

Valid Inequalities and Facets for the Steiner Problem in a Directed Graph

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Abstract

In this paper, we describe the facial structure of the steiner problem in a directed graph by formulating it as a set covering problem. We first characterize trivial facets and derive a necessary condition for nontrivial facets. We also introduce a class of valid inequalities with 0-1 coefficients and show when such inequalities define facets.

Key words: Steiner problem in a directed graph, valid inequalities, facets.

1. Introduction

The Steiner Problem in a Directed Graph (SPDG) is defined as follows: Given a directed graph $G=(V,E)$ with a set of nodes $V=\{r\} \cup I \cup J$ and a set of arcs E , and a set of nonnegative arc costs c_a , the problem is to find a minimum cost subgraph spanning a given subset $\{r\} \cup I$ of nodes such that there exists a directed path from the root node r to every member of I , called a *demand node*. Any node in J , called a *steiner node*, may or may not be included in that subgraph. So far, a great research attention has been paid to this problem and its counterpart for an undirected graph, the Steiner Problem in an Undirected Graph (SPUG). The excellent surveys on the proposed formulations and the solution approaches can be found in Winter[13] and Maculan[11]

A last decade has witnessed an explosive growth in the polyhedral study on combinatorial

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optimization problems. And its theoretical and practical significance is well recognized. However, a little surprisingly, we can hardly find such an effort for the steiner problem in graphs, especially for the SPDG. Only a notable exception is the recent work given by Chopra and Rao[5, 6]. They develop some classes of valid inequalities and facets which are applicable to the SPUG with specific structure and extend those results to its bidirected version. It is well known that the SPUG can be transformed to the equivalent SPDG, a bidirected version of the SPUG. Moreover, the linear programming relaxation of the latter formulation provides a lower bound at least as good as one from that of the former formulation. Thus any valid inequality of the SPDG can also be used to obtain the sharp lower bounds for the SPUG. For more details about the existing formulations for the steiner problem in graphs and their linear programming relaxations, see Goemans and Myung[10].

In this paper, we describe a class of inequalities with 0-1 integer coefficients which are valid with respect to the SPDG regardless of the structure of a given graph. In the next section, we introduce some notation used throughout the study and formulate the SPDG as a set covering problem. In section 3, we characterize trivial facets of the SPDG, and derive a necessary condition for nontrivial facets along with some sufficient conditions for a class of trivial facets. In section 4, we derive a class of valid inequalities with 0-1 coefficients of the SPDG and show when those inequalities define facets. Also shown are some necessary conditions for nontrivial facets with 0-1 coefficients.

2. Notation and Problem Formulation

Let $G=(V,E)$ be a directed graph with $V=\{r\} \cup I \cup J$ where I is the demand node set, J is the steiner node set, and r is the root node. For a subset $U \subset V$, $\delta^+(U)$ denotes the set of arcs $\{(ij) \in E | i \in U, j \in V \setminus U\}$, $\delta^-(U)=\delta^+(V \setminus U)$, and $E(U)=\{(ij) \in E | i \in U, j \in U\}$. We use $\delta^+(i)$ ($\delta^-(i)$) instead of $\delta^+(\{i\})$ ($\delta^-(\{i\})$) for some $i \in V$. We represent subsets of arcs by their incidence vectors $x \in \{0,1\}^{|E|}$. A *steiner arborescence rooted at r* of G , which we will simply call a steiner arborescence, is defined as a subgraph of G whose underlying graph is a tree spanning $\{r\} \cup I$ and in which every node in I has a directed path from r . Since we assume nonnegative cost coefficients, a steiner arborescence becomes an optimal solution of the SPDG.

Let U_i be a subset of V such that $U_i \cap I \neq \emptyset$ and $r \in V \setminus U_i$, then $\delta^-(U_i)$ is called a directed steiner cut. Let $|E|=n$ and let $M=\{1, \dots, m\}$ be the index set of such U_i 's. And also

let A be an $m \times n$ 0-1 incidence matrix, each row of which has a one to one correspondence with the incidence vector of each directed steiner cut. Then the SPDG can be formulated as the following set covering problem:

$$\begin{aligned}
 (P) \quad & \min \sum_{e \in E} c_e x_e \\
 \text{s.t.} \quad & Ax \geq 1 \\
 & x_e = 0 \text{ or } 1, e \in E.
 \end{aligned}$$

This set covering formulation is first given by Aneja[1]. As far as the optimal objective value of its LP relaxation is concerned, this formulation along with the flow formulation given by Wong[14] is the strongest one among the formulations proposed to date for the SPDG[10, 11]. Throughout the study, we assume that every steiner cut has at least two arcs. If a given graph has a steiner cut with a single arc, this arc must be contained in the optimal solution. So we can transform the graph to the one satisfying the assumption by contraction. Furthermore, we also assume that there exists at least one directed path from r to each node. If the problem is feasible, there exists at least one directed path from r to each node in I . Moreover, unless some steiner node j has a directed path from r , we can delete that node and all the arcs incident to it from G . Let $F(A)$ denote the convex hull of the feasible solutions of (P) .

For any 0-1 matrix A with no zero row, we define the undirected bipartite incidence graph $B=(N, M, E_A)$ associated with it. B has a node $i \in M$ for each row of A , a node $j \in N$ for each column of A and an edge between nodes $i \in M$ and $j \in N$ if and only if $a_{ij}=1$ in the matrix A . Given $M' \subseteq M$, a set of nodes $T \subseteq N$ is called a *cover* of M' if every $i \in M'$ is adjacent to at least one node of T . We will also call a cover of M one of A . $\beta(M')$ denotes the minimum cardinality of a cover of M' . We also denote by $\beta(A)$ the minimum cardinality of a cover of A and refer to $\beta(A)$ as the *covering number* of A . Therefore, $\beta(A) = \beta(M)$.

As already mentioned, Chopra and Rao[5,6] also considered the facial structure of the SPDG where a given graph has specific structure and only bidirected arcs. Precisely speaking, the feasible region they dealt with is not exactly $F(A)$ but its dominant, $F^+(A) = \{x' | x' \geq x \text{ for some } x \in F(A)\}$. Therefore, any facet of $F(A)$ except $x_e \leq 1$ for all $e \in E$ also defines a facet of $F^+(A)$, but the reverse is not necessarily true. As shown in [5], under $F^+(A)$ every facet of a contraction minor of G is also a facet of G as far as only the bidirected arcs are contracted. However, neither is this necessarily true under $F(A)$.

3. Trivial Facets

We first characterize trivial facets of $F(A)$.

Theorem 1 *For a given graph $G=(V,E)$, the following statements are valid:*

(i) $x_e \geq 0$ defines a facet of $F(A)$ if and only if every directed steiner cut in G includes at least two arcs other than e .

(ii) $x_e \leq 1$ defines a facet of $F(A)$.

(iii) Suppose that $\delta^-(U_i)$ is a directed steiner cut. Then $\sum_{e \in \delta^-(U_i)} x_e \geq 1$ defines a facet of $F(A)$ if and only if (a) it is a minimal steiner cut and (b) for each $e \in E \setminus \delta^-(U_i)$, some steiner arborescence in G doesn't contain e , but contains exactly one arc in $\delta^-(U_i)$.

(iv) The only facet defining inequalities for $F(A)$ with integer coefficients and right hand side equal to 1 are those of the system $Ax \geq 1$.

Proof. By our assumption that any steiner cut has more than one arc, $F(A)$ is full dimensional. Therefore, (i),(ii),(iii), and (iv) are the direct consequences of the results for the set covering polytope[2,12]. □

Remark 1 Let $ax \geq \alpha_0$ define a facet of $F(A)$. Except for inequalities of the form $x_e \leq 1$, all $\alpha_e \geq 0$. And except for inequalities of the form $x_e \geq 0$, $\alpha_0 \neq 0$. Therefore, without loss of generality we can describe all other facets as $ax \geq 1$ with $\alpha \geq 0$.

In the remaining section, we describe a necessary condition for nontrivial facets. In order to do that, we first prove some sufficient characterizations of the trivial facets of the form

$$\sum_{e \in \delta^-(U_i)} x_e \geq 1.$$

Theorem 2 *For a given graph $G=(V,E)$, let $ax \geq 1$ define a facet of $F(A)$ and $\delta^-(U_i)$ be a directed steiner cut. Then $ax = \sum_{e \in \delta^-(U_i)} x_e$ if*

(i) $\alpha_e > 0$ for all $e \in \delta^-(U_i)$; and

(ii) for any two distinct nodes $v_1, v_2 \in U_i$ such that $\delta^-(u_k) \cap \delta^-(U_i) \neq \emptyset$ for $k=1,2$, at least one directed path containing only the arcs with $\alpha_e = 0$ exists either from v_1 to v_2 or from v_2 to v_1

Proof. Suppose that (i) and (ii) of the theorem are satisfied and that $ax \neq \sum_{e \in \delta^-(U_i)} x_e$. Then some feasible vector of $F(A)$ satisfies $ax = 1$ and the property that $x_{e_1} = x_{e_2} = 1$ for at least two $e_1, e_2 \in \delta^-(U_i)$. Now our aim is to induce a contradicting result by showing that some feasible vector x' of $F(A)$ satisfies $ax' < 1$. Let v_1 and v_2 be the heads of e_1 and e_2 , respectively. If $v_1 = v_2$, one of x_{e_1} , and x_{e_2} can be set equal to zero without violating the feasibility. This is be-

cause each node in a steiner arborescence has at most one incoming arc. In this case, x' can be constructed from x by setting either x_{e_1} or x_{e_2} equal to zero. Suppose $v_1 \neq v_2$. Without loss of generality, by (ii) of the theorem, we assume that there exists a directed path from v_1 to v_2 which consists of the arcs with $\alpha_e=0$. Let E_p be the set of arcs in such a directed path and T be the subgraph of G whose incidence vector is x . If there doesn't exist a directed path from r to v_1 in T , we construct x' from x by setting x_{e_1} equal to zero. Otherwise, construct x' as follows:

$$x' = \begin{cases} 0, & \text{if } e=e_1, \\ 1, & \text{if } e \in E_p, \\ x_e, & \text{otherwise,} \end{cases}$$

Corollary 1 For a given graph $G=(V,E)$, let $\alpha x \geq 1$ define a facet of $F(A)$. Then, $\alpha x = \sum_{e \in \delta^-(i)} x_e$ for some $i \in I$ if and only if $\alpha_e > 0$ for all $e \in \delta^-(i)$.

The following theorem shows one particular case of Theorem 2.

Theorem 3 For a given graph $G=(V,E)$, let $\alpha x \geq 1$ define a facet of $F(A)$ and $\delta^-(U_i)$ be a directed steiner cut. Then $\alpha x = \sum_{e \in \delta^-(U_i)} x_e$ if $\alpha_e > 0$ for all $e \in \delta^-(U_i)$, $\delta^-(U_i) \subseteq \delta^-(U_i \cap I)$, and $\alpha_e=0$ for all $e \in \bigcup_{j \in U_i \cap J} \delta^-(j)$.

Proof. We prove the theorem by induction on $|U_i|$. The case $|U_i|=1$ was treated in Corollary 1. Suppose that the theorem is true when $|U_i| \leq p$ for some positive integer p . Consider the case $|U_i|=p+1$. We also suppose the followings: $\alpha_e > 0$ for all $e \in \delta^-(U_i)$; $\delta^-(U_i) \subseteq \delta^-(U_i \cap I)$; and $\alpha_e=0$ for all $e \in \bigcup_{j \in U_i \cap J} \delta^-(j)$. It is sufficient to show that the condition (ii) of Theorem 2 is satisfied. Let v_1 and v_2 be two distinct nodes in U_i such that $\delta^-(v_k) \cap \delta^-(U_i) \neq \emptyset$ for $k=1,2$. Suppose that there exists no directed path from v_1 to v_2 which consists of the arcs with $\alpha_e=0$. Then, there exists some $W \subset V$ such that $v_1 \notin W$, $v_2 \in W$, and $\alpha_e > 0$ for all $e \in \delta^-(W)$. Let $W' = W \cap U_i$. The $\alpha_e > 0$ for all $e \in \delta^-(W')$ because $\alpha_e > 0$ for all $e \in \delta^-(U_i) \cup \delta^-(W')$. Moreover, $\delta^-(W') \subseteq \delta^-(W' \cap I)$ and $\alpha_e = 0$ for all $e \in \bigcup_{j \in W' \cap J} \delta^-(j)$ because $\alpha_e = 0$ for all $e \in \bigcup_{j \in U_i \cap J} \delta^-(j)$. By the assumption that the theorem holds for $|U_i| \leq p$ $\alpha x = \sum_{e \in \delta^-(W')} x_e$. This contradicts the fact that $\alpha_e > 0$ for $e \in \delta^-(v_1) \cap \delta^-(U_i)$. \square

From Theorem 3, we can derive the following necessary condition of nontrivial facets of $F(A)$

Theorem 4 For a given graph $G=(V,E)$, let $\alpha x \geq 1$ define a nontrivial facet of $F(A)$. Then $\alpha_e > 0$ for at least one $e \in \bigcup_{j \in J} \delta^-(j)$.

Proof. Suppose that $\alpha_e=0$ for all $e \in \bigcup_{j \in J} \delta^-(j)$. Let $E^+ = \{e \in E | \alpha_e > 0\}$. Then, for some arc $e' = (v, v') \in E^+$, $v' \in I$ by the assumption. Moreover, some feasible vector of $F(A)$ satisfies $\alpha x=1$ and $x_{e'}=1$. Otherwise, all the feasible vectors satisfying $\alpha x=1$ also satisfy $x_{e'}=0$, which

contradicts the fact that $\alpha x=1$ defines a nontrivial facet. And there exists a directed path from r to v' in G which consists of the arcs in $E \setminus E^+$. If such a path doesn't exist, there must exist $U_i \subseteq I \cup J$ such that $v' \in U_i$ and $\delta^-(U_i) \subseteq E^+$. Moreover, $\delta^-(U_i) \subseteq \delta^-(U_i \cap I)$ because $\alpha_e=0$ for all $e \in \bigcup_{j \in J} \delta^-(j)$. By Theorem 3, this contradicts the fact that $\alpha x=1$ defines a nontrivial facet. Let E_p be the set of arcs in a directed path from r to v' defined before. Consider the new feasible vector of $F(A)$ constructed as follows:

$$x'_e = \begin{cases} 0, & \text{if } e=e', \\ 1, & \text{if } e \in E_p, \\ x_e, & \text{otherwise,} \end{cases}$$

Then $\alpha x' < 1$, and this contradicts the fact that $\alpha x \geq 1$ defines a facet. \square

If $J=\emptyset$, (P) formulates the problem of finding a minimum cost arborescence rooted at r . For this problem, by slightly modifying Theorems 3 and 4, we can show that all the trivial facet defining inequalities completely describe the convex hull of all the feasible solutions of (P). In [9], Fulkerson has shown that $F^+(A) = \{x \in R^{|E|} \mid Ax \geq 1, x_e \geq 0, \text{ for all } e \in E\}$ for the minimum cost arborescence problem. Therefore, the modifications of Theorems 3 and 4 also provide a new proof of his result.

4. Valid Inequalities with 0-1 Coefficients

In this section, we derive a class of valid inequalities with 0-1 coefficients of $F(A)$. For a given graph $G=(V,E)$ and $E' \subseteq E$, let $G'=(V,E')$. From G' , $I^S \subseteq I$, $J^S \subseteq J$, and the corresponding 0-1 matrix S are defined through the following procedure, which we call Procedure S.

Procedure S

- (i) For each $i \in I$, if i has no directed path from r in G' , set $i \in I^S$
- (ii) For each $j \in J$, if there exists at least one directed path from j to some $i \in I^S$ in G' , set $j \in J^S$
- (iii) Construct an $|I^S| \times |J^S|$ 0-1 matrix S such that for each $i \in I^S$ and $j \in J^S$, $s_{ij}=1$, if at least one directed path from j to i exists in G' and 0, otherwise.

Example. Figure 1 shows a given graph G . $I=\{1,2,3,4\}$ and $J=\{5,6,7,8\}$. The arcs in E' are dotted. $I^S=\{1,2,3\}$ and $J^S=\{5,6,7\}$. The corresponding 0-1 matrix S is as follows:

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

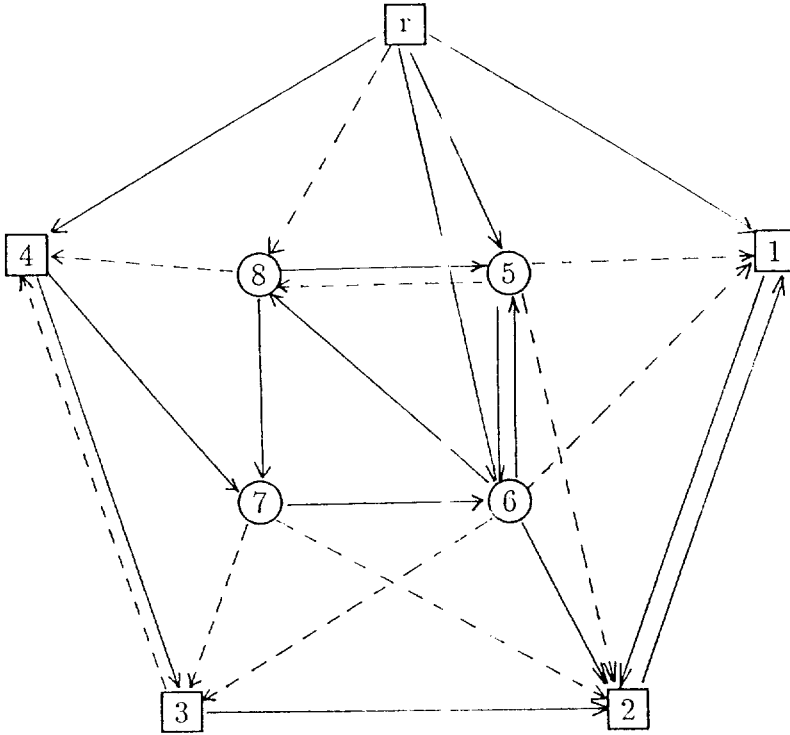
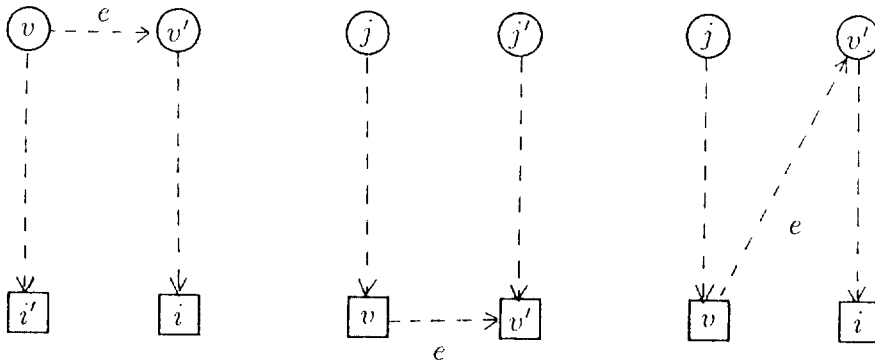


Figure 1

Remark 2 Let $I_j^S = \{i \in I^S \mid s_{ij} = 1\}$ for $j \in J^S$, $J_i^S = \{j \in J^S \mid s_{ij} = 1\}$ for $i \in I^S$, and $V^S = I^S \cup J^S$. From the definition of Procedure S, the following statements are valid.

- (i) S has no zero column.
- (ii) $\delta^-(V^S) \cap E' = \emptyset$. So, there is no directed path from r to any $v \in V^S$ in G' .
- (iii) For some $(v, v') \in E(J^S)$, if $(v, v') \in E'$, then $I_v^S \subseteq I_{v'}^S$.
- (iv) For some $(v, v') \in E(I^S)$, if $(v, v') \in E'$, then $J_v^S \subseteq J_{v'}^S$.
- (v) For some (v, v') with $v \in I^S$ and $v' \in J^S$, if $(v, v') \in E'$, then $I_v^S \subseteq I_j^S$ for all $j \in J_{v'}^S$ and $J_v^S \subseteq J_i^S$ for all $i \in I_{v'}^S$.

The last three cases are illustrated in Figure 2 where only the arcs in E' are shown and dotted.



Case (iii)

Case (iv)

Case (v)

Figure 2

For a given graph G and $E' \subseteq E$, consider the following inequality with 0-1 coefficients.

$$\sum_{e \in E'} x_e \geq k \tag{1}$$

Then the following theorem provides a necessary condition for (1) to define a nontrivial facet of $F(A)$.

Theorem 5 For a given graph $G=(V,E)$ and $E' \subseteq E$, if (1) defines a nontrivial facet of $F(A)$, then E' produces a 0-1 matrix S with no zero row by Procedure S.

Proof. Obviously, I^S is not empty. If I^S is empty, k should be zero for (1) to be valid with respect to $F(A)$. This contradicts the fact that (1) defines a nontrivial facet. Suppose that some $i \in I^S$ has no directed path from any $j \in J$ in $G'=(V,E')$. Since E' for any $e \in \delta^-(I^S) \cap \delta^+(\{i\} \cup I^S)$, there exists $I^0 \subseteq I^S$ such that $\delta^-(I^0) \cap E' = \emptyset$ and $i \in I^0$. By Theorem 3, this contradicts the fact that (1) defines a nontrivial facet of $F(A)$.

Therefore, from now on, when we consider a subset E' of E , we always assume that E' produces a 0-1 matrix S with no zero row and no zero column by Procedure S. Now we introduce a class of valid inequalities with 0-1 coefficients of $F(A)$.

Theorem 6 For a given graph $G=(V,E)$ and $E' \subseteq E$, let S be a 0-1 matrix generated from E' by Procedure S. Then the inequality

$$\sum_{e \in E'} x_e \geq \beta(S) \tag{2}$$

is valid with respect to $F(A)$.

Proof. It is sufficient to show that any steiner arborescence of G contains at least $\beta(S)$ arcs of $E \setminus E'$. By (ii) of Remark 2, there is no directed path from r to each $i \in I^S$ in $G' = (V, E')$. Let $e = (v, v') \in E \setminus E'$, then e corresponds to one of the following cases: (a) $v' \notin V^S$; (b) $v \in J^S$ and $v' \notin I^S$; (c) $v, v' \in J^S$; (d) $v, v' \in I^S$; (e) $v \in I^S$ and $v' \in J^S$; and (f) $v \notin V^S$ and $v \in V^S$. Even though all the arcs of the form (a) are added to G' , the resulting graph has no directed path from r to each $\hat{v} \in V^S$. If any arc of the form (b),(c),(d), or (e) is added to G' , the resulting graph still has no directed path from r to each $i \in I^S$. Moreover, the 0-1 matrix generated from this graph by Procedure S, say S_e , has the same row and column sets as those of S and satisfies $\beta(S_e) \geq \beta(S) - 1$. Note that any $i \in I^S$ has a directed path from at least one $j \in J^S$ in G' . If any arc of the form (f) is added to G' , it creates the directed paths from r to each $i \in I' \subseteq I^S$ such that $I' \subseteq I_j^S$ for some $j \in J^S$. Therefore, in order to construct a directed path from r to each $i \in I^S$, at least $\beta(S)$ arcs of $E \setminus E'$ should be added to G' . \square

Several different subsets E' of E can produce the same matrix S , so the strength of the cut (2) is dependant on the choice of E' . For a given $|I^S| \times |J^S|$ 0-1 matrix S with no zero row and no zero column, E^S is defined as the subset of E which consists of the following arcs:

- (i) $e \in \delta^-(v)$ for all $v \in V \setminus V^S$;
- (ii) $e = (v, v')$ for any $v \in J^S$ and $v' \in I^S$ such that $s_{vv'} = 1$;
- (iii) $e = (v, v') \in E(J^S)$ such that $I_v^S \subseteq I_{v'}^S$;
- (iv) $e = (v, v') \in E(I^S)$ such that $J_v^S \subseteq J_{v'}^S$; and
- (v) $e = (v, v')$ for any $v \in I^S$ and $v' \in J^S$ such that $I_v^S \subseteq I_{j'}^S$ for all $j \in J_v^S$.

Then the following theorem holds.

Theorem 7 For a given graph $G = (V, E)$ and an $|I^S| \times |J^S|$ 0-1 matrix S with no zero row and no zero column, the following inequality

$$\sum_{e \in E \setminus E^S} x_e \geq \beta(S) \tag{3}$$

is valid with respect to $F(A)$. Furthermore, (3) dominates any inequality of the form (2) such that E' produces S by Procedure S.

Proof. By the definition of E^S , if some directed path from $j \in J^S$ to $i \in I^S$ of G only consists of the arcs in E^S , $s_{ji} = 1$. And $\delta^-(V_S) \cap E^S = \emptyset$. Therefore, if we let S' be the 0-1 matrix produced from E^S by Procedure S, $\beta(S') \geq \beta(S)$. So, (3) is valid for $F(A)$ by Theorem 6. Moreover, by Remark 2 and the definition of E^S , $E' \subseteq E^S$ for any $E' \subseteq E$ such that E' produces S by Procedure S. \square

The following theorem shows when the right hand side of (2) is tight.

Theorem 8 For a given graph $G = (V, E)$ and $E' \subseteq E$, let S be the matrix generated from E' by Procedure S. Suppose that there exists a minimum cardinality cover $\hat{J} \subseteq J^S$ of S such

that for each $j \in \hat{J}$, $G'_e = (V, E' \cup \{e\})$ for some $e \in E \setminus E'$ contains a directed path from r to j . Then (I) is a valid inequality of $F(A)$ if and only if $k \leq \beta(S)$.

Proof. (\Leftarrow) is the direct consequence of Theorem 6.

(\Rightarrow) Suppose $k > \beta(S)$. By the definition of Procedure S, $G' = (V, E')$ contains not only at least one directed path from r to each $i \in I \setminus I^S$ but also at least one directed path from j to i for $j \in J_i^S$ and $i \in I^S$. Moreover, from the assumption of the theorem, G' , if $|\hat{J}| (= \beta(S))$ arcs in $E \setminus E'$ are added to it, contains a steiner arborescence of G . Hence, we have a contradiction. \square

5. Nontrivial Facets with 0-1 Coefficients

Now we investigate when those inequalities derived in the previous section are facet inducing for $F(A)$. In order to do that we need the following terminology. We say that a 0-1 matrix S with no zero row and no zero column is a β -maximal adjacency matrix if

- (a) the bipartite incidence graph of S is connected;
- (b) there exists at least one zero element in each row of S ; and
- (c) changing a zero element of S to one decreases its covering number, $\beta(S)$.

The central concept of β -maximal adjacency matrices is used not only for describing the simple plant location polytope [3,4] but also for developing a necessary and sufficient condition of the facet for the set covering polytope [8]. Some specific β -maximal adjacency matrices are shown in [3,4,7]. Moreover, Cho et al. [4] show that β -maximal adjacency matrices satisfy the following properties.

- (i) $\beta(S) \geq 2$.
- (ii) For each row and column, there exist at least $\beta(S) - 1$ zero elements. Furthermore, if $s_{rr} = 0$, some $\beta(S) - 1$ elements of $J^S \setminus J_i^S$ including j^* cover $I^S \setminus \{i^*\}$.
- (iii) The support of any column cannot be a proper subset of the support of another column.
- (iv) The support of any row cannot be a subset of the support of another row.
- (v) $|J_i^S| \geq 2$ for all $i \in I^S$ and $|I_j^S| \geq 2$ for all $j \in J^S$.
- (vi) Let S' be a submatrix of S such that all the identical columns are reduced to one. Then S' is also a β -maximal adjacency matrix.

Remark 3 By the definition of Procedure S and the properties of β -maximal adjacency matrices, if any $E' \subseteq E$ produces a β -maximal adjacency matrix S by Procedure S, the following statements are valid.

- (i) For some (v,v') with $v,v' \in J^S$, if $(v,v') \in E'$, then $I_v^S = I_{v'}^S$. This is from Remark 2(iii) and the property (iii) of β -maximal adjacency matrices.
- (ii) For any (v,v') with $v,v' \in I^S$, $(v,v') \notin E'$ by Remark 2(iv) and the property (iv) of β -maximal adjacency matrices.
- (iii) For any (v,v') with $v \in I^S$ and $v' \in J^S$, $(v,v') \notin E'$ by Remark 2(v) and the properties (iv) and (v) of β -maximal adjacency matrices.

For any $W \subseteq V$, we define a *directed W -path* as a directed path which has no intermediate node in W . Then the following theorem shows when (2) defines a facet of $F(A)$.

Theorem 9 For a given graph $G=(V,E)$ and $E' \subseteq E$, suppose that E' produces S by Procedure S . Then (2) defines a nontrivial facet of $F(A)$ if

- (i) $E' = E^S$;
- (ii) S is a β -maximal adjacency matrix;
- (iii) G contains at least one directed V^S -path from r to each $j \in J$; and
- (iv) for each $e \in E(V \setminus V^S)$, there exists at least one minimum cardinality cover $J_e \subseteq V^S$ of S such that $G_e = (V, E \setminus \{e\})$ contains not only a directed V^S -path from r to each $j \in J_e$ but also a directed $(J^S \setminus J_e)$ -path from r to each $i \in I \setminus I^S$.

To prove the theorem, we need the following results for the set covering polytope[8,12] Let $B=(N,M,E_A)$ be the undirected bipartite incidence graph of a 0-1 matrix A and let $Q(B)$ be the convex hull of the incidence vectors of all the covers of M . Then $Q(B)$ coincides with $F(A)$. For notational convenience, we also use $Q(N,M)$ instead of $Q(B)$. The critical graph of B is denoted by $B^*=(N,E^*)$ where E^* is the set of critical edges defined as

$$E^* = \{ \{n_i, n_j\} \mid \beta(M \setminus M_{ij}) < \beta(M) \}$$

where $M_{ij} \subseteq M$ is the set of the common neighbors of n_i and n_j .

Theorem 10 (Sassano[12]) If B^* is connected, then the inequality

$$\sum_{j \in N} x_j \geq \beta(M)$$

defines a facet of $Q(N,M)$.

Theorem 11 (Sassano[12]) Let $B'=(N',M,E'_A)$ be the subgraph of $B=(N,M,E_A)$ induced by $N' \cup M$ where $N' \subseteq N$, and suppose that the inequality

$$\sum_{j \in N} \alpha_j x_j \geq \alpha_0$$

defines a facet of $Q(N',M)$. Then for each $k \in N \setminus N'$,

$$\sum_{j \in N} \alpha_j x_j + (\alpha_0 - z_k) x_k \geq \alpha_0$$

defines a facet of $Q(N' \cup \{k\}, M)$ where

$$z_k = \min \left\{ \sum_{j \in N} x_j x_k \mid x \in Q(N' \cup \{k\}, M), x_k = 1 \right\}.$$

(Proof of Theorem 9)

Let $B=(E, M, E_A)$ be the bipartite incidence graph of a 0-1 matrix A for (P) . Note that A is the incidence matrix of directed steiner cuts versus arcs. Therefore, when we consider the bipartite incidence graph B , we use the same notation $e \in E$ in G to represent a node corresponding to a column of A . Let $E_1 = \delta^-(V^S)$, then $E_1 \cap E^S = \emptyset$. Also let $E_2 = E(V^S) \setminus E^S$, then $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E \setminus E^S$. In other words, E_1 and E_2 are the partition of $E \setminus E^S$.

Consider the following subset of inequalities in (P) :

$$\sum_{e \in \delta^-(U_i)} x_e \geq 1, \quad \text{for all } i \in I^S$$

where $U_i = J_i^S \cup \{i\}$. Let $M^S \subseteq M$ be a subset of nodes for the rows of A corresponding to the incidence vectors of $\delta^-(U_i)$ for $i \in I^S$. Furthermore, without loss of generality, we assume that $i \in I^S$ also indexes a node in M^S corresponding to the row of A which is the incidence vector of $\delta^-(U_i)$.

Consider the following inequality

$$\sum_{e \in \hat{E}_1} x_e \geq \beta(S) \tag{4}$$

The outline of the proof is as follows. We first show that (4) defines a facet of $Q(E_1, M^S)$, one of $Q(E_1 \cup E^S, M^S)$, and one of $Q(E_1 \cup E^S, M)$. Then we show that (2) is a lifting of (4) for $Q(E, M)$.

Claim 1 (4) defines a nontrivial facet of $Q(E_1, M^S)$.

Proof. Let B' denote the subgraph of B induced by E_1 and M^S . From (iii) of the theorem, for each $j \in J^S$ there exists at least one $e = (vj) \in E_1$ such that $v \notin J^S$. We let \hat{E} be a subset of E_1 which contains exactly one such arc for each $j \in J^S$. Consider the submatrix A' of A whose bipartite incidence graph is B' . In A' , the support of any column is a subset of the support of some column corresponding $e \in \hat{E}$. Moreover, the collection of the columns in A' corresponding $e \in \hat{E}$ is exactly the same as S , so $\beta(A') = \beta(S)$. By Theorem 10 it is sufficient to show that the critical graph of B' is connected.

By the definition of E_1 , every node of E_1 in B' is adjacent to at least one node $i \in M^S$. If any two nodes in E_1 are adjacent to the same $i \in M^S$, a critical edge exists between them in the critical graph of B' . This can be known by the following observation. Note that there exist $|J^S|$ nodes in \hat{E} , each of which is adjacent to all $i \in I_j^S \subseteq M^S$ for some $j \in J^S$. Moreover, by the property (ii) of β -maximal adjacency matrices, $\beta(M^S \setminus \{i\}) = \beta(M^S) - 1$ for each $i \in I^S$. There-

fore, the critical graph of B' contains $|I^S|$ subgraphs each of which is completely connected. The remaining thing to prove is to show the connectedness among those subgraphs. Note that the subgraph of B' induced by \hat{E} and M^S is exactly the bipartite incidence graph of S . Therefore, the connectedness of the bipartite incidence graph of S also implies that the critical graph is connected. Moreover, since $\beta(S) \geq 2$, (4) defines a nontrivial facet.

Claim 2 (4) defines a nontrivial facet of $Q(E_1 \cup E^S, M^S)$.

Proof. By (ii) and (iii) of Remark 3, any $(v, v') \in E^S$ corresponds to one of the following cases: (a) $v' \notin V^S$; (b) $v \in J^S$ and $v' \in J^S$; and (c) $v, v' \in J^S$. In any case, $(v, v') \notin \delta^-(U_i)$, for all $i \in I^S$. In Cases (a) and (b), it is obvious. In Case (c), it is also true by (i) of Remark 3. Therefore, any $e \in E^S$ can't be adjacent to any $i \in M^S$. So the lifting coefficient of any x_e for $e \in E^S$ is equal to zero and the claim holds by Theorem 11.

Claim 3 (4) defines a nontrivial facet of $Q(E_1 \cup E^S, M)$.

Proof. Let $q = |E_1 \cup E^S|$, then $q = |E_1| + |E^S| = n - |E_2|$. Since (4) is a facet of $Q(E_1 \cup E^S, M^S)$, there exist q covers of M^S whose incidence vectors are linearly independent and satisfy (4) as an equality. Let D_1, \dots, D_q be such covers. Then we can show that $D_t \cup E^S, t = 1, \dots, q$ can cover M . By (iii) of the theorem and the definitions of E^S and Procedure S , a subgraph $G^S = (V, E^S)$ of G contains not only at least one directed path from r to each $v \in (I \cup J) \setminus V^S$ but also at least one directed path from j to i for all $j \in J_i^S$ and $i \in I^S$. Since any $e \in E^S$ is not adjacent to any $i \in M^S$, D_t contains at least one arc (v, v') such that $v \notin V^S$ and $v' \in U_i$ for each $i \in I^S$. Therefore, for each t , some steiner arborescence of G can be constructed by only using the arcs in $D_t \cup E^S$. Moreover, from q such covers, we can obtain $|E_1|$ covers of M whose incidence vectors are linearly independent and satisfy (4) with equality.

Next, we construct the additional $|E^S|$ covers of M which are linearly independent. Let $G_e^S = (V, E^S \setminus \{e\})$ for each $e \in E^S$. We will show that G_e^S if $\beta(S)$ arcs in E_1 are added to it, contains a steiner arborescence of G . This can be explained by the following observation. Let $e = (v, v') \in E^S$. First, suppose that $e \notin E(V \setminus V^S)$. By Remarks 2 and 3, e corresponds to one of the following cases: (a) $v \in V^S$ and $v' \notin V^S$; (b) $v, v' \in J^S$; and (c) $v \in J^S$ and $v' \in I^S$. In any case, G_e^S has still contains all the directed paths from r to each $i \in I \setminus I^S$ and at least one directed path from r to each $j \in J \setminus J^S$ of G^S by (ii) of Remark 3 and (iii) of the theorem, respectively. Moreover, there exists a subset J' of J^S with $|J'| = \beta(S)$ such that G_e^S has a directed path to each $i \in I^S$ from some $j \in J'$. In Case (a), it is obvious. In Case (b), such J' exists by (i) of Remark 3. In Case (c), by the properties (v) and (vi) of β -maximal adjacency matrices, $s_{ij} = 1$ for some $j \in J^S$ such that $I_j^S \neq I_v^S$. Therefore, by the property (ii) of β -maximal matrices, such J' exists. So, G_e^S , if $\beta(S)$ arcs in E_1 , each of which is an incoming arc to each $j \in J'$, are added to it, contains a steiner arborescence of G .

Now suppose that $e \in E(V \setminus V^S)$. In this case, all the directed paths from each $j \in J^S$ to all $i \in I^S$ of G^S also exist in G_e^S . Let J_e be a cover of I^S as defined in (iv) of the theorem and E_e be a subset of E_1 which is a collection of exactly one incoming arc to each $j \in J_e$. Then G_e^S , if $|J_e| (= \beta(S))$ additional arcs of E_e are added to it, has a directed path from r to each $i \in I^S$ by (iv) of the theorem. Let $\hat{G}_e^S = (V, (E^S \setminus \{e\}) \cup E_e)$. Then we can easily show that \hat{G}_e^S also has a directed path from r to each $i \in I \setminus I^S$. By (iv) of the theorem, for each $i \in I \setminus I^S$, at least one directed $(J^S \setminus J_e)$ -path from r to i , say P_e , exists in G_e . If P_e is a directed V^S -path, \hat{G}_e^S contains P_e . Otherwise, P_e has at least one intermediate node in $J_e \cup I^S$. Let \hat{v} be the node which P_e passes last among $J_e \cup I^S$. Since \hat{G}_e^S has a directed path from r to \hat{v} , \hat{G}_e^S also has a directed path from r to i . Likewise, for each $e \in E^S$, we can construct a cover $(E^S \setminus \{e\}) \cup E_e$ of M , whose incidence vector satisfies (4) with equality. Furthermore, those $|E^S|$ covers along with the previously selected $|E_1|$ covers are linearly independent.

Claim 4 (2) is a nontrivial facet of $Q(E, M)$.

Proof. It is sufficient to show that the lifting coefficient of x_e for all $e \in E_2$ is equal to one. Remind that $G^S = (V, E^S)$ contains at least one directed path not only from r to $v \in (I \cup J) \setminus V^S$ but also at least one directed path from j to i for all $j \in J^S$ and $i \in I^S$. Moreover, for each $j \in J^S$, G^S , if some single arc in E_1 is added to it, contains a directed path from r to j . Let $e = (v, v') \in E_2$.

Case 1. $v, v' \in J^S$. By the definition (iii) of I^S , $I_v^S \not\subseteq I_{v'}^S$. Therefore, there exists some $i \in I^S$ such that $s_{iv} = 1$ and $s_{iv'} = 0$. And by the property (ii) of β -maximal adjacency matrices, some $\beta(S) - 1$ elements including v cover $I^S \setminus \{i\}$. Therefore, the lifting coefficient of x_e is 1.

Case 2. $v, v' \in I^S$. By the property (ii) of β -maximal adjacency matrices, some $j \in J^S$ with $s_{vj} = 0$ along with the additional $\beta(S) - 2$ elements of J^S can cover $I^S \setminus \{v'\}$ including v . Therefore, the lifting coefficient of x_e is 1.

Case 3. $v \in J^S, v' \in I^S$. By the definition (ii) of E^S , $s_{vv'} = 0$. Therefore, some $\beta(S) - 1$ elements of J^S including v cover $I^S \setminus \{v'\}$. So, the lifting coefficient of x_e is 1.

Case 4. $v \in J^S, v' \in J^S$. Since $|I_v^S| \geq 2$, there exists at least one $i \in I_v^S$ with $i \neq v$. Moreover, $\beta(S) - 1$ elements of J^S other than $\{v'\}$ cover $I^S \setminus \{i\}$ including v . Therefore, the lifting coefficient of x_e is 1.

Although we didn't explicitly show that the lifting coefficients are no more than one, it can be easily known by the definition of the covering number of S .

This completes the proof of the theorem. □

Now we show some necessary conditions for nontrivial facets with 0-1 coefficients.

Theorem 12 For a given graph $G = (V, E)$ and $E' \subseteq E$, suppose that (2) defines a nontrivial facet of $F(A)$ and that E' produces S by Procedure S. Then

- (i) the bipartite incidence graph of S is connected; and
- (ii) there exists no arc $e=(ij) \in E \setminus E'$ such that $\beta(S_e) = \beta(S)$ where S_e is a 0-1 matrix generated from $E' \cup \{e\}$ by Procedure S .

Proof. Note that S has no zero row and no zero column. Suppose that the bipartite incidence graph of S is not connected. Let $S_t, t \in T$ be a submatrix of S which corresponds to a connected component of the bipartite incidence graph of S . Also let $I_t \subseteq I^S$ and $J_t \subseteq J^S$ be the index sets of rows and columns of S_t , respectively. By the fact that $\delta^-(I^S \cup J^S) \cap E' = \emptyset$ and the disconnectedness assumption, $\delta^-(I_t \cup J_t) \cap E' = \emptyset$ for all $t \in T$. Let $E_t = \bigcup_{v \in I_t \cup J_t} \delta^-(v) \setminus E'$, then $\sum_{e \in E_t} x_e \geq \beta(S_t)$ is valid with respect to $F(A)$. Moreover, $\bigcup_{t \in T} E_t \subseteq E \setminus E'$, $\sum_{t \in T} \beta(S_t) = \beta(S)$, and $E_t \cap E_l = \emptyset$ for any pair of $t, l \in T$. This contradicts the assumption that (2) defines a facet. Moreover, if (ii) is violated, we can include such an arc to E' without increasing the right hand side of (2). This also contradicts the fact that (2) defines a facet. \square

The following corollary is the direct consequence of Theorem 8 and Theorem 12.

Corollary 2 For a given graph $G=(V,E)$ and $E' \subseteq E$, let S be the matrix generated from E' by Procedure S . Suppose that there exists a minimum cardinality cover $\hat{J} \subseteq J^S$ of S such that for each $j \in \hat{J}$, $G'_e=(V, E' \cup \{e\})$ for some $e \in E \setminus E'$ contains a directed path from r to j . And suppose that (1) defines a nontrivial facet of $F(A)$. Then $k = \beta(S)$, $E' = E_s$, and the bipartite incidence graph of S is connected.

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