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Constrained Integer Multiobjective Linear Fractional Programming Problem

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Abstract

In this paper an algorithm based on cutting plane approach is developed which constructs all the efficient p-tuples of multiobjective integer linear fractional programming problem. The integer soultion is constrained to satisfy any h out of n additional constraint sets. A numerical illustration in support of the proposed alogrithm is developed.

I. Introduction

Many interesting applications of integer linear fractional programming problems have been given in literature. Fixed charge problems, plant location problems, Job shop scheduling problems are some of the examples of integer linear fractional programming problem. Integer linear fractional programming problem have been studied by many authors [7, 8, 9, 10]. But by considering only one criteria does not serve our purpose because in practical life two or more objectives are associated with a problem. In this paper we construct the pareto-optimal set of integer solutions to enable the decision maker to choose a solution according to his constraints. Further we check that the solution so obtained, satisfy additional constraints or not.

We can find all the efficient solutions of an integer linear fractional programming problem by using Dantzig cut also but this may be a long process because its rate of convergence is very slow and at each step it studies all the dominated and non dominated solutions. The procedure developed in this paper is also hased on cutting plane technique but introduce

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deeper cuts than the Dantzig cuts. In this procedure we ifnd all the feasible soulutions of the relaxed problem (i.e. the problem without the integer constraints). Eligible directions leading to potentially efficient solutions are identified and once a point or a region is scanned it is deleted in order to truncate the current region. The portion which is deleted once, does not reapear thus leading to convergence in a finite number of steps.

II. Theoretical Development

The Constrained Integer Multiobjective Linear Fractional Programming Problem (CIMLFPP) is :

(p-1): $\text{Maximize}(z_1, z_2, \cdots, z_p)$

$$\text{where } z_r = \frac{C^r X + \alpha^r}{D^r X + \beta^r}, \quad r = 1, 2, \cdots, p$$

subject to

$$X \in S = \{X \in R^n \mid AX = b,\ X \geqq 0 \text{ is an integer point}\}$$

S is a closed and bounded convex polyhedron over which $D^r X = \beta^r > 0$, C^r, $D^r \in R^n$, $\alpha^r, \beta^r \in R$, r-1,2,...,p and

$$X \in F = \bigcup_{[j_1, j_2, \cdots j_h] \in K} (F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_h})$$

Without loss of generality one can assume that the components of C^r, D^r, $r = 1, 2, \cdots p$ are all integers.

$$F_i = \{X \mid g_i(X) \leqq = \geqq 0,\ X \geqq 0 \text{ is an integer point}\}$$

where $g_i(X) \leqq = \geqq 0$ is the set of secondary constraints in F_i $\forall$ $i \in I = \{1, 2, \cdots, n\}$.

$$K = \{\{j_1, j_2, \cdots, j_h\} \mid \{j_1, j_2, \cdots, j_h\} \subset I\}$$

is the set of all subsets of I taking h of its elements at a time.

DEFINITION :

1. Efficient Point : A point $X^0 \in S$ is said to be an efficient point iff there does not exist another point $X^1 \in S$ such that

$z_r(X^1) \geqq z_r(X^0)$, $\quad r \in R = \{1, 2, \cdots p\}$

with strict inequality sign holding for at least one $r \in R$.

If there exists such a point X^1 then the point X^0 is said to be dominated by X^1 and the p-tuple $\{z_1, z_2, \cdots, z_p\}$ corresponding to the soultion X^0 is called a dominated p-tuple.

Consider problem (PZ_1)

$$(PZ_1): \text{Max } (Z_1) = \frac{C^1X+\alpha^1}{D^1X+\beta^1}$$

subject to

$X \in S = \{X \in R^n \mid AX=b,\ X \geqq 0 \text{ is an integer point}\}$

and $X \in F$.

Let $S^1 = \{X \in R^n \mid AX=b,\ X \geqq 0\}$

S^1 is the feasible region of the relaxed problem i.e. the problem without the integer constraints.

$X^1 = \{\{x_{1j}\}$ is the optimal integer solution of problem (PZ_1) with value of

$$Z_1 = Z_1^1 = \frac{C^1X^1+\alpha^1}{D^1X^1+\beta^1}$$

Z_r^k = Value of Z_r, $r=2,3,\cdots p$ corresponding to solution X^k of (PZ_1)

$\{Z_1^k, Z_2^k, \cdots, Z_p^k\}$ is the efficient p-tuple.

B_k = Basis associated with X^k

$X^k = \{X_j^k\}$ is the optimal integer solution obtained after applying the cut

$$\sum_{j \in N_{k-1} - \{j_{k-1}\}} X_j \geq 1,\quad j_{k-1} \in T_{k-1}$$

a_j^k = activity vector of x_j^k

$Y_j^k = (B_j)^{-1} a_j^k,\ j \in J_k$

$a_k = \{j \mid a_j^k \in B_k$

$N^k = \{j \mid a_j^k \notin B_k\}$

$Z_{rj}^k = C_{B_k}^r Y_j^k, \qquad r \in R = \{1,2,\cdots,p\}$

where $C_{B_k}^r$ is the numerator of the rth objective function corresponding to the basis B_k.

$$L_{rj}^k = C_{B_k}^r\, Y_j^k, \qquad r \in R$$

where $C_{B_k}^r$ is the denominator of the rth objectove fimctopm cprresponding to the basis B_k.

$n_k^r = C_{B_k}^r X^k + \alpha^r, \qquad r=1,2,\cdots,p$

$1_r^k = C_{B_k}^r X^k + \beta^r, \qquad r=1,2,\cdots,p$

$\Delta_{rj}^{k}=n_r^k(L_{rj}^k-d_{rj}^k)-1_r^k(z_{rj}^k-c_{rj}^k)$

$T_k=\{j \mid j\in N_k$ and $\Delta_{rj}^k\le 0$ and $\Delta_{rj}^k>0$ for at least one $r\in R'=\{2,3,\cdots,p\}\}$

$J_k=\{j \mid j\in N_k,\ \Delta_{rj}^k\le 0,\ r\in R'\}$

For $J_k\in T_k'$ the edge E_{j_k} incident at solution x^k is defined as follows :

$$E_{j_k}=\left\{x=(x_1,x_2,\cdots,x_n)\ \middle|\ \begin{array}{ll} x_i=x_i^k-\theta_{j_k}\ y_{ij_k}^k, & i\in I_k \\ x_{j_k}=\theta_{j_k} & \\ x_v=0,\ v\in n_k-\{j_k\} & \end{array}\right\}$$

where $0<\theta_{j_k}\le \min\left\{\dfrac{x_i^k}{y_{ij_k}^{k'}}\ y_{ij_k}^k>0\right\}$

θ_{j_k} is an integer and $\theta_{j_k}\ y_{ij_k}^k$ is also an integer for every $i\in I_k$.

2. **Dominated Edge** : An edge incident at an integer feasible point is said to be dominated if all the solutions along that edge yield dominated p-tuples $\{z_1,z_2,\cdots z_p\}$.

Theorem I. An integer feasible solution of problem (PZ_1) not on an edge E_{j_k}, $j_1\in T_k$, $k\ge 1$ through x^k in the truncated region S^1 lies in the closed half space

$$\sum_{j\in N_k-\{j_1\}} x_j\ge 1 \tag{1}$$

Proof. Let $\hat{X}=(\hat{x}_j)$ be an integer feasible solution of problem (PZ_1) not on an edge E_{j_k}, which does not satisfy (1).

Then $\hat{x}_j=0$ for all $j\in N_K-\{j_1\}$ and $\hat{x}_{j_1}$ is an integer such that

$$0<\hat{x}_{j_1}<\min_{i\in I_k}\left\{\frac{x_i^k}{y_{ij_1}^{k'}}\ y_{ij_1}^k>0\right\}$$

then $\hat{X}$ lies on Edge E_{j_1} which is not the case.

If $x_{j_1}>\min_{i\in I_k}\left\{\dfrac{x_i^k}{y_{ij_1}^{k'}}\ y_{ij_1}^k>0\right\}$ then it leads to an infeasible point. Therefore $\hat{x}_j>0$ for some $j\in N_K-\{j_1\}$ which implies that $\hat{x}_j\ge 1$, since $\hat{X}$ is an integer feasible solution

Hence, the result.

Remark 1. The cut (1) i.e.

$\sum_{j\in N_k-\{j_1\}} x_j\ge 1$ is a generalization of the Dantzig cut as when T_k $(k\ge 1)$ is empty, the corresponding cut reduce to the Dantzig cut

$$\sum_{j\in N_k} x_j \geq 1$$

This cut (1) is preferable to the Dantzig cut as it truncates whole edge while Dantzig cut truncates only a point.

Remark 2. If for some $j_1 \in T_k (\neq \phi)$

$$\theta = \min_{i\in I_k} \left\{ \frac{x_i^k}{y_{ij_1}^{k'}} \; y_{ij_1}^k > 0 \right\} < 1$$

corresponding to solution x^k, then no integer feasible solution can be obtained on edge E_{j_1}.

Theorem II. All integer feasible solution for problem (PZ_I) through x^k, which lie in the closed half space

$$\sum_{j\in J_k} x_j \geq 1$$

are dominated.

Proof. The integer feasible solution $\bar{x}$ on edge E_k incident at solution x^k is defined as

$$x = \left\{ \bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \;\middle|\; \begin{array}{l} \bar{x}_i = x_i^k - \theta_{j_k} \; y_{ij_k}^k \\ \bar{x}_{j_k} = \theta_{j_k} \\ \bar{x}_v = 0, \;\; v \in N_k - \{j_k\} \end{array} \right.$$

where $0 < \theta_{j_k} \leq \min_{i\in I_k} \left\{ \frac{x_i^k}{y_{ij_k}^{k'}} \; y_{ij_k}^k > 0 \right\}$

θ_{j_k} and $\theta_{j_k} \; y_{ij_k}^k$ are integers for every $i \in I_k$, solution $\bar{x}$ lies in the closed half space

$$\sum_{j\in J_k} x_j \geq 1$$

For such a solution $\bar{x}$,

$$Z_r(\bar{x}) - Z_r(x^k) = \frac{\sum_{i\in I_k} C_i^r \bar{x}_i + C_{j_k}^r \bar{x}_{j_k} + \alpha^r}{\sum_{i\in I_k} d_i^r \bar{x}_i + d_{j_k}^r \bar{x}_{j_k} + \beta^r} - \frac{n_r^k}{1_r^k}$$

$$= \frac{\sum_{i\in I_k} C_i^r (x_i^k - \theta_{j_k} y_{ij_k}^k) + C_k^{\,r} \theta_{j_k} + \alpha^r}{\sum_{i\in I_k} d_i^r (x_i^k - \theta_{j_k} y_{ij_k}^k) + d_k^{\,r} \theta_{j_k} + \beta^r} - \frac{n_r^k}{1_r^k}$$

$$= \frac{\sum_{i\in I_k} C_i^r x_i^k + \alpha^r - \theta_{j_k} (\sum_{i\in I_k} C_i^r y_{ij_k}^k - C_{j_k}^r)}{\sum_{i\in I_k} d_i^r x_i^k - \beta^r - \theta_{j_k} (\sum_{i\in I_k} d_i^r y_{ij_k}^k - d_{j_k}^r)} - \frac{1_r^k}{1_r^k}$$

$$= \frac{n_r^k - \theta_{j_k} (z_{rj_k}^k - c_{j_k}^r)}{1_r^k - \theta_{j_k} (L_{rj_k}^k - d_{j_k}^r)} - \frac{n_r^k}{1_r^k}$$

$$=\frac{\theta_{jk}\{-1_r(z^k_{rjk}-c^r_{jk})+n^k_r(L^k_{rjk}-d^r_{jk})\}}{[1^k_r-\theta_{jk}(L^k_{rjk}-d^r_{jk})1^k_r}-\frac{n^k_r}{1^k_r}$$

$$=\frac{\theta_{jk}\,\Delta^k_{rjk}}{\overline{1}^r\,1^k_r}\leq 0 \text{ as } \Delta^k_{rjk}\leq 0 \text{ for } j_k\in J_k$$

where $\overline{1}^r=1^k_r-\theta_{jk}(L^k_{rjk}-d^r_{jk})$

Hence, integer feasible solution $\overline{x}$ which lies in the closed half space $\sum_{j\in J_k} x_j\geq 1$, is a dominated solution.

Similarly, all other integer feasible solutions which can be derived from x^k by moving in direction x_j, $j\in J_k$ lie in the closed half space $\sum_{j\in J_k} x_j\geq 1$ and are also dominated.

Ⅲ. Procedure to solve the problems :

Step 1. Solve problem (PZ_1). Note that in place of problem (PZ_1) one can similarly construct problem (PZ_r) for any r=2,3,···,p and proceed with any problem. Let x be an optimal solution of problem (PZ_1). Record the corresponding p-tuple $(Z^1_1, Z^2_1, \cdots Z^1_p)$ and B_1, N_1, T_1.

Step 2. Choose any $j_1\in T_1$. Find the corresponding minimum ratio θ of the pivot operation.

(a) If $\theta<1$, ignore j_1 and choose $j\in T_{1,}\ j\neq j_1$. Rename it as j_1.

(b) If $\theta<1$, determine all integer feasible solutions along edge E_{j1}. Each such solution give rise to new potentially efficient p tuples of the form $(z_1, z_2, \cdots, z_p)$ with $z_1\leq z^1_1$ and $z_r > z^1_1$ for at least one r=2,3,···,p.

Let S_2' be the collection of all p-tuples recorded upto step 1.

Remove all the dominated p-tuples from S_2' and let S_2 be the remaining set.

Thus, S_2 is the set of all potentially efficient p-tuples at the end of step 1. Test if any one satisfies F. If not, go to step 3.

Step 3. Truncate edge E_{j1} by the following cut

$$\sum_{j\in N_1-\{j_1\}} x_j\geq 1$$

Record all non-dominated solutions and check whether any one of it satisfies F or not. If it satisfies F stop otherwise repeat the process.

General kth Stage. Choose $j_k \in T_k$, determine all integer feasible solutions (if any) along edge E_{j_k}. Read the corresponding p-tuples and augment set S_k with these p-tuples to construct set S_{k+1}. Remove the dominated p-tuple from S'_{k+1} to obtain S_{k+1}.

Truncate edge E_{j_k} by the cut

$$\sum_{j \in N_k - \{j_k\}} x_j \geq 1$$

and determine the optimal integer solution x^{K+1} in the truncated region. Read the corresponding p-tuple. If it is non dominated then augment set S_{k+1} by adding this p-tuple to it and rename the augmented set as S_{k+1} again Check, if any one of the solution satisfy F or not. If not, repeat the process.

Terminal Stage. The process terminates after an nth stage when either

(1) $T_n = \phi$

(2) $Tn \neq \phi$ with

(a) any $j_n \in T_n$ yields dominated edge only

or (b) The cut

$$\sum_{j \in N_n - \{j_n\}} x_j \geq 1$$

leads to an infeasible solution in the truncated region for some $j_n \in T_n$.

The current set of stored p-tuple is the desired set of efficient p-tuples.

Theorem Ⅲ. In any efficient p-tuple$(z_1, z_2, \cdots, z_p)$, z_1 is the maximum value of the first objective corresponding to (p-1) tuple $(z_2, z_3, \cdots, z_p)$ and $(z_2, z_3, \cdots, z_p)$ is a non dominated (p-1) tuple corresponding to the value z_1 of the first objective.

Proof. Let $(z_1^*, z_2^*, \cdots, z_p^*)$ be an efficient p-tuple corresponding to X^* and z_1^* is not the maximum value of z_1. Suppose that there exist a solution X^{**} yielding value z_1^{**} of z_1 such that

$$z_1^{**} > z_1^*$$

and $$z_i^{**} = z_i^*, \; i \in R' = \{2, 3, \cdots, p\}$$

Then, $(z_1^*, z_2^*, \cdots, z_p^*)$ is dominated by the p-tuple $(z_1^{**}, z_2^*, \cdots, z_p^*)$, which is a contradiction.

Conversely. Consider an efficient p-tuple $(z_1^1, z_2^1, \cdots, z_p^1)$ and assume that there exists a solution $\hat{x}$ yielding value $\hat{z}_i$ of z_i, $i \in R = \{1, 2, \cdots, p\}$ such that $\hat{z}_1 = z_1^1$ and $(z_1^2, \cdots, z_p^1)$ is dominated by $(\hat{z}_2, \cdots, \hat{z}_p)$. Then $(z_1^1, z_1^2, \cdots, z_p^1)$ gets dominated by the p tuple $(z_1^1, \hat{z}_2, \cdots, \hat{z}_p)$ which is again a contradiction.

Thus, $(z_1^2,\cdots,z_1^p)$ is a non dominated (p-1) tuple corresponding to the value z_1^1 of the first objective.

Hence, the result.

Ⅳ. Numerical Result

Consider the problem

$$\text{Maximize } (z_1, z_2, z_3)$$

where

$$z_1 = \frac{3x_1+5x_2}{2x_1+3x_2+1}$$

$$z_2 = \frac{4x_1+x_2}{x_1+2x_2+2}$$

$$z_3 = \frac{9x_1+2x_2}{5x_1+x_2+1}$$

subject to

$$4x_1+5x_2 \le 22$$

$$5x_1+3x_2 \le 15$$

$$x_1,\ x_2 \ge 0 \text{ and integer}$$

and $X \in F = (F_1 \cap F_2) \cup (F_2 \cap F_3) \cup (F_3 \cap F_1)$

where $F_1 = \{(x_1,x_2) : x_1^2+x_2^2 \le 16,\ x_1+x_2 \ge 2\}$

$F_2 = \{(x_1,x_2) : 2x_1+x_2 \le 5,\ 3x_1+5x_2 \le 10\}$

$F_3 = \{(x_1,x_2) : 4x_1+5x_2 \le 10,\ 3x_1+4x_2 \ge 15\}$

Consider the problem (PZ_1) give by

(PZ_1) : Maximize $z_1 = \dfrac{3x_1+5x_2}{2x_1+3x_2+1}$

subject to $4x_1+5x_2 \le 22$

$5x_1+3x_2 \le 15$

$x_1,\ x_2 \ge 0$ and integer.

The optimal solution of the relaxed problem i.e. without the integer condition is $x_1=0$, $x_2=\frac{22}{5}$.

Applying Gomory cut,

$$-\frac{4}{5}x_1 - \frac{1}{5}x_3 + x_5 = -\frac{2}{5}$$

The optimal integer solution is given by Table-1.

$x^1=(0,4)$ is the efficient solution of the given problem yielding the triad$(\frac{20}{13}, \frac{2}{5}, \frac{8}{5})$. But x^1 does not satisfy F, therefore we find another efficient solution from, table-1.

$$I_1=\{2,3,4\},\ N_1=\{1,5\},\ T_1=\{1,5\}$$

Choose $j_1=1\in T_1$, $\theta=\min\{\frac{3}{5}, \frac{2}{4}\}=\frac{1}{2} < 1$

As $\theta < 1$, ignore $j_1=1$.

Truncate edge E_1 by the cut

$$\sum_{j\in N_1-\{j_1\}} x_j \geq 1$$

i.e. $x_5 \geq 1$ or $-x_5+x_6=1-1$.

Appending this cut in table-1 and solving we get table-2.

Here $x^2=(0,3)$ is an efficient solution with the triad $(\frac{3}{2}, \frac{3}{8}, \frac{3}{2})$. As x^2 does not satisfy F, therefore we find another efficient solution.

From table-2, $I_2=\{2,3,4,5\}$, $N_2=\{1,6\}$, $T_2=\{1,6\}$

Choosing $j=1\in T_2$, $\theta=\min\{\frac{6}{5}, \frac{7}{4}\}=\frac{6}{5} > 1$.

For $\theta=1$, new solution is given by $x_2=3$, $x_1=0$, i.e. $x=(0,3)$ yielding triad $(\frac{3}{2}, \frac{3}{8}, \frac{3}{2})$ which is same as x^2. If we choose $j=6\in T_2$, $\theta=\min\{3\}=3>1$.

For $\theta=1$, new solution is $x_2=2$, $x_1=0$

i.e. $X=(0,2)$ yielding triad $(\frac{10}{7}, \frac{1}{3}, \frac{4}{3})$ is another efficient solution and X satisfy F.

Therefore, the efficient solution of the problem is (0,2) yielding the triad $(\frac{10}{7}, \frac{1}{3}, \frac{4}{3})$.

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REFERENCES

1. Chandra, S., Chandramohan M., "A note on Linear Fractional Programming", Naval Research Logistics Quarterly, 27(1980) 171-174.

2. Chandra, S., Chandramohan M., "An improved Branch and Bound Method for Mixed Integer Linear Fractional Programs", Zeitschrift fiie Angewandte Mathematik and Mechanik, 59 (1979), 575-577.

3. Ecker, J.G. and Koueda, I.A., "Finding efficient points for linear multiple objective programs", Mathematical Programming 8(1975), 375-377.

4. Kornbluth, J.S.H., Steuer R.E., "Multiple Objective Linear Fractional Programming", Management Science, 27(1981) 9, 1024-1039.

5. Nykowski, I., Zolkiewski, Z., "A compromise procedure for the Multiple Objective Linear Fractional Programming Problem", European Journal of Operational Research, 19(1985), 91-97.

6. Salkin, H., "Integer Programming", Addison Wesley, Reading Mass 1975.

7. Seshan, C.R., Tibekar, V.G., "Algorithms for Integer Fractional Programming", Journal of the Indian Institute of Scinece, Section B : Physical and Chemical Series, 62(1980), 9-16.

8. Swarup, K., "Some Aspects of Linear Fractional Functionals Programming", Australian Journal of Statistics, 7(1965), 90-104.

9. Verma V., "Constrained Integer Linear Fractional Programming Problem", Optimization 21 (1990), 5, 749-757.

10. Verma V., Bakshi H.C. and Puri M.C., "Ranking in Integer Linear Fractional Programming Problems", Methods and Models of Operations Research, Vol. 34, (1990), 325-334.

Table-1

							$c_j^2 \rightarrow$	3	5	0	0	0
							$d_j^1 \rightarrow$	2	3	0	0	0
							$c_j^2 \rightarrow$	4	1	0	0	0
							$d_j^2 \rightarrow$	1	2	0	0	0
							$c_j^3 \rightarrow$	9	2	0	0	0
							$d_j^3 \rightarrow$	5	1	0	0	0
D_B^3	D_B^3	D_B^2	D_B^2	D_B^1	D_B^1	X_B	b	a_1	a_2	a_3	a_4	a_5
1	2	2	1	3	5	x_2	4	0	1	0	0	1
0	0	0	0	0	0	x_4	3	5	0	0	1	−3
0	0	0	0	0	0	x_3	2	4	0	1	0	5
$Z_3 = \frac{8}{5}$,		$Z_2 = \frac{2}{5}$,		$Z_1 = \frac{20}{13}$,								
$Z_{33}^3 = 8$,		$Z_{22}^2 = 4$,		$Z_{11}^1 = 20$								
$L_{33}^3 = 5$,		$L_{22}^2 = 10$,		$L_{11}^1 = 13$								
							$Z_{1j}^1 - c_j^1 \rightarrow$	−3	0	0	0	5
							$L_{1j}^1 - d_j^1 \rightarrow$	−2	0	0	0	3
							$\Delta_j^1 \rightarrow$	−1	0	0	0	−5
							$Z_{2j}^2 - c_j^2 \rightarrow$	−4	0	0	0	1
							$L_{2j}^2 - d_j^2 \rightarrow$	−1	0	0	0	2
							$\Delta_j^2 \rightarrow$	36	0	0	0	−2
							$Z_{3j}^3 - c_j^3 \rightarrow$	−9	0	0	0	2
							$L_{3j}^3 - d_j^3 \rightarrow$	−5	0	0	0	1
							$\Delta_j^3 \rightarrow$	5	0	0	0	−2

Table-2

							$c_j^1 \rightarrow$	3	5	0	0	0	0
							$d_j^1 \rightarrow$	2	3	0	0	0	0
							$c_j^2 \rightarrow$	4	1	0	0	0	0
							$d_j^2 \rightarrow$	1	2	0	0	0	0
							$c_j^3 \rightarrow$	9	2	0	0	0	0
							$d_j^3 \rightarrow$	5	1	0	0	0	0
D_B^3	D_B^3	D_B^2	D_B^2	D_B^1	D_B^1	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6
1	2	2	1	3	5	x_2	3	0	1	0	0	0	1
0	0	0	0	0	0	x_4	6	5	0	0	1	0	−3
0	0	0	0	0	0	x_3	7	4	0	1	0	0	−5
0	0	0	0	0	0	x_5	1	0	0	0	0	1	−1
		$Z_3 = \frac{3}{2}$,		$Z_2 = \frac{3}{8}$,		$Z_1 = \frac{3}{2}$,							
		$Z_{33}^3 = 6$,		$Z_{22}^2 = 3$,		$Z_{11}^1 = 15$							
		$L_{33}^3 = 4$,		$L_{22}^2 = 8$,		$L_{11}^1 = 10$							
							$Z_{1j}^1 - c_j^1 \rightarrow$	−3	0	0	0	0	5
							$L_{1j}^1 - d_j^1 \rightarrow$	−2	0	0	0	0	3
							$\Delta_j^1 \rightarrow$	0	0	0	0	0	−5
							$Z_{2j}^2 - c_j^2 \rightarrow$	−4	0	0	0	0	1
							$L_{2j}^2 - d_j^2 \rightarrow$	−1	0	0	0	0	2
							$\Delta_j^2 \rightarrow$	29	0	0	0	0	−2
							$Z_{3j}^3 - c_j^3 \rightarrow$	−9	0	0	0	0	2
							$L_{3j}^3 - c_j^3 \rightarrow$	−5	0	0	0	0	1
							$\Delta_j^3 \rightarrow$	6	0	0	0	0	−2