Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 1159–1174

LAPLACIAN SPECTRA OF GRAPH BUNDLES

JU YOUNG KIM

ABSTRACT. The spectrum of the Laplacian matrix of a graph gives an information of the structure of the graph. For example, the product of non-zero eigenvalues of the characteristic polynomial of the Laplacian matrix of a graph with n vertices is n times of the number of spanning trees of that graph. The characteristic polynomial of the Laplacian matrix of a graph tells us the number of spanning trees and the connectivity of given graph. In this paper, we compute the characteristic polynomial of the Laplacian matrix of a graph bundle when its voltages lie in an abelian subgroup of the full automorphism group of the fibre; in particular, the automorphism group of the fibre is abelian. Also we study a relation between the characteristic polynomial of the Laplacian matrix of a graph G and that of the Laplacian matrix of a graph bundle over G. Some applications are also discussed.

1. Introduction

Let G be a finite simple connected graph with vertex set $V(G) = \{u_1, u_2, \cdots, u_n\}$ and edge set $E(G)$. We denote the set of vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the *neighborhood* of a vertex v. Denote the degree of a vertex u by $d(u)$. Let

$$D(G) = \text{Diag}[d(u_1), d(u_2), \cdots, d(u_n)]$$

be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $C(G) = D(G) - A(G)$, where $A(G)$ is the $(0,1)$-adjacency matrix of G. The *characteristic polynomial* of a graph G is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$, denoted by $\Phi(G;\lambda)$. A zero of $\Phi(G;\lambda)$

Received April 27, 1996. Revised August 15, 1996.
1991 AMS Subject Classification: 05C05.
Key words and phrases: Laplacian, graph bundles, spanning trees .
This research was supported by a grant from Catholic University of Taegu Hyosung in 1994.

is called an *eigenvalue* of G. We denote the characteristic polynomial $\det(\lambda I - C(G))$ of the Laplacian matrix of G by $\Psi(G;\lambda)$. By $|X|$, we denote the cardinality of a finite set X. Convert G to a digraph $\overrightarrow{G}$ by replacing each edge e of G with a pair of oppositely directed edges, say e^+ and e^-. We denote the set of directed edges of $\overrightarrow{G}$ by $E(\overrightarrow{G})$. Note that the adjacency matrix of the graph G is the same as that of the digraph $\overrightarrow{G}$. Now, we introduce the notion of a graph bundle. By e^{-1} we mean the reverse edge to an edge $e \in E(\overrightarrow{G})$. Denote the directed edge e of G by uv if the initial and the terminal vertices of e are u and v, respectively. For a finite group Γ, a Γ-*voltage assignment* of G is a function $\phi : E(\overrightarrow{G}) \longrightarrow \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\overrightarrow{G})$. We denote the set of all Γ-voltage assignments of G by $C^1(G;\Gamma)$. Let F be another finite graph and let $\phi \in C^1(G;\mathrm{Aut}(F))$, where $\mathrm{Aut}(F)$ denotes the group of all graph automorphisms of F. Now, we construct a new graph $G \times^{\phi} F$ as follows: $V(G \times^{\phi} F) = V(G) \times V(F)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times^{\phi} F$ if either $u_1u_2 \in E(\overrightarrow{G})$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. We call $G \times^{\phi} F$ the *F-bundle over G associated with* ϕ and the first coordinate projection $p^{\phi} : G \times^{\phi} F \longrightarrow G$ the *bundle projection* . We also call G and F the *base* and the *fibre* of the bundle $G \times^{\phi} F$, respectively. Moreover, if $F = \overline{K}_n$ the complement of the complete graph K_n on n vertices, then an F-bundle over G is just an n-fold covering graph of G. If $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in E(\overrightarrow{G})$, then $G \times^{\phi} F$ is just the cartesian product of G and F.

2. Laplacian matrices of graph bundles

Let F be a finite graph and let ϕ be an $\mathrm{Aut}(F)$-voltage assignment of G. For each $\gamma \in \mathrm{Aut}(F)$, let $\overrightarrow{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\overrightarrow{G}$ whose directed edge set is $\phi^{-1}(\gamma)$, so that the digraph $\overrightarrow{G}$ is the edge-disjoint union of spanning subgraphs $\overrightarrow{G}_{(\phi,\gamma)}$, $\gamma \in \mathrm{Aut}(F)$. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(F) = \{v_1, v_2, \cdots, v_m\}$. We define an order relation $\leq$ on $V(G \times^{\phi} F)$ as follows: for any two vertices (u_i, v_k) and (u_j, v_ℓ) of $G \times^{\phi} F$, $(u_i, v_k) \leq (u_j, v_\ell)$ if and only if either $k < \ell$ or $k = \ell$ and $i \leq j$. Let $P(\gamma)$ denote the $m \times m$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$. Here, the tensor product $A \otimes B$ of matrices A and B is considered as the matrix B having the element b_{ij} replaced by the matrix Ab_{ij}.

It is known [6] that the adjacency matrix of a graph bundle $G \times^{\phi} F$ is

$$A(G \times^{\phi} F) = \left(\sum_{\gamma \in \mathrm{Aut}(F)} A(\overrightarrow{G}_{(\phi,\gamma)}) \right) + I_{|V(G)|} \otimes A(F),$$

where $P(\gamma)$ is the permutation matrix associated with γ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$, and $I_{|V(G)|}$ is the identity matrix of order $|V(G)|$.

To find the diagonal matrix $D(G \times^{\phi} F)$ of vertex degrees, we recall that two vertices (u_i, v_k) and (u_j, v_ℓ) are adjacent in $G \times^{\phi} F$ if either $u_i u_j \in E(\overrightarrow{G})$ and $v_\ell = \phi(u_i u_j) v_k$ or $u_i = u_j$ and $v_k v_\ell \in E(F)$. Hence the degree of (u_i, v_k) is the sum of the degree of u_i in $V(G)$ and the degree of v_k in $V(F)$. It implies that the degree of (u_i, v_k) is $(n(k-1)+i, n(k-1)+i)$-entry of

$$D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F).$$

That is,

$$D(G \times^{\phi} F) = D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F).$$

Now, the Laplacian matrix $C(G \times^{\phi} F)$ of the bundle $G \times^{\phi} F$ is given as follows.

$$\begin{aligned}
&D(G \times^{\phi} F) - A(G \times^{\phi} F) \\
&= \left[D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F) \right] \\
&\quad - \left[\left(\sum_{\gamma \in \mathrm{Aut}(F)} A(\overrightarrow{G}_{(\phi,\gamma)}) \otimes P(\gamma) \right) + I_{|V(G)|} \otimes A(F) \right] \\
&= D(G) \otimes I_{|V(F)|} \\
&\quad - \left[\left(\sum_{\gamma \in \mathrm{Aut}(F)} A(\overrightarrow{G}_{(\phi, \gamma)}) \otimes P(\gamma) \right) + I_{|V(G)|} \otimes (A(F) - D(F)) \right]
\end{aligned}$$

$$=D(G)\otimes I_{|V(F)|}$$
$$-\left[\left(\sum_{\gamma\in \mathrm{Aut}(F)} A(\overrightarrow{G}_{(\phi,\gamma)})\otimes P(\gamma)\right)+I_{|V(G)|}\otimes(-C(F))\right].$$

We summarize our discussions in the following theorem.

THEOREM 1. *The Laplacian matrix $C(G\times^{\phi}F)$ of the graph bundle $G\times^{\phi}F$ is*

$$D(G\times^{\phi}F)-A(G\times^{\phi}F)=D(G)\otimes I_{|V(F)|}-\sum_{\gamma\in \mathrm{Aut}(F)} A(\vec{G}_{(\phi,\gamma)})\otimes P(\gamma)$$
$$+I_{|V(G)|}\otimes C(F).$$

If the fibre F of the graph bundle $G\times^{\phi}F$ is $\overline{K_n}$, then $G\times^{\phi}F$ is an n-fold covering graph of G and the adjacency matrix $A(F)$ is the zero matrix. Hence, we get the following corollary.

COROLLARY 1. *The Laplacian matrix $C(G\times^{\phi}\overline{K_n})$ of an n-fold covering $G\times^{\phi}\overline{K_n}$ of G is*

$$D(G)\otimes I_n-\sum_{\gamma\in S_n} A(\vec{G}_{(\phi,\gamma)})\otimes P(\gamma).$$

Since the cartesian product $G\times F$ of two graphs G and F is just the F-bundle over G associated with the trivial voltage assignment ϕ, *i.e.*, $\phi(e)$ = the identity for all $e\in E(\vec{G})$, and $A(G)=A(\vec{G})$, we get the following corollary.

COROLLARY 2. *The Laplacian matrix $C(G\times F)$ of the cartesian product $G\times F$ of two graphs G and F is*

$$C(G)\otimes I_{|V(F)|}+I_{|V(G)|}\otimes C(F).$$

From now on, we consider a voltage assignment ϕ of G whose image lies in an abelian subgroup Γ of $\mathrm{Aut}(F)$. Since the permutation matrices $P(\gamma),\gamma\in\Gamma$ and the Laplacian matrix $C(F)$ of the fibre are all diagonalizable and commute with each other, they are simultaneously diagonalizable. In other words, there exists an invertible matrix M_Γ such that $M_\Gamma P(\gamma) M_\Gamma^{-1}$ and $M_\Gamma C(F) M_\Gamma^{-1}$ are diagonal matrices for all $\gamma \in \Gamma$.

For convenience, we write

$$M_\Gamma P(\gamma) M_\Gamma^{-1} = \text{Diag}\left[\lambda_{(\gamma,1)}, \cdots, \lambda_{(\gamma,|V(F)|)}\right]$$

for $\gamma \in \Gamma$, and

$$M_\Gamma C(F) M_\Gamma^{-1} = \text{Diag}\left[\lambda_{(F,1)}, \cdots, \lambda_{(F,|V(F)|)}\right].$$

That is, $\lambda_{(\gamma,1)}, \cdots, \lambda_{(\gamma,|V(F)|)}$ are the eigenvalues of the permutation matrix $P(\gamma)$, and $\lambda_{(F,1)}, \cdots, \lambda_{(F,|V(F)|)}$ are the eigenvalues of the Laplacian matrix $C(F)$. Then, Theorem 1 gives

$$\begin{aligned}&(I_{|V(G)|} \otimes M_\Gamma) C(G \times^\phi F)(I_{|V(G)|} \otimes M_\Gamma)^{-1} \\ &= \bigoplus_{i=1}^{|V(F)|} \left\{ D(G) - \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_{|V(G)|} \right) \right\}\end{aligned}$$

Now, we have

THEOREM 2. *Let Γ be an abelian subgroup of $\text{Aut}(F)$. Then, for any Γ-voltage assignment ϕ of G , the Laplacian matrix of the graph bundle $G \times^\phi F$ is similar to*

$$\bigoplus_{i=1}^{|V(F)|} \left\{ D(G) - \left(\sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_{|V(G)|} \right) \right\}.$$

COROLLARY 3. *Let Γ be an abelian subgroup of the symmetric group S_n. Then, for any Γ-voltage assignment ϕ of G, the Laplacian matrix of an n-fold covering $G \times^\phi \overline{K}_n$ of G is similar to*

$$\bigoplus_{i=1}^{n} \left\{ D(G) - \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\vec{G}_{(\phi,\gamma)}) \right\}.$$

COROLLARY 4. *The Laplacian matrix of the cartesian product $G \times F$ of two graphs G and F is similar to*

$$\bigoplus_{i=1}^{m} \left\{ C(G) + \lambda_{(F,i)} I_n \right\}.$$

3. Regular coverings

A covering $p : \tilde{G} \to G$ is said to be *regular* if there is subgroup A of the automorphism group $\mathrm{Aut}(\tilde{G})$ of $\tilde{G}$ acting freely on $\tilde{G}$ such that $\tilde{G}/A$ is isomorphic to G.

The *graph* $G \times_\phi \Gamma$ *derived from* a voltage assignment $\phi : E(\overrightarrow{G}) \to \Gamma$ has as its vertex set $V(G) \times \Gamma$ and as its edge set $E(G) \times \Gamma$, so that an edge of $G \times_\phi \Gamma$ joins a vertex (u, γ) to $(v, \phi(e)\gamma)$ for $e = uv \in E(\overrightarrow{G})$ and $\gamma \in \Gamma$. A vertex (u, γ) is denoted by u_γ, and an edge (e, γ) by e_γ. The voltage group Γ acts on $G \times_\phi \Gamma$ as follows: for every $\gamma \in \Gamma$, let $\Phi_\gamma : G \times_\phi \Gamma \to G \times_\phi \Gamma$ denote the graph automorphism defined by $\Phi_\gamma(v_{\gamma'}) = v_{\gamma'\gamma^{-1}}$ on vertices and $\Phi_\gamma(e_{\gamma'}) = e_{\gamma'\gamma^{-1}}$ on edges. Then the natural map $G \times_\phi \Gamma \to (G \times_\phi \Gamma)/\Gamma \cong G$ is a regular $|\Gamma|$-fold covering projection.

From now on, we assume that the voltage group Γ is a finite abelian group. Then Γ is isomorphic to a product of cyclic groups. Say, $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$. For all $\alpha = 1, \cdots, \ell$, let ρ_α denote a generator of the cyclic group $\mathbb{Z}_{n_\alpha}$ so that $\mathbb{Z}_{n_\alpha} = \{\rho_\alpha^0, \rho_\alpha^1, \cdots, \rho_\alpha^{n_\alpha - 1}\}$.

We define an order relation $\leq$ on $\mathbb{Z}_{n_\alpha}$ by $\rho^\ell \leq \rho^m$ if and only if $\ell \leq m$. This order relation gives the relation as in Section 2 on Γ. For any $\gamma \in \Gamma$, let $P(\gamma)$ be the permutation matrix associated with γ under the above order. We note that the set of vertices of $G \times_\phi \Gamma$ also has the corresponding order relation if an order relation on $V(G)$ is given.

Chae and Lee computed the adjacency matrix of the covering graph $G \times_\phi \Gamma$ as follows [3]:

$$A(G \times_\phi \Gamma) = \sum_{(k_1, \cdots, k_\ell)} A(\overrightarrow{G}_{(\phi, (\rho_1^{k_1}, \cdots, \rho_\ell^{k_\ell}))}) \otimes P(\rho_1^{k_1}, \cdots, \rho_\ell^{k_\ell}),$$

where $P(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell})$ is the permutation matrix associated with $(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell})$. Moreover, the adjacency matrix $A(G\times_\phi\Gamma)$ is similar to

$$\sum_{(k_1,\cdots,k_\ell)} A\left(\overrightarrow{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}\right)\otimes\left(D(\rho_1)^{k_1}\otimes\cdots\otimes D(\rho_\ell)^{k_\ell}\right),$$

where $D(\rho_\alpha)$ is the $n_\alpha\times n_\alpha$ matrix

$$\begin{pmatrix} 1 & & & & \\ & \zeta_\alpha & & 0 & \\ & & \zeta_\alpha^2 & & \\ & 0 & & \ddots & \\ & & & & \zeta_\alpha^{n_\alpha-1}\end{pmatrix}$$

and $\zeta_\alpha=\exp(\frac{2\pi}{n_\alpha}i)$ for $1\le\alpha\le\ell$.

Let $\mathbb{C}$ denote the field of complex numbers, $\Gamma=\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_\ell}$, and let ϕ be a Γ-voltage assignment of G. For each $(s_1,\cdots,s_\ell)\in\Gamma$ with $0\le s_\alpha<n_\alpha$ and $1\le\alpha\le\ell$, we define a weight function $\omega_{(s_1,\cdots,s_\ell)}(\phi):E(\overrightarrow{G})\to\mathbb{C}$ by

$$\omega_{(s_1,\cdots,s_\ell)}(\phi)(e)=\prod_{\alpha=1}^{\ell}(\zeta_\alpha^{k_\alpha})^{s_\alpha}\quad\text{for}\quad\phi(e)=\prod_{\alpha=1}^{\ell}\rho_\alpha^{k_\alpha}.$$

Then, we have

$$\sum_{(k_1,\cdots,k_\ell)} A(\overrightarrow{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))})\otimes(D(\rho_1)^{k_1}\otimes\cdots\otimes D(\rho_\ell)^{k_\ell})$$
$$=\bigoplus_{(s_1,\cdots,s_\ell)} A(\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)}).$$

To find the diagonal matrix $D(G\times_\phi\Gamma)$ of vertex degrees, we recall that an edge of $G\times_\phi\Gamma$ joins a vertex (u,γ) to $(v,\phi(e)\gamma)$, for $e=uv\in E(G)$ and $\gamma\in\Gamma$. It implies that $D(G\times_\phi\Gamma)=D(G)\otimes I_{|\Gamma|}$. Hence, we get the following

THEROEM 3. *Let $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$ and let ϕ be a Γ-voltage assignment of G. Then, the Laplacian matrix $C(G \times_\phi \Gamma)$ of a regular covering graph $G \times_\phi \Gamma$ is*

$$D(G) \otimes I_{|\Gamma|} - \sum_{(k_1,\cdots,k_\ell)} A(\overrightarrow{G}_{(\phi,(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}))}) \otimes P(\rho_1^{k_1},\cdots,\rho_\ell^{k_\ell}).$$

Moreover, it is similar to

$$\bigoplus_{(s_1,\cdots,s_\ell)} \{D(G) - A(\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)})\}.$$

4. Computational formulas

Let $\mathbb{C}$ denote the field of complex numbers, and let D be a digraph. A *vertex-and-edge weighted digraph* (in short, VEW digraph) is a pair $D_\omega = (D, \omega)$, where $\omega : E(D) \cup V(D) \to \mathbb{C}$ is a function on the set $E(D)$ of edges in D and the set $V(D)$ of vertices in D. We call D the *underlying digraph* of D_ω and ω the *vertex-and-edge weight function* on D_ω. Moreover, if $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$, for each edge $e \in E(D)$, we say ω is a *symmetric vertex-and-edge weight function* and D_ω a *symmetrically vertex-and-edge weighted digraph.*

Given any VEW digraph D_ω, the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij} = \begin{cases} \omega(v_i) & if \quad i = j, \\ \omega(v_i v_j) & if \quad v_i v_j \in E(D), \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial of VEW digraph D_ω is that of its adjacency matrix $A(D_\omega)$. Now, for any Γ-voltage assignment ϕ of G, with notations as in Section 2, let $\omega_i(\phi) : E(\overrightarrow{G}) \cup V(\overrightarrow{G}) \to \mathbb{C}$ be the function defined by $\omega_i(\phi)(e) = -\lambda_{(\phi(e),i)}$ for $e \in E(\vec{G})$ and $\omega_i(\phi)(v_j) = d(v_j)$, the degree of v_j in G, so the adjacency matrix of a VEW digraph $(\vec{G}, \omega_i(\phi))$ is the matrix

$$D(G) - \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\overrightarrow{G}_{(\phi,\gamma)}),$$

for each $i = 1, 2, \cdots, |V(F)|$. Then, we can obtain the characteristic polynomial of the Laplacian matrix of the graph bundle $G \times^{\phi} F$ from Theorem 2 as follows.

THEOREM 4. *Let Γ be an abelian subgroup of* $\mathrm{Aut}(F)$ and let ϕ be a Γ*-voltage assignment* of G. *Then the characteristic polynomial of the Laplacian matrix $C(G \times^{\phi} F)$* of $G \times^{\phi} F$ is

$$\Psi(G \times^{\phi} F; \lambda) = \prod_{i=1}^{|V(F)|} \Phi(\vec{G}_{\omega_i(\phi)}; \lambda - \lambda_{(F,i)}).$$

COROLLARY 5. (1) *If Γ be an abelian subgroup of S_n and ϕ a Γ-voltage assignment of G, then the characteristic polynomial of the Laplacian matrix $C(G \times^{\phi} \overline{K_n})$ of an n-fold covering graph of G is*

$$\Psi(G \times^{\phi} \overline{K_n}; \lambda) = \prod_{i=1}^{n} \Phi(\overrightarrow{G}_{\omega_i(\phi)}; \lambda).$$

(2) *The characteristic polynomial of the Laplacian matrix $C(G \times F)$ of the cartesian product $G \times F$ of two graphs G and F is*

$$\Psi(G \times F; \lambda) = \prod_{i=1}^{|V(F)|} \Psi(G; \lambda - \lambda_{(F,i)}).$$

Corollary 5.(2) shows that the Laplacian eigenvalues of the cartesian product $G \times F$ of graphs G and F are equal to all the possible sums of eigenvalues of two factors: $\lambda_{(G,j)} + \lambda_{(F,i)}$, where $\lambda_{(G,j)}$, $j = 1, 2, \cdots, |V(G)|$ and $\lambda_{(F,i)}$, $i = 1, 2, \cdots, |V(F)|$, are the eigenvalues of $C(G)$ and $C(F)$, respectively.

Now, we need to calculate the characteristic polynomials $\Phi(\overrightarrow{G}_{\omega_i(\phi)}; \lambda)$ of a VEW digraph $\overrightarrow{G}_{\omega_i(\phi)}$ for $i = 1, 2, \cdots, |V(F)|$.

An undirected simple graph S is called a *basic figure* if each of its components is either K_1 or K_2 or a cycle $C_m (m \geq 3)$. We denote by $B_j(G)$ the set of all subgraphs of G which are basic figures with j vertices. Then, the characteristic polynomial of a VEW digraph $\overrightarrow{G}_{\omega_i(\phi)}$ is given as follows:

Let Γ be an abelian subgroup of $\mathrm{Aut}(F)$. Then, for any Γ-voltage assignment ϕ of G, we have

$$\Phi(\overrightarrow{G}_{\omega_i(\phi)};\lambda) = \lambda^{|V(G)|} + \sum_{j=1}^{|V(G)|}\Big(\sum_{S\in B_j(G)}(-1)^{\kappa(S)}\times\prod_{u\in I_v(S)}\omega_i(\phi)(u)$$
$$\times\prod_{e\in K_2(S)}\omega(e^+)\omega(e^-)\times\prod_{C\in C(S)}(\omega_i(\phi)(C^+)+(\omega_i(\phi)(C^+))^{-1})\Big)\lambda^{|V(G)|-j}.$$

In this equation, $\kappa(S)$ denotes the number of components of $S, K_2(S)$ the subgraph of S consisting of all components isomorphic to K_2, $C(S)$ the set all cycle $C_m(m\geq 3)$ in S, and $I_v(S)$ does the set of all isolated vertices in S. If a component of S in G is a cycle C, C^+ and C^- are two linear directed cycle and $\omega_i(\phi)(C^+) = \prod_{C\in E(C^+)}\omega(e)$.

Now, we calculate the characteristic polynomial of a regular covering. For any Γ-voltage assignment ϕ of G, with notations as in Section 3, let $\omega_{(s_1,\cdots,s_\ell)}(\phi) : E(\overrightarrow{G})\cup V(\overrightarrow{G})\to\mathbb{C}$ be the function defined by $\omega_{(s_1,\cdots,s_\ell)}(\phi)(e) = -\prod_{\alpha=1}^{\ell}(\zeta_\alpha^{k_\alpha})^{s_\alpha}$ for $\phi(e) = \prod_{\alpha=1}^{\ell}\rho_\alpha^{k_\alpha}, e\in E(\overrightarrow{G})$ and $\omega_{(s_1,\cdots,s_\ell)}(\phi)(v_j) = d(v_j)$, the degree of v_j in G.

Then, the following comes from Theorem 3.

THEOREM 5.

$$\Psi(G\times_\phi\Gamma;\lambda) = \Phi(C(G\times_\phi\Gamma);\lambda) = \prod_{(s_1,\cdots,s_\ell)}\Phi(\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};\lambda).$$

Now, we need to calculate the characteristic polynomial $\Phi(\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};\lambda)$ of a VEW digraph $\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)}$.

Finally, we compute the characteristic polynomial $\Phi(\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)};\lambda)$ of a VEW digraph $\overrightarrow{G}_{\omega_{(s_1,\cdots,s_\ell)}(\phi)}$ for a pseudograph G as a generalization.

In an undirected pseudograph, two elementary configurations S_1 and S_2 are equivalent if the identity map of vertex set $V(G)$ induces an isomorphism between S_1 and S_2. We denote the set of equivalence classes of $B_j(G)$ by $[B_j(G)]$ for $j = 1, \cdots, |V(G)|$. Let $[S]$ be an element of $[B_j(G)]$. Then $[S]$ is an equivalence class of K_1 or K_2 or cycles. Let $E(K_2[S])$ be the equivalence classes of the copies of K_2 and $E(C[S])$ the equivalence classes of the cycles in $[S]$. Note every copy of K_2 in G induces two directed edges in $\overrightarrow{G}$, say e^+ and e^-, and every loop is a cycle of length 1. Then we can get the following theorem.

THEOREM 6. *Let Γ be a finite abelian group and let ϕ be a Γ-voltage assignment of G. Let ω be one of $\omega_{(s_1,\cdots,s_\ell)}(\phi)$. Then, for each $[S] \in [B_j(G)]$, the contribution of $[S]$ in the coefficient of $\lambda^{|V(G)|-j}$ of $\Phi(\overrightarrow{G}_\omega; \lambda)$ is*

$$(-1)^{\kappa(S)} \prod_{u \in I_v(S)} \omega(u) \prod_{[e] \in E(K_2[S])} \left(\sum_{e \in [e]} \omega(e^+) \right) \left(\sum_{e \in [e]} (\omega(e^+))^{-1} \right) 2^{|E(C[S])|}$$
$$\times \prod_{[C] \in E(C[S])} \left(\sum_{C \in [C]} Re(\omega(C^+)) \right),$$

where $Re(\omega(C^+))$ is the real part of $\prod_{e \in C^+} \omega(e)$ and S is a representative of $[S]$

5. Applications

Let n be a positive integer. The *wrapped butterfly WB_n of order n* has vertex set

$$V(WB_n) = \mathbb{Z}_n \times \mathbb{Z}_2{}^n,$$

and each vertex

$$< \ell, \beta_0\beta_1 \cdots \beta_{\ell-1}\alpha\beta_{\ell+1} \cdots \beta_{n-1} >$$

is adjacent to each of the vertices

$$< \ell + 1 \pmod n, \beta_0\beta_1 \cdots \beta_{\ell-1}\omega\beta_{\ell+1} \cdots \beta_{n-1} >$$

for $\omega \in \mathbb{Z}_2$. For example, WB_3 can be drawn as follows:

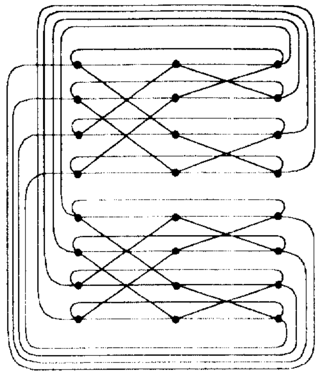


FIGURE 1. The wrapped butterfly WB_3.

Let WG_n be the pseudograph with vertex set $V(WG_n) = \{v_0, \cdots, v_{n-1}\}$, edge set $E(WG_n) = \{d_1, \cdots, d_n; e_1, \cdots, e_n\}$, where both d_i and e_i having the same endpoints v_{i-1} and v_i for $1 \leq i \leq n$.

It is known [7] that a wrapped butterfly WB_n can be represented as a covering graph $WG_n \times_\phi \mathbb{Z}_2^n$ with a $\mathbb{Z}_2^n$-voltage assignment ϕ; $\phi(e_i) = 0\cdots0$ for all $i = 1, 2, \cdots, n$ and $\phi(d_1) = 10\cdots0, \cdots, \phi(d_n) = 0\cdots01$. Then $\omega_{(s_1,\cdots,s_n)}(\phi)(e_i) = -1$ for all $i = 1, 2, \cdots, n$ and

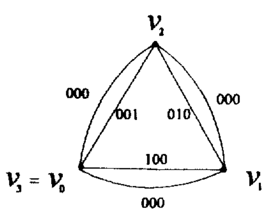
$$\omega_{(s_1,\cdots,s_n)}(\phi)(d_i) = \begin{cases} 1 & \text{if } s_i = 1, \\ -1 & \text{otherwise,} \end{cases}$$

and

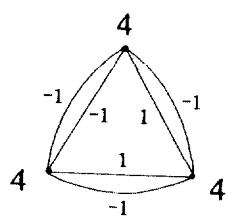
$$\omega_{(s_1,\cdots,s_n)}(\phi)(v_i) = 4$$

for all $i = 1, 2, \cdots, n$.

For example, if $G = WG_3$ and $(s_1, s_2, s_3) = (1, 1, 0)$, then we get the following figures.



G



$\vec{G}_{\omega_{(1,1,0)}(\phi)}$

The adjacency matrix $A(\overrightarrow{G}_{\omega_{(1,1,0)}(\phi)})$ is

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} A(\overrightarrow{G}_{\omega_{(1,1,0)}(\phi)}) &= (\,4\,) \oplus \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \\ &= (4D(P_1) - 2A(P_1)) \oplus (4D(P_2) - 2A(P_2)), \end{aligned}$$

and

$$\begin{aligned} \Phi(\overrightarrow{G}_{\omega_{(1,1,0)}(\phi)};\lambda) &= \det((\lambda-4)I_1 + 2A(P_1))\det((\lambda-4)I_2 + 2A(P_2)) \\ &= (-1)^3 2^3 \det\left(\frac{4-\lambda}{2}I_1 - A(P_1)\right)\det\left(\frac{4-\lambda}{2}I_2 - A(P_2)\right) \\ &= (-1)^3 2^3 \Phi\left(P_1;\frac{4-\lambda}{2}\right)\Phi\left(P_2;\frac{4-\lambda}{2}\right). \end{aligned}$$

In general, if P_k is a path on k vertices and C_k is a cycle of length k, then

$$\Phi(WG_{n_{\omega_{(s_1,\cdots,s_n)}(\phi)}};\lambda) = \begin{cases} (-1)^n 2^n \Phi(C_n;\frac{4-\lambda}{2}) & \text{if } \ (s_1,\cdots,s_n) = (0,\cdots,0), \\ (-1)^n 2^n \Phi(P_{k_1};\frac{4-\lambda}{2})\cdots\Phi(P_{k_r};\frac{4-\lambda}{2}) & \text{otherwise}, \end{cases}$$

where $\{k_1,\cdots,k_r\} \subset \{1,\cdots,n\}$.

Let $1 \le r \le n-1$. Identify d_i with d_j and s_i with s_j if $i \equiv j \pmod n$. Then $\Phi(P_r;\frac{4-\lambda}{2})$ is a factor of $\Phi(WG_{n_{\omega_{(s_1,\cdots,s_n)}(\phi)}};\lambda)$ if and only if

$$\omega_{(s_1,\cdots,s_n)}(\phi)(d_{i+1}) = \cdots = \omega_{(s_1,\cdots,s_n)}(\phi)(d_{i+r-1}) = -1$$

and

$$\omega_{(s_1,\cdots,s_n)}(\phi)(d_i) = \omega_{(s_1,\cdots,s_n)}(\phi)(d_{i+r}) = 1$$

for some $i \in \{1,2,\cdots,n\}$. This is equivalent to say that

$$s_{i+1} = \cdots = s_{i+r-1} = 0, \qquad s_i = s_{i+r} = 1$$

for some $i \in \{1, 2, \cdots, n\}$. Since

$$\left|\{(s_1, \cdots, s_n) \in \mathbb{Z}_2^n s_{i+1} = \cdots = s_{i+r-1} = 0, s_i = s_{i+r} = 1\}\right| = 2^{n-r-1}$$

for each $1 \leq i \leq n$, the multiplicity of $\Phi(P_r; \frac{4-\lambda}{2})$ in $\Phi(C(WB_n); \lambda)$ is $n \cdot 2^{n-r-1}$.

Now, $\Phi(P_n; \frac{4-\lambda}{2})$ is a factor of $\Phi(WG_{n_{\omega_{(s_1,\cdots,s_n)}(\phi)}}; \lambda)$ if and only if for some $i \in \{1, 2, \cdots, n\}$, $\omega_{(s_1,\cdots,s_n)}(\phi)(d_i) = 1$ and $\omega_{(s_1,\cdots,s_n)}(\phi)(d_k) = -1$ for all $k \neq i$. Hence the multiplicity of $\Phi(P_n; \frac{4-\lambda}{2})$ in $\Phi(C(WB_n); \lambda)$ is n.

Cleary, there exists only one factor of $\Phi(C_n; \frac{4-\lambda}{2})$ in $\Phi(C(WB_n); \lambda)$. Therefore

$$\begin{aligned}
\Psi(WB_n; \lambda) = \Phi(C(WB_n); \lambda) &= \prod_{(s_1,\cdots,s_n)} \Phi(WG_{n_{\omega_{(s_1,\cdots,s_n)}(\phi)}}; \lambda) \\
&= ((-1)^n 2^n)^{2^n} \prod_{r=1}^{n-1} \left[\Phi\left(P_r; \frac{4-\lambda}{2}\right)\right]^{n \cdot 2^{n-r-1}} \\
&\quad \times \left[\Phi\left(P_n; \frac{4-\lambda}{2}\right)\right]^n \Phi\left(C_n; \frac{4-\lambda}{2}\right).
\end{aligned}$$

To get the number of all spanning trees of WB_n, we need to calculate the product of all non-zero roots of both

$$\Phi\left(P_r; \frac{4-\lambda}{2}\right) = 0 \quad \text{and} \quad \Phi\left(C_n; \frac{4-\lambda}{2}\right) = 0.$$

Since $\exp(\frac{2\pi}{n}i)$ is the root of the equation $x^n - 1 = 0$,

$$\left\{1 - \exp\left(\frac{2\pi}{n}i\right)\right\} \times \cdots \times \left\{1 - \exp\left(\frac{2(n-1)\pi}{n}i\right)\right\} = n.$$

If $1 \leq k \leq n-1$,

$$\begin{aligned}
1 - \exp\left(\frac{2k\pi}{n}i\right) &= \exp\left(\frac{k\pi}{n}i\right)\left\{\exp\left(\frac{-k\pi}{n}i\right) - \exp\left(\frac{k\pi}{n}i\right)\right\} \\
&= (-2i)\exp\left(\frac{k\pi}{n}i\right)\sin\frac{k\pi}{n}.
\end{aligned}$$

Hence,

$$\prod_{k=1}^{n-1} \sin^2(\frac{k\pi}{n}) = \frac{n^2}{4^{n-1}}.$$

The spectrum of a path P_n consists of the numbers $2\cos\frac{k\pi}{n+1}(k = 1, \cdots, n)$. Put $\frac{4-\lambda}{2} = 2\cos\frac{k\pi}{n+1}$. Then $\lambda = 4(1 - \cos\frac{k\pi}{n+1}) \neq 0$ for $k = 1, \cdots, n$ and

$$\begin{aligned}\prod_{k=1}^{n} 4\left(1 - \cos\frac{k\pi}{n+1}\right) &= 4^n \prod_{k=1}^{n}\left(1 - \cos\frac{k\pi}{n+1}\right) \\ &= 4^n \left(\prod_{k=1}^{n}\left(1 - \cos\frac{k\pi}{n+1}\right)^2\right)^{\frac{1}{2}} \\ &= 4^n \left(\prod_{k=1}^{n}\left(1 - \cos\frac{k\pi}{n+1}\right)\left(1 - \cos\frac{((n+1)-k)\pi}{n+1}\right)\right)^{\frac{1}{2}} \\ &= 4^n \left(\prod_{k=1}^{n} \sin^2\frac{k\pi}{n+1}\right)^{\frac{1}{2}} \\ &= (n+1)2^n.\end{aligned}$$

The spectrum of a cycle C_n consists of the numbers $2\cos\frac{2k\pi}{n}(k = 1, \cdots, n)$.
From $\frac{4-\lambda}{2} = 2\cos\frac{2k\pi}{n}$, we can get $\lambda = 4 - 4\cos\frac{2k\pi}{n} \neq 0$ for $k = 1, \cdots, n-1$. Hence

$$\begin{aligned}\prod_{k=1}^{n-1}\left(4 - 4\cos\frac{2k\pi}{n}\right) &= 4^{n-1}\prod_{k=1}^{n-1}\left(1 - \cos\frac{2k\pi}{n}\right) \\ &= 4^{n-1}\prod_{k=1}^{n-1}\left(2\sin^2\frac{2k\pi}{n}\right) \\ &= 4^{n-1}2^{n-1}\prod_{k=1}^{n-1}\sin^2\frac{2k\pi}{n} \\ &= n^2 \cdot 2^{n-1}.\end{aligned}$$

We summarize our discussions in the following theorem.

THEOREM 7. *The number $t(WB_n)$ of spanning trees of the wrapped butterfly WB_n is* $n(n+1)^n 2^{n^2-1+n2^n \sum_{r=1}^{n-1} r2^{-r-1}} \prod_{r=1}^{n-1}(r+1)^{n\cdot 2^{n-r-1}}$.

PROOF. Let $t(G)$ denote the number of spanning trees contained in a graph G. Then it is well known that $t(G) = \frac{1}{n}\prod \lambda$, where λ runs through all non-zero eigenvalues of the Laplacian matrix of G. Hence

$$\begin{aligned} t(WB_n) &= \frac{1}{n\cdot 2^n}\prod_{r=1}^{n-1}((r+1)2^r)^{n2^{n-r-1}}\{(n+1)\cdot 2^n\}^n n^2\cdot 2^{n-1} \\ &= n(n+1)^n 2^{n^2-1+n2^n\sum_{r=1}^{n-1} r2^{-r-1}}\prod_{r=1}^{n-1}(r+1)^{n\cdot 2^{n-r-1}}. \end{aligned}$$

References

1. Biggs, N., *Algebraic Graph Theory 2nd ed.*, Cambridge University Press, 1995.
2. Chae, Y., Kwak, J. H. and Lee, J., *Characteristic polynomials of some graph bundle*, J. Korean Math. Soc. **30** (1993), 229-249.
3. Chae, Y. and Lee, J., *Regular coverings and manifold crystallization*, Comm. Korean Math. Soc. **6** (1991), 229-310.
4. Cvetkovic, D. M., Doob, M. and Sacks, H., *Spectra of Graphs*, Academic Press, New York, 1979.
5. Gross, J. L. and Tucker, T. W., *Topological Graph Theory*, John Wiley and Sons, New York, 1989.
6. Kwak, J. H. and Lee, J., *Characteristic polinomials of some graph bundles II*, Linear and Multilinear Algebra **32** (1992), 61-73.
7. ———, *A construction method of network mappings*, J. Korea Information Science Soc. **32** (1995), 1239-1249.
8. ———, *Isomorphism classes of graph bundles*, Canad. J. Math. **XLII** (1990), 747-761.
9. Sohn, M. Y. and Lee, J., *Characteristic polynomials of some weighted graph bundles and its application to links*, Internat. J. Math. & Math. Sci. **17 No. 3** (1994), 503-510.

Department of Mathematics
Catholic University of Taugu-Hyosung
Kyongsan 713-702, Korea