A Formula for Computing the Autocorrelations of the AR Process

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ABSTRACT

In this paper, we propose a formula to compute the exact autocorrelations of the autoregressive (AR) process. For an arbitrary value of N, we first review the Yule-Walker equation and some basic properties of the AR model. We then modify the Yule-Walker equation to construct a new system of $N+1$ linear equations that can be used to solve for the $N+1$ autocorrelation coefficients for lags 0, $1, \cdots, N$, provided that the AR parameters of order N and the power of the white noise of the AR process are given.

I. Introduction

The AR model has been widely used as a valuable tool to fit a variety of practical data in several different applications, such as geophysics, parametric spectral estimation, and speech processing. One of the reasons why the AR model becomes so popular in many different fields is that the parameters of the AR model can be computed in a very efficient manner by solving the well-known Yule-Walker equation [1]-[4].

Sometimes, however, for given AR parameters, it is necessary to compute the autocorrelation coefficients of the output data of the AR filter. For example, consider the adaptive predictor model in which the AR process output is used as the input to the predictor. We suppose that the popular least mean square (LMS) algorithm [5], [6] is employed for the adaptation proccess. To ensure the mean and mean-squared convergence of the LMS algorithm, we must select the convergence parameter (or adaptive step-size) in the coefficient update equation properly according to certain conditions. For this, we have to know the eigenvalues of the input autocorrelation matrix, which in turn requires knowledge of the autocorrelation values. Since these autocorrelations are usually one of the unknowns, it is very difficult to find the exact eigenvalues of the matrix. In practice, we often estimate the autocorrelations by means of the time-average formula, but they are still only estimates.

In this paper, for an arbitrary value of N, we first review the Yule-Walker equation and basic properties of the AR system, for which a set of $N+1$ simultaneous linear equations is derived. Here, the $N+1$ autocorrelation values of the AR process for lags 0, $1, \cdots, N$ are used as the known quantities and the AR parameters of order N and the input white noise power as the unknowns. We then modify the Yule-Walker equation to construct a set of new $N+1$ linear equations where, this time, the AR parameters of order N and the input power of the AR process are the known values and the $N+1$ autocorrelations for lags 0, $1, \cdots, N$ are the unknowns.

II. Preliminaries

The AR process of order N can be expressed as

$$x(n)=\xi(n)+\sum_{i=1}^{N} a_i\, x(n-i), \tag{1}$$

where $\xi(n)$ and $x(n)$ denote the input and output processes of the AR filter of order N, respectively, and a_i $\leq i \leq N$, denotes the i-th AR parameter. The transfer function $H(z)$ of the AR process is

$$H(z)=\frac{1}{1-\sum_{i=1}^{N} a_i\, z^{-i}}. \tag{2}$$

We assume that a_i's take on all real values and that $\xi(n)$ is zero-mean, wide-sense stationary, and white with variance σ_ξ^2.

Let $r_x(k)$ denote the autocorrelation of $x(n)$ defined as

$$r_x(k)=E\{x(n)\,x(n-k)\},\ 0 \leq k \leq N, \tag{3}$$

where $E\{\cdot\}$ denotes the statistical expectation of $\{\cdot\}$. Now, substituting (1) in (3) yields

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\begin{aligned}
r_x(k) &= E\left\{\left[\xi(n)+\sum_{i=1}^{N} a_i\, x(n-i)\right] x(n-k)\right\} \\
&= E\{\xi(n)\, x(n-k)\} + \sum_{i=1}^{N} a_i\, r_x(k-i) \\
&= \sigma_\xi^2\, \delta(k) + \sum_{i=1}^{N} a_i\, r_x(k-i) \qquad (4)
\end{aligned}
$$

for $k = 0, 1, \cdots, N$, where

$$\delta(k) = \begin{cases} 1 \text{ if } k=0 \\ 0 \text{ otherwise.} \end{cases} \qquad (5)$$

From (4), the variance of the white noise process $\xi(n)$ that is used to excite the AR model, can be expressed as

$$\sigma_\xi^2 = r_x(0) - \sum_{i=1}^{N} a_i\, r_x(i). \qquad (6)$$

The following matrix equation is readily obtained:

$$\mathbf{R}\underline{\mathbf{a}} = \mathbf{P}, \qquad (7)$$

where

$$\mathbf{R} = \begin{bmatrix} r_x(0) & r_x(1) & \cdots & r_x(N-1) \\ r_x(1) & r_x(0) & \cdots & r_x(N-2) \\ \vdots & \vdots & & \vdots \\ r_x(N-1) & r_x(N-2) & \cdots & r_x(0) \end{bmatrix}, \qquad (8)$$

$$\mathbf{P} = \begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(N) \end{bmatrix}, \qquad (9)$$

and

$$\underline{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}. \qquad (10)$$

Equation (7) is referred to as the Yule-Walker equation. The Yule-Walker equation is, in fact, of exactly the same as the normal equation for the forward predictor [7]. Moreover, equation (6) which represents the variance of the input process to the AR filter, is also of the same mathematical form as the minimum mean-squared forward prediction error. For an AR process for which the model order N is known, we thus can state that the tap weights of a forward predictor take on the same value as the corresponding AR parameters once the predictor is optimized in the mean-squared sense.

We can combine (6) and (7) into a single augmented matrix relation as

$$\begin{bmatrix} r_x(0) & \mathbf{P}^T \\ \mathbf{P} & \mathbf{R} \end{bmatrix} \begin{bmatrix} 1 \\ -\underline{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \sigma_\xi^2 \\ \underline{0} \end{bmatrix}, \qquad (11)$$

where $\underline{0}$ indicates the N-by-1 null vector, and $[\cdot]^T$ represents the transpose of $[\cdot]$.

One of the most celebrating things here is the existence of an efficient and elegant way for computing the AR parameters a_i's and the input white noise power σ_ξ^2, i.e., the Levinson-Durbin algorithm [3], [4]. This method is recursive in nature and makes particular use of the Toeplitz structure of the matrix **R**. It also has been widely employed in problems of finding partial correlation (PARCOR) coefficients or reflection coefficients of the lattice structure [8].

Ⅲ. Formulation of a New Linear System

For given σ_ξ^2 and a_i's, $1 \le i \le N$, the problem is now to construct the $N+1$ linear equations by modifying (6) and (7) to solve for $r_x(k)$, $0 \le k \le N$.

Define the normalized autocorrelation coefficient $r_x'(k)$ as

$$r_x'(k) = \frac{r_x(k)}{r_x(0)}, \; 0 \le k \le N, \qquad (12)$$

so that $r_x'(0) = 1$: Using (12) in (6) leads to

$$r_x(0) = \sigma_\xi^2 + r_x(0) \sum_{i=1}^{N} a_i\, r_x'(i). \qquad (13)$$

Hence

$$r_x(0) = \frac{\sigma_\xi^2}{1 - \sum_{i=1}^{N} a_i\, r_x'(i)}. \qquad (14)$$

Usign (12) once again in (7), we get

$$\mathbf{R}'\,\underline{\mathbf{a}} = \mathbf{P}', \qquad (15)$$

$$\mathbf{R}' = \begin{bmatrix} 1 & r_x'(1) & \cdots & r_x'(N-1) \\ r_x'(1) & 1 & \cdots & r_x'(N-2) \\ \vdots & \vdots & & \vdots \\ r_x'(N-1) & r_x'(N-2) & \cdots & 1 \end{bmatrix} \qquad (16)$$

and

$$\mathbf{P}'=\begin{bmatrix} r_x'(1) \\ r_x'(2) \\ \vdots \\ r_x'(N) \end{bmatrix}. \tag{17}$$

We are now ready to modify equations (15) to construct, together with (13), the new $N+1$ simultaneous linear equations, for which a_i's of order N and σ_ξ^2 are the known quantities and $r_x(k)$'s for lage 0, $1,\cdots,N$ are the unknowns. The construction procedure of this new system is straightforward, but nonsystematic and tedious particularly when the order N becomes large. Here, we just give the final simple result.

We will only consider the modification of the system given in (15), since one of the unknowns, $r_x(0)$, can be immediately evaluated using (14) after having the modified system solved. To compute the remaining N unknowns, the new system should be of a form

$$\mathbf{AP}'=-\mathbf{a}, \tag{18}$$

where $\mathbf{A}$ is the N-by-N matrix whose elements consist of some functions of a_i's Any subroutine for linear systems can now be invoked to solve (18). Once (18) is solved for $\mathbf{P}'$, we use the resultant $r_x'(k)$ values in (12) to get $r_x(k)$ for $k=1, 2,\cdots,N$.

Let $\mathbf{A}_{ij}$ denotes the $(i,\ j)$-th element of the matrix $\mathbf{A}$. For any value of N, it can be shown that

$$\mathbf{A}_{ij}=a_{i+j}+a_{i-j}-\delta(i-j), \tag{19}$$

in which

$$a_{i+j}=0 \text{ whenever } i+j\geq N+1, \tag{20}$$

$$a_{i-j}=0 \text{ whenever } i-j\leq 0, \tag{21}$$

and

$$\delta(i-j)=0 \text{ whenever } i\neq j. \tag{22}$$

For example, when $N=6$, it follows from (19)-(22) that

$$\mathbf{A}=\begin{bmatrix} a_2-1 & a_3 & a_4 & a_5 & a_6 & 0 \\ a_1+a_3 & a_4-1 & a_5 & a_6 & 0 & 0 \\ a_2+a_4 & a_1+a_5 & a_6-1 & 0 & 0 & 0 \\ a_3+a_5 & a_2+a_6 & a_1 & -1 & 0 & 0 \\ a_4+a_6 & a_3 & a_2 & a_1 & -1 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & -1 \end{bmatrix}. \tag{23}$$

Equations (12), (14), and (18), along with (19)-(20), collectively the new system.

For the AR process described in (1) to be convergent, the AR parameters a_i's should be given in such a way that all the poles of the transfer function $H(z)$ lie strictly inside the unit circle in the Z-plane. Morever, since a_i's are assumed to be all real values, the locations of the poles are on the real axis or in complex conjugate pairs. Under these conditions, it is easy to conjecture that the matrix $\mathbf{A}$ is nonsingular for any value of N so that the solution vector $\mathbf{P}'$ always exists and is unique. Any rigorous proof, however, has not yet been made.

Ⅳ. Conclusion

In this paper, a method of computing the exact values for the autocorrelation function of the AR process is proposed when the AR parameters and the power of the white noise process are given. For an arbitrary value of N, we first review the Yule-Walker equation and some basic properties of the AR model. We then modify the Yule-Walker equation to construct a new linear system consisting of $N+1$ linear equations, where the AR parameters of order N and the input white noise power of the AR process are used as the known values, and the $N+1$ autocorrelation coefficients for lags 0, $1,\cdots,N$ as the unknowns. The matrix in this new linear system is observed to be always nonsingular if the AR parameters are selected in such a way that the AR process is stable.

In the future, efforts should be made in finding efficient ways of solving the system, if any. It will be also interesting to explore some properties of the matrix in the system in conjuction with the Jury's stability test [9].

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