk}{N})$$
$$+a_2^*(\alpha)\sum_{n=0}^{N-1}X_\alpha(k)\,\delta[(n+k)_N]$$
$$+\frac{a_1^*}{\sqrt{N}}\sum_{n=0}^{N-1}X_\alpha(k)\exp(j\frac{2\pi nk}{N}) \qquad (14)$$

Ⅲ. Implementation of the DRFT

After some steps of algebra for equations (8)-(11), we can represent those angular coefficients as complex number given below.

$$a_i(\alpha)=\frac{1}{2}\,[\cos(\alpha-\frac{\pi i}{2})+\cos^2(\alpha-\frac{\pi i}{2})$$
$$+j(-1)^i\sin(\alpha)\cos(\alpha)] \qquad (15)$$

Also, using the relations

$$x(k)=\sum_{n=0}^{N-1}x[n]\,\delta[n-k]$$

$$X(k)=\sum_{n=0}^{N-1}x[n]\exp(-j\frac{2\pi nk}{N})$$

$$x[(-k)_N]=\sum_{n=0}^{N-1}x[n]\,\delta[(n+k)_N]$$

$$X((-k)_N)=\sum_{n=0}^{N-1}x[n]\exp(j\frac{2\pi nk}{N}),$$

we can rewrite the equation (4) as follow:

$$X_\alpha(k)=a_0(\alpha)\,x[k]$$
$$+a_1(\alpha)\,X[k]$$
$$+a_2(\alpha)\,x[(-k)_N]$$
$$+a_3(\alpha)\,X((-k)_N) \qquad (16)$$

The computation flow of the DRFT is show in Figure 1.

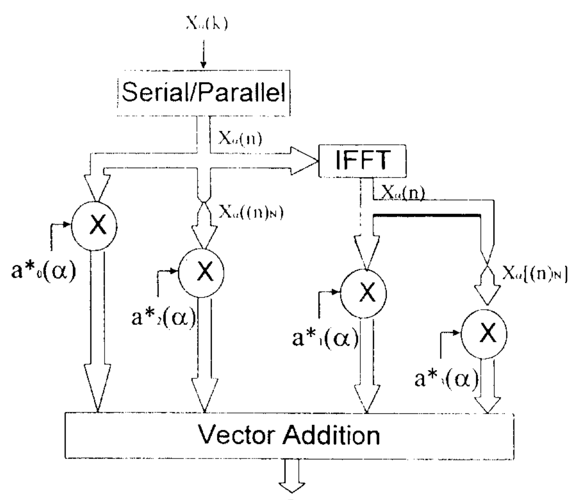

Figure 1. Computation of the DRFT

Similarly, the inverse DRFT is given by

$$x[n]=a_0^*(\alpha)\,X_\alpha(n)$$
$$+a_3^*(\alpha)\,X_\alpha[(-n)_N]$$
$$+a_2^*(\alpha)\,X_\alpha((-n)_N)$$
$$+a_1^*(\alpha)\,X_\alpha[n] \qquad (17)$$

The computation flow of the inverse DRFT is show in Figure 2.

Thus, the complexity of the DRFT is $o(16N+N\log_2 N)$ assuming one DFT is the complexity of $o(N\log N)$ because it needs a DFT computation and four complex multiplications (one multiplication of two complex numbers needs four multiplications) for each $k=0, 1, 2, \cdots, N-1$.

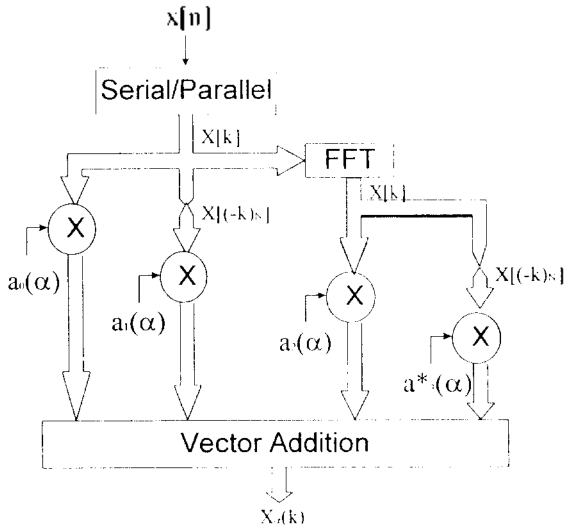

Figure 2. Computation of the inverse DRFT

Ⅳ. Conclusions

In this paper, the DRFT which is an angular generalization of the DFT, and its inverse transform was implemented. Namely, the computation algorithm that avoids eigenvalues decomposition was efficiently simplified and its complexity was in the order of $16N + N\log N$, assuming the complexity of a DFT is $N\log N$.

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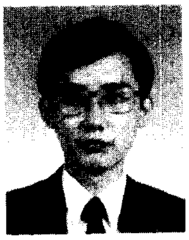

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