⊙ 硏究論文

# Robust Stability of a Servosystem with Multiplicative Uncertainty

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# 곱셈형 불확실성을 갖는 서보계의 강인한 안정성

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**Key words** : Two - degree - of - freedom servosystem(2 자유도 서보계), Integral compensation(적분 보상), Multiplicative uncertainty(곱셈형 불확실성), Robust stability (강인한 안정성)


## Abstract

In order to reject the steady - state tracking error, it is common to introduce integral compensators in servosystems for constant reference signals. However, the mathematical model of the plant is exact and no disturbance input exists, the integral compensation is not necessary. From this point of view, a two - degree - of - freedom(2DOF) servosystem has been proposed, in which the integral compensation is effective only when there is a modeling error or a disturbance input. The present paper considers robust stability of this 2DOF servosystem to the unstructured uncertainty of the controlled plant. A robust stability condition is obtained using Riccati inequality, which is independent of the gain of the integral compensator. An example is presented, which demonstrates that the tracking response of the 2DOF servosystem with uncertainty becomes faster when the integral gain made larger under the robust stability condition.


## 1. Introduction

One of the most fundamental objectives required of control systems is the robust servo-property, that is, tracking reference signals in the steady state robustly against the plant uncertainties and disturbance inputs. To reject the steady - state tracking error to constant

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reference signals, it is common to introduce integral compensators. However, the mathematical model of the plant is exact and no disturbance input exists, the integral compensation is not necessary as implied by the Intermal Model Principle[1]. From this point of view, a two - degree - of - freedom (2DOF) servosystem has been proposed[2,3] in the context of LQ regulator theory, in which the integral compensation is effective only when there is a modeling error or a disturbance input.

A similar problem has been considered in4,5). In the recent literatures, stability condition of the 2DOF servosystem to the structured uncertainty has been proposed[6,7], which is independent of the gain of the integral compensator.

This paper considers robust stability of the 2DOF servosystem to the unstructured uncertainty of the controlled plant. A robust stability condition is obtained using Riccati inequality, which is independent of the gain of the integral compensator. An example is presented, which demonstrates that the tracking response of the 2DOF servosystem with uncertainty becomes faster when the integral gain made larger under the robust stability condition.

## 2. Two - Degree - of - Freedom - Servosystem

Let us consider a linear time invariant plant described by the state equation

$$\dot{x}(t)=Ax(t)+Bu(t)$$
$$y(t)=Cx(t) \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$ are the state, control input, controlled output, respectively, and $A$, $B$, $C$ are real constant matrices of proper dimensions. We require this plant to track a step - type reference signal

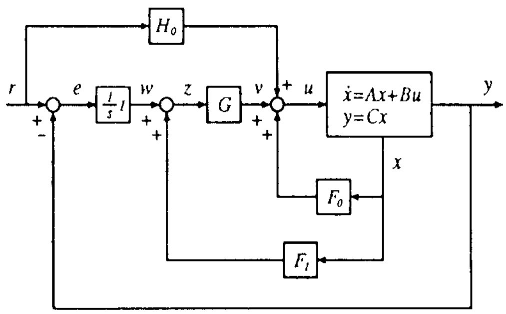


**Fig.1 A two degree freedom servosystem**

$$r(t)=\begin{cases} r'\ (t \geq 0) \\ r\ (t<0) \end{cases} \tag{2}$$

in the steady - state with no error. For this, we assume that the pair $(A, B)$ is stabilizable,

$$\text{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n+m \tag{3}$$

and the state x is measurable.

The 2DOF servosystem proposed in[2,3] is illustrated by Fig. 1, the state equation of which is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} A+B(F_0+GF_1) & BG \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} BH_0 \\ I \end{bmatrix} r(t)$$
$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{4}$$

Here, we consider the gain $F_0$ as a stabilizing one for $(A, B)$, while it is chosen as an optimal regulator gain for a quadratic performance index in2,4). The gains $F_1$ and $H_1$ are defined by

$$F_1 = C(A+BF_0)^{-1}$$
$$H_0 = [-C(A+BF_0)^{-1}B]^{-1} \tag{5}$$

and $G$ is any gain such that the closed - loop system (4) is stable.

The system matrix of (4) is represented as

$$\begin{bmatrix} A+B(F_0+GF_1) & BG \\ -C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ -F_1 & I \end{bmatrix} \begin{bmatrix} A+BF_0 & BG \\ 0 & F_1BG \end{bmatrix} \begin{bmatrix} I & 0 \\ -F_1 & I \end{bmatrix}^{-1} \tag{6}$$

which means that stability of the closed-loop system(4) is equivalent to that $F_1BG$[4], is stable matrix A choice of $G$ so that $F_1BG$ is a stable matrix, is

$$G = -R^{-1}(F_1B)^T W \tag{7}$$

where $R$ and $W$ are arbitrary positive definite matrices. This is seen from

$$F_1BG = W^{-1/2}[-W^{1/2}F_1BR^{-1}(F_1B)^T W^{1/2}]W^{1/2} \tag{8}$$

that is, $F_1BG$ is similar to a negative definite matrix, where $F_1B$ is nonsingular[4,5].

The gain (7) is optimal for the pair of a quadratic perfomance index and the augmented system composed of the plant (1) and integral compensators, when the gain $F_0$ is chosen as an optimal regulator gain for the plant. We use this particular gain extensively in the present paper.

## 3. Robust Stability

The above stability analysis is valid when the system description (1) of a plant is an exact model. However, system descriptions we deal with are generally only nominal models of plants, and we should suppose that there are modeling errors. In the recent literatures, stability condition of the 2DOF servosystem to the structured uncertainty has been proposed[6,7], which is independent of the gain of the integral compensator. In this section, we present a robust stability condition on unstructured uncertainty.

Let us describe the state equation of an uncertain plant as

$$\begin{aligned} \dot{x}(t) &= Ax(t)+B\tilde{u}(t) \\ y(t) &= Cx(t) \\ \tilde{u}(t) &= u(t)+u_\delta(t) \\ u_\delta(s) &= \Delta(s)u(s) \end{aligned} \tag{9}$$

where $\Delta(s)$ is an input multiplicative perturbation as the unstructured uncertainty defined as

$$\| \Delta(s) \|_\infty \leq \gamma \tag{10}$$

and $\tilde{u} \in R^{m}$ is input of perturbed plant, $u_\delta \in R^m$ is output of perturbation. For symplicity, no distrubance input is considered. Then, the 2DOF servosystem (4) which is shown in Fig. 2 is represented as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} A+B(F_0+GF_1) & BG \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} BH_0 \\ I \end{bmatrix} r(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u_\delta(t) \tag{11}$$

where $G = G_0W$.

Here, we transform the 2DOF system (11) using

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} I & O \\ F_1 & I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{12}$$

to obtain the representation

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A+BF_0 & BG \\ 0 & F_1BG \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ F_1B \end{bmatrix} u_\delta(t)$$

$$u(t) = \begin{bmatrix} F_0 & G \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{13}$$

This is the system representation from $u_\delta$ to $u$ for $r=0$. And we define the transfer function

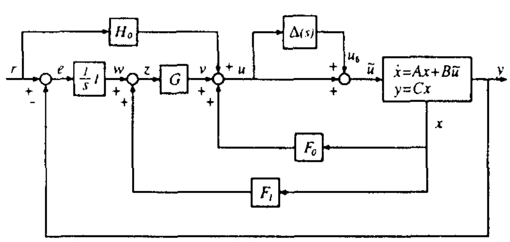


**Fig. 2 A two degree freedom servosystem with multiplicative uncertainty**

from $u_\delta$ to $u$ as $T_{uu\delta}(s)$.

**Lemma** The necessary and sufficient condition for the 2DOF servosystem (9) shown in Fig. 2 to be robustly stable for the uncertainty is that the closed loop system is stable when $\Delta(s)=0$ and

$$\| T_{uu\delta}(s) \|_\infty < \frac{1}{\gamma} \tag{14}$$

holds[5].

Using this lemma, we now consider robust stability of the system. From the system representation (13), define

$$\tilde{A}(W)=\begin{bmatrix} A+BF_0 & BG_0W \\ 0 & F_1BG_0W \end{bmatrix}$$
$$\tilde{B}=\begin{bmatrix} B \\ F_1B \end{bmatrix}$$
$$\tilde{C}(W)=\begin{bmatrix} F_0 & G_0W \end{bmatrix} \tag{15}$$

and positive definite symmetric matrix $\tilde{P}(W)$

$$\tilde{P}(W)=\begin{bmatrix} P & 0 \\ 0 & W \end{bmatrix} \tag{16}$$

where $P$ is a positive definite solution of Lyapunov inequality

$$P(A+BF_0)+(A+BF_0)^TP<0 \tag{17}$$

The existence of $P$ is guaranteed by the stability of $A+BF_0$. Here, from equations (15) and (16), we describe $\tilde{A}(W)$, $\tilde{C}(W)$, and $P(W)$ as $\tilde{A}(I)$, $\tilde{C}(I)$ and $\tilde{P}(I)$ when $W=\mathrm{I}$, respectively.From this, assume

$$\tilde{P}(I)\tilde{A}(I)+\tilde{A}^T(I)\tilde{P}(I)<0 \tag{18}$$

holds[6].

The following theorem can now be demonstrated.

**Theorem** For the structured uncertainties $\Delta A$, $\Delta B$, $\Delta C$ and $r>0$, there exist a positive definite matrix $\tilde{P}(I)$ and positive definite number $\mu$ such that Riccati ineguality

$$\tilde{P}(I)\tilde{A}(I)+\tilde{A}^T(I)\tilde{P}(I)+\frac{\gamma^2}{\mu}\cdot\tilde{P}(I)\tilde{B}\tilde{B}^T\tilde{P}(I)$$
$$+\mu\tilde{C}^T(I)\tilde{C}(I)<0 \tag{19}$$

holds, then the servosystem (9) is robustly stable independently of the tuning parameter $W$.

***Proof*** : Equations (15) and (16) imply that $\tilde{A}(W)$, $\tilde{C}(W)$ and $\tilde{P}(W)$ are represented as

$$\tilde{A}(W)=\tilde{A}(I)\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}$$
$$\tilde{C}(W)=\tilde{C}(I)\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}$$
$$\tilde{P}(W)=\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}\tilde{P}(I) \tag{20}$$

Using these relations and assumption (18), Lyapunov inequality

$$\tilde{P}(W)\tilde{A}(W)+\tilde{A}^T(W)\tilde{P}(W)+\frac{\gamma^2}{\mu}\tilde{P}(W)\tilde{B}\tilde{B}^T$$
$$\tilde{P}(W)+\mu\tilde{C}^T(W)\tilde{C}(W)=\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}$$
$$\cdot[\tilde{P}(I)\tilde{A}(I)+\tilde{A}^T(I)\tilde{P}(I)+\frac{\gamma^2}{\mu}\tilde{P}(I)\tilde{B}\tilde{B}^T\tilde{P}(I)$$
$$+\mu\tilde{C}^T(I)\tilde{C}(I)]\cdot\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}<0 \tag{21}$$

is obtained which concludes robust stability of the system (11) to the unstructured uncertainty $\Delta(s)(\|\Delta(s)\|_\infty\le\gamma)$ for any $W>0$.

Now, show that there always exists a positive definite matrix $\tilde{P}(I)$ for which the assumption (18) holds[6]. For this, condsider

$$\tilde{\Gamma}=\tilde{P}(I)\tilde{A}(I)+\tilde{A}^T(I)\tilde{P}(I) \tag{22}$$

the negative definiteness of which is equivalent to (18). We decompose $\tilde{\Gamma}$ into four blocks as

$$\tilde{\Gamma}=\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \tag{23}$$

where

$$\Gamma_{11}=P(A+BF_0)+(A+BF_0)^TP+PBG_0F_1 +(PBG_0F_1)^T+F_1^TF_1BG_0F_1+(F_1^TF_1BG_0F_1)^T$$
$$\Gamma_{12}=PBG_0+F_1^TF_1BG_0+(F_1BG_0F_1)^T$$
$$\Gamma_{21}=(PBG_0)^T+(F_1^TF_1BG_0)^T+F_1BG_0F_1$$
$$\Gamma_{22}=F_1BG_0+(F_1BG_0)^T \quad (24)$$

In (24), $\Gamma_{22}$ is negative definite since $F_1BG_0 = -F_1BR^{-1}(F_1B)^T$. Therefore, negative definiteness of $\tilde{\Gamma}$ is equivalent to that of

$$\Gamma_{11}-\Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21}=P(A+BF_0)+(A+BF_0)^TP +(1/2)PBR^{-1}B^TP \quad (25)$$

Here, the sum of first and second terms in the right side is negative definite as seen from (17). The third term is positive semidefinite, but square in $P$. This implies that, by choosing a sufficiently small $P$ satisfying (17) which always exists, we can make (25) negative definite and conclude negative definiteness of $\tilde{\Gamma}$ of (22).

When we choose an optimal regulator gain for the nominal plant as the stabilizing gain $F_0$ for $(A, B)$, the positive definite solution $P$ of the Riccati equation

$$PA+A^TP-PBR^{-1}B^TP+Q=0 \quad (26)$$

where $Q$ and $R$ are positive definite, always makes $\Gamma$ of (22) negative definite. Acturally, for

$$F_0=-R^{-1}B^TP \quad (27)$$

the right-side of (25) becomes negative definite $-Q-(1/2)PBR^{-1}B^TP$.

## 4. A Numerical Example

We present an example to illustrate the change of behaviors of the 2DOF servosystem (9) when we increase the tuning parameter W in the gain G of (7). Let the matrices A, B, C of the plant (1) be

$$A=\begin{bmatrix}-1.5 & 1\\ 0 & 2.0\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},\ C=\begin{bmatrix}1 & 0\end{bmatrix} \quad (28)$$

We adopt an optimal regulator gain for stabiliz-ing$(A, B)$ so that the assumption(18) holds automatically. We use the Riccati equation (26) with

$$R=I,\ Q=\mathrm{diag}\{0.5,\ 1.5\} \quad (29)$$

The positive definite solution of (26) is

$$P=\begin{bmatrix}0.17 & 0.04\\ 0.04 & 4.36\end{bmatrix} \quad (30)$$

From this, the gains are com- puted as

$$F_0=[-0.04 \quad -4.36]$$
$$F_1=[-0.66 \quad -0.28]$$
$$H_0=3.58$$
$$G_0=0.28 \quad (31)$$

These results satify the assumption (18). And the condition (19) in theorem is holds when $\mu=0.5$, $\gamma=0.45$. Thus it is clear that the 2DOF servosystem (9) is robustly stable for the unstructured uncertainty which is in the class denoted by

$$\|\Delta(s)\|_\infty\leq\gamma=0.45 \quad (32)$$

From this, consider an input multiplicative perturbation as the unstructured uncertainty

$$\Delta(s)=-0.45\frac{(s+0.22)}{(s+1)} \quad (33)$$

The reference signal is $r_+=1$ and the initial state $x_0$ and the initial value $w_0$ of the integral-compensator are 0.

Fig. 3 is the simulated results corresponding to the cases of $\alpha=1$, 10 and 100 in $W$, respectively. Where the dashdot line is the behavior of the controlled output in case of $\alpha=1$, the dashed line is that of $\alpha=10$, and the solid line

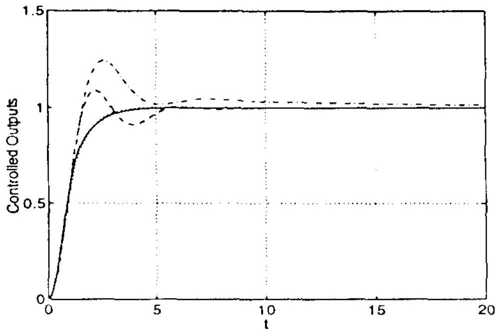


**Fig.3 Step response**

is that of $\alpha = 100$. And the dotted line shows the nominal behavior of the 2DOF servosystem(9).We see that we can achieve a fast tracking response by increasing $\alpha$.

## 5. Concluding Remarks

In this paper, a robust stability condition for a 2DOF servosystem to the unstructured uncertainty of controlled plant, which is independent of the gain of the integral compensator is presented. It was demonstrated that we can carry out high-gain integral compensation preserving robust stability to achieve fast transient responses on the untructured uncertainty. State feedback was used extensively in this paper. It is interesting to expend these results to the output feedback case.

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