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비정체형 2차원 다공성 매질의 대수투수계수-수두 교차공분산에 관한 연구

A Study on Logconductivity-Head Cross Covariance in Two-Dimensional Nonstationary Porous Formations

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Abstract

An expression for the cross covariance of the logconductivity and the head in nonstationary porous formation is obtained. This cross covariance plays a key role in the inverse problem, i.e., in inferring the statistical characteristics of the conductivity field from head data. The nonstationary logconductivity is modeled as superposition of definite linear trend and stationary fluctuation and the hydraulic head in saturated aquifers is found through stochastic analysis of a steady, two-dimensional flow. The cross covariance with a Gaussian correlation function is investigated for two particular cases where the trend is either parallel or normal to the head gradient. The results show that cross covariances are stationary except along separation distances parallel to the mean flow direction for the case where the trend is parallel to head gradient. Also, unlike the stationary model, the cross covariance along distances normal to flow direction is non-zero. From these observations we conclude that when a trend in the conductivity field is suspected, this information must be incorporated in the analysis of groundwater flow and solute transport.

요 지

본 논문에서는 다공성 매질의 투수율이 비정체형인 경우 대수투수계수-수두 교차공분산에 관한 식을 유도하였으며, 이 교차공분산은 수두분포로부터 투수장의 통계학적 특성을 유추하는데(inverse problem) 매우 중요한 역할을 담당한다. 비정체형 대수투수계수는 일정한 선형경향과 정체형인 미소 변동의 합으로 구성되었으며, 2차원 포화대수층에서 정상 유동문제를 추계학적으로 해석하여 수두분포를 얻었고 이로부터 교차공분산을 유도하였다. 투수계수의 상관함수가 가우스분포를 가지고 그 경향이 수두 경사와 평행이거나 직교하는 두 가지 경우에 대하여 교차공분산을 살펴 본 결과, 투수장의 경향이 주 흐름방향과 평행한 경우 흐름방

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향 쪽만 제외하고는 정체형임이 밝혀졌다. 또한, 흐름방향과 직교하는 쪽으로의 교차공분산은 정체형 모델 결과와 달리 영이 아님을 알 수 있었다. 따라서 지하수 유동이나 오염물질 확산문제를 다룰 경우, 투수계수장에 어떤 경향이 존재한다고 의심될 때에는 반드시 그 경향을 해석과정에 포함시켜야 한다.

1. Introduction

The problem of water flow and solute transport in aquifers is being studied with increasing intensity as concerns about water quality and pollution continue to grow. The movement of water and solute through natural porous formations depends not only on the subsurface flow conditions but also on the hydrogeologic properties through which the flow occurs and it is common to find that earth materials have highly variable hydrogeologic properties. It is this highly variable nature coupled with the scarcity of actual field measurements in general that led to the development of stochastic methods in study of flow and transport in porous formations.

Stochastic approach regards aquifer properties such as hydraulic conductivity K and its logarithm, the logconductivity, $Y=\ln K$, as spatial random variables characterized by probability distributions. Field findings, such as Hoeksema and Kitanidis (1985), tend to indicate that Y is normal and most of the stochastic study is based on the assumption that it is also stationary, i.e., its spatial mean is a constant. Although this assumption of stationarity greatly simplifies the mathematical analysis and may be applicable in many situations, it is by no means universal. Recent reports have found that there may be definite trends in the conductivity fields such as Woodbury and Sudicky (1991) who investigated the possibility of a trend at the Borden site and Rehfeldt et al. (1992) who found good indications that a trend in the conductivity did influence the results of tracer experiment at the Columbus AFB.

There have also been studies by such authors as Rajaram and McLaughlin (1990) and Loaiciga et al. (1993) who systematically incorporated the nonstationarity of the logconductivity field in the stochastic analysis of subsurface flow. The problem of transport in a nonstationary conductivity field was treated by Rubin and Seong (1994) who provided the first-order analysis when there is a definite linear trend. The purpose of this study is to further explore the phenomena of subsurface flow along their line of analysis by studying the effects of nonstationarity on the cross covariance of the logconductivity and the head.

The logconductivity–head cross covariance, C_{YH}, is widely used in the inverse problem. The inverse problem is one where appropriate spatial distribution of the logconductivity is sought from the rather extensive head data and limited information on conductivity. Detailed information on logconductivity in field applications is seldom available as drilling wells and performing pumping tests are costly, whereas measurements of the head obtained from piezometers, which are much simpler and cheaper to operate, are more easily obtainable. This inverse or identification problem which has been the subject of much research in subsurface flow is beyond the scope of our present study and readers are referred to Carrera and Neuman (1986) for a comprehensive review.

2. Mathematical Statement and Perturbation Solution

We consider here steady, two-dimensional groundwater flow in an aquifer lying horizontally and without recharge. Following field studies, $Y(\boldsymbol{x}) = \ln K(\boldsymbol{x})$ is modeled as a log-normal space random function (SRF) and in order to investigate the effects of nonstationarity, Y is assumed to be made up of a spatially varying mean and a small-scale local fluctuation:

$$Y(\boldsymbol{x}) = \langle Y(\boldsymbol{x}) \rangle + Y'(\boldsymbol{x}) \tag{1}$$

where angle brackets denote expected-value operator. Here and subsequently, bold letters represent vectors and $\boldsymbol{x}$, $\boldsymbol{y}$ are the location vectors. In this study we assume that the expected value of Y is a linear function of space coordinate:

$$\langle Y(\boldsymbol{x}) \rangle = m_0 + \boldsymbol{a}\cdot\boldsymbol{x} \tag{2}$$

where m_0 and $\boldsymbol{a}$ are constants. The local fluctuation, Y', of Eq. (1) is stationary, i.e., with a zero mean and a covariance, C_Y:

$$C_Y(\boldsymbol{x},\boldsymbol{y}) = \langle Y'(\boldsymbol{x}) Y'(\boldsymbol{y}) \rangle = C_Y(|\boldsymbol{r}|) = \sigma_Y^2 \rho_Y(|\boldsymbol{r}|) \tag{3}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, σ_Y^2 the variance of Y and ρ_Y is the correlation function.

The hydraulic head H and the specific discharge $\boldsymbol{q}$ follow the continuity and Darcy's law.

$$\nabla\cdot\boldsymbol{q}(\boldsymbol{x}) = 0\ ;\ \boldsymbol{q}(\boldsymbol{x}) = -K(\boldsymbol{x})\nabla H(\boldsymbol{x}) \tag{4}$$

Eliminating $\boldsymbol{q}$ from Eq. (4) and combining Eqs. (1) and (2) result in the following:

$$\boldsymbol{a}\cdot\nabla H(\boldsymbol{x}) + \nabla^2 H(\boldsymbol{x}) = -\nabla Y'\cdot\nabla H(\boldsymbol{x}) \tag{5}$$

which due to the randomness of Y' is a stochastic PDE. For the boundary condition we assume that the head gradient, $-\boldsymbol{J}$, at some point $\boldsymbol{\zeta}$ in the flow domain, which is unbounded, is given as

$$\langle \nabla H(\boldsymbol{\zeta}) \rangle = -\boldsymbol{J} = (-J_0, 0) \tag{6}$$

impling that our coordinate is set up such that x_1-axis is aligned with the main flow direction. This assumption of an infinite domain boils down practically to a requirement that the flow domain be much larger than the logconductivity correlation scale, roughly 10 integral scales (Shapiro and Cvetkovic 1990). Although our solution is based on an infinite flow domain, whereas actual aquifers are obviously bounded, it can be applied to actual situations as long as they are sufficiently removed from the boundaries (Rubin and Dagan, 1988).

When H is expanded in terms of σ_Y,

$$H = H_0[1] + H_1[\sigma_Y] + H_2[\sigma_Y^2] + O[\sigma_Y^3] \tag{7}$$

and terms of same order collected, Eq. (5) becomes in ascending order of magnitude up to $O[\sigma_Y^2]$:

$$O[1]: \boldsymbol{a}\cdot\nabla H_0(\boldsymbol{x}) + \nabla^2 H_0(\boldsymbol{x}) = 0 \tag{8a}$$

$$O[\sigma_Y]: \boldsymbol{a}\cdot\nabla H_1(\boldsymbol{x}) + \nabla^2 H_1(\boldsymbol{x}) = -\nabla Y'(\boldsymbol{x})\cdot\nabla H_0(\boldsymbol{x}) \tag{8b}$$

$$O[\sigma_Y^2]: \boldsymbol{a}\cdot\nabla H_2(\boldsymbol{x}) + \nabla^2 H_2(\boldsymbol{x}) = -\nabla Y'(\boldsymbol{x})\cdot\nabla H_1(\boldsymbol{x}) \tag{8c}$$

Eqs. (8a) to (8c) together with the boundary condition as stated in Eq. (6) constitute the entire set of equations describing the flow in a nonstationary porous formation whose nonstationarity is manifested by a spatially varying mean described in Eq. (2). Although a small parameter expansion of the type used in our analysis is usually strictly valid for parameters much less than unity, recent studies such as Saladin and Rinaldo (1990) indicate that these approximations are quite accurate for σ_Y^2 of the order of unity, thus making our results applicable to many aquifers.

Solutions to water flow and solute transport as defined by Eqs. (6) and (8) are provided by Rubin and Seong (1994) and readers are referred to *op. cit.* for more detailed derivations. Here we seek the cross covariance of logconductivity and head,

$$C_{YH}(\boldsymbol{x},\boldsymbol{y}) = <Y'(\boldsymbol{x})H_1Y(\boldsymbol{y})> \qquad (9)$$

In the most general case, the mean head gradient $\boldsymbol{J}$ and the logconductivity trend $\boldsymbol{a}$ will form some arbitrary angle in the horizontal flow plane. However we limit our analysis to two particular cases: the case of $\boldsymbol{a}$ parallel to $\boldsymbol{J}$ and the case where $\boldsymbol{a}$ is orthogonal to $\boldsymbol{J}$. With our coordinate system set up as explained earlier, the former case results in $\boldsymbol{a}=(a_1, 0)$ which we will refer to as the a_1-case and the latter $\boldsymbol{a}=(0, a_2)$ which will be referred to as the a_2-case.

3. Logconductivity-Head Cross Covariance

3.1 The a_1-case: $\boldsymbol{a}=(a_1, 0)$

For the case of $a_2 = 0$, the mean head gradient, $\nabla <H_0>$, from Eq. (8a) is

$$\frac{\partial <H_0(\boldsymbol{x})>}{\partial x_1} = -\boldsymbol{J}_0 e^{-a_1(x_1-\xi_1)} \quad \text{and}$$

$$\frac{\partial <H_0(\boldsymbol{x})>}{\partial x_2} = 0 \qquad (10)$$

and H_1 from Eq. (8b) can be found as

$$H_1(\boldsymbol{x}) = -\boldsymbol{J}_0 e^{a_1\xi_1} \int G(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial Y'(\boldsymbol{x}')}{\partial x_1'} e^{-\frac{1}{2}a_1(x_1'+x_1)} d\boldsymbol{x}' \qquad (11)$$

where integration is performed over the infinite domain and G is the Green's function to the modified Helmholtz equation (see Arfken, 1985). Eq. (11) relates the two SRF's H_1 and Y' explicitly and since H_1 results from a linear operation on Y', it is also normal.

We can now derive the logconductivity-head cross covariance, $C_{YH}(\boldsymbol{x},\boldsymbol{y})$ by multiplying $H_1(\boldsymbol{x})$ by $Y'(\boldsymbol{y})$ and applying the averaging operator as follows:

$$C_{YH}(\boldsymbol{x},\boldsymbol{y}) = <Y'(\boldsymbol{x})H_1(\boldsymbol{y})>$$
$$= \boldsymbol{J}_0 e^{a_1\xi_1} e^{\frac{1}{2}a_1y_1} \sigma_Y^2 \int \frac{\partial}{\partial x_1}\rho_Y(\boldsymbol{x}-\boldsymbol{x}') e^{-\frac{1}{2}a_1x_1'} G(\boldsymbol{y}-\boldsymbol{x}')d\boldsymbol{x}' \qquad (12)$$

where we have used the fact that C_Y is stationary as expressed in Eq. (3).
After change of variables to $\boldsymbol{z}=\boldsymbol{x}'-\boldsymbol{y}$ and using convolution theorem of the Fourier transform, Eq. (12) can be expressed in terms of $\hat{\rho}_Y$ and $\hat{g}$ which are the Fourier transforms of the correlation function ρ_Y and $e^{-\frac{1}{2}a_1z_1}(\frac{\partial G}{\partial z_1}-\frac{1}{2}a_1G)$:

$$C_{YH}(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{J}_0 e^{a_1\xi_1} e^{-a_1y_1} \sigma_Y^2 \int_{-\infty}^{\infty} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \hat{\rho}_Y(-\boldsymbol{k})\hat{g}(-\boldsymbol{k})d\boldsymbol{k} \qquad (13)$$

where i denotes the imaginary unit and $\hat{g}(-\boldsymbol{k})$ is found to be the following.

$$\hat{g}(-\boldsymbol{k})=-\frac{1}{2\pi}\frac{\mathrm{i}k_1}{k^2-\mathrm{i}a_1k_1} \tag{14}$$

Finally when the cross covariance C_{YH} is non-dimensionalized by $\boldsymbol{J}_0 \; e^{a_1\xi_1}$, C_{YH} is expressed in terms of the correlation function $\hat{\rho}_Y$, which need not be isotropic, and the trend represented by the gradient of the logconductivity field a_1.

$$\frac{C_{YH}(\boldsymbol{x},\boldsymbol{y})}{\sigma_Y^2}=-e^{-a_1y_1}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\rho}_Y(\boldsymbol{k}) \frac{\mathrm{i}k_1}{k^2-\mathrm{i}a_1k_1}e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}\,d\boldsymbol{k} \tag{15}$$

3.2 The a_2-case: $\boldsymbol{a}=(0, a_2)$

For the case of $a_1 = 0$, mean head gradient from Eq. (8a) is found as

$$\frac{\partial\langle H_0(\boldsymbol{x})\rangle}{\partial x_1}=-\boldsymbol{J}_0 \quad\text{and}\quad \frac{\partial\langle H_0(\boldsymbol{x})\rangle}{\partial x_2}=0 \tag{16}$$

which makes Eq. (8b) amenable to a Fourier transform solution for H_1 resulting in

$$\hat{H}_1(\boldsymbol{k})=\boldsymbol{J}_0\frac{\mathrm{i}k_1}{k^2+\mathrm{i}a_2k_2}\hat{Y}'(\boldsymbol{k}) \tag{17}$$

By expressing $H_1(\boldsymbol{y})$ in terms of $\hat{H}_1(\boldsymbol{k})$ through an inverse Fourier transform, the cross covariance (non-dimensionalized by J_0) is again found as an integral of the correlation function and the trend of the logconductivity field a_2.

$$\frac{C_{YH}(\boldsymbol{x},\boldsymbol{y})}{\sigma_Y^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\rho}_Y(\boldsymbol{k})\frac{\mathrm{i}k_1}{k^2+\mathrm{i}a_2k_2}e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}\,d\boldsymbol{k} \tag{18}$$

With final expressions obtained in simple forms as in Eqs. (15) and (18), they could serve as a useful benchmark in testing of numerical codes that are being developed to handle flow and transport problems in nonstationary conductivity fields.

4. Results and Discussions for a Gaussian Covariance

To study the effects of a linear trend in the logconductivity on the cross covariance C_{YH}, the two-dimensional Gaussian correlation function will be used.

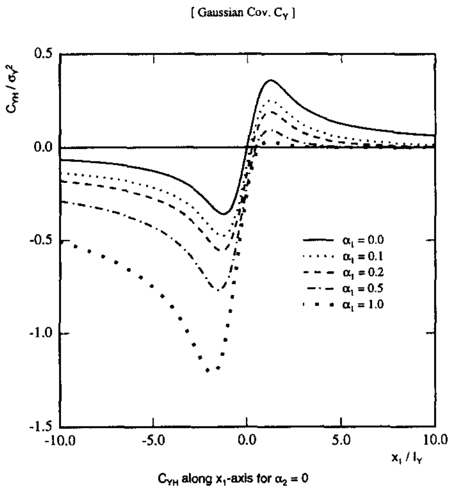
$$\rho_Y(\boldsymbol{r})=\exp[-\frac{r^2}{a^2}]\;;\; a/I_Y=2\sqrt{\pi} \tag{19}$$

Also, we will concentrate on the cross covariances along separation distances in the mean flow direction and one perpendicular to it. Before going into the particulars, it would be worthwhile to analyze the general characteristics of C_{YH} as expressed in Eqs. (15) and (18).

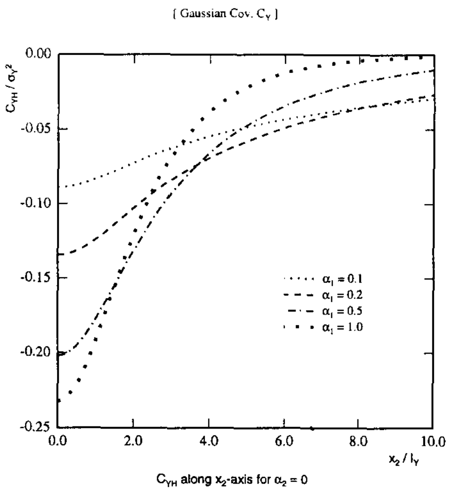
Regarding the stationarity, it should be noted that C_{YH} for the a_1-case is nonstationary, i.e., it not only depends on the distance $\boldsymbol{y}-\boldsymbol{x}$ but also on the location of the points themselves through the exponential term $e^{-a_1y_1}$. Hence, its nonstationarity is in the mean flow direction (along x_1-axis) and a closer examination of Eq. (15) reveals that the situation is simplified by the relation,

$$C_{YH}(\boldsymbol{x},\boldsymbol{y})=e^{-a_1x_1}C_{YH}(0, \boldsymbol{y}-\boldsymbol{x}) \tag{20}$$

Unlike the a_1-case, C_{YH} for the a_2-case turns out



(a) Along Mean Flow Direction



(b) Normal to Mean Flow Direction

Fig. 1. Logconductivity-Head Cross Covariance for $\boldsymbol{a} = (a_1, 0)$

to be stationary, i.e., it depends only on the separation vector $\boldsymbol{y}-\boldsymbol{x}$, similar to the results for a stationary logconductivity field.

Cross covariances for the a_1-case with a Gaussian ρ_Y along x_1 and x_2-axis are obtained through numerical quadratures and are given in Figs. 1(a) and 1(b), respectively. It should be noted that because of its nonstationarity along x_1-axis, C_{YH} is obtained between (0,0) and (x_1, 0) in Fig. 1(a). This model clearly depicts the antisymmetric profile for $a_1 = 0$, reproducing the stationary model (see Gelhar, 1993) and it is seen that due to the exponential term, it becomes skewed with magnitude increasing in the upstream direction as the trend intensifies.

Nonstationary cross covariance is symmetric along the x_2-axis as plotted in Fig. 1(b), whereas the stationary model predicts an identically zero C_{YH} (see Dagan, 1989). It can be seen that as a_1 increases, magnitude near the origin increases with the correlation length decreasing rapidly.

In Fig. 2, C_{YH} for the a_2-case is presented along the mean flow direction, which is antisymmetric. The cross covariance vanishes for any separation distance normal to the mean

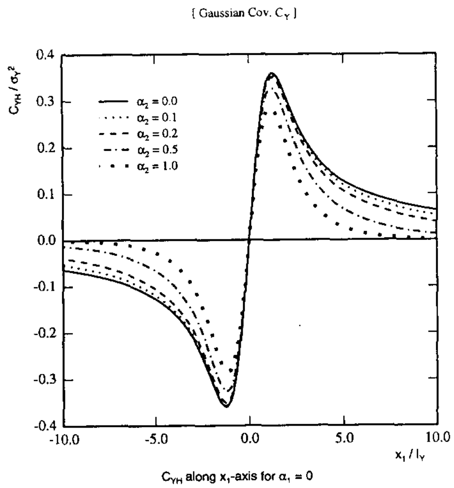


Fig. 2. Logconductivity-Head Cross Convariance for $\boldsymbol{a} = (0, a_2)$ along Mean Flow Direction.

flow direction as does the stationary model. Unlike the a_1–case, the magnitude uniformly decreases along with correlation length as the trend intensifies. From these results, one can clearly observe not only the quantitative effects of the logconductivity trend on the cross covariance but also different qualitative characteristics compared to the case where the conductivity field is stationary.

5. Conclusions

In this paper, expressions were derived for the logconductivity–hydraulic head cross covariance, C_{YH}, in two–dimensional heterogeneous porous formations whose logconductivity shows a definite linear trend. The study was motivated by the fact that this cross covariance plays a key role in the so called "inverse problem", i.e., the problem of inferring the conductivity field characteristics from head measurements and also by recent findings that report actual field situations where the existence of trends in the logconductivity is strongly suspected.

The cross covariance is developed from a linearized flow equation followed by a perturbation type expansion. The flow equation is solved up to $O[\sigma_Y^2]$ for two particular cases, where the conductivity trend is parallel to (a_1–case) or normal to the mean flow direction (a_2–case). The cross covariance is found as an integral over the Fourier wave number space in terms of the parameters which characterize the logconductivity field and the average head gradient.

For applications, results for separation distances along (x_1–axis) and normal to the mean flow direction (x_2–axis) for a conductivity field with Gaussian covariance are presented. Our results indicate that C_{YH} is stationary except for the a_1–case along x_1–axis. Also, C_{YH} along x_2–axis for the a_1–case is non–zero, unlike the results for stationary conductivity fields where it is identically zero. Considering these qualitative differences as compared to the stationary case, the stationary model would lead to erroneous results when a trend in the logconductivity field is suspected, especially when it is used in characterizing the conductivity field statistics.

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