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HAMILTONIANS IN STEINHAUS GRAPHS

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ABSTRACT. A Steinhaus graph is a labelled graph whose adjacency matrix $A = (a_{i,j})$ has the Steinhaus property: $a_{i,j} + a_{i,j+1} \equiv a_{i+1,j+1} \pmod 2$. We consider random Steinhaus graphs with n labelled vertices in which edges are chosen independently and with probability $\frac{1}{2}$. We prove that almost all Steinhaus graphs are Hamiltonian like as in random graph theory.

1. Introduction

Since its introduction by Erdős and Rényi ([9]), the theory of random graphs has been greatly developed and many properties of a random graph have been studied in detail [1], [3], [11] and [13] etc. One of the important questions Erdős and Rényi [9] raised in their fundamental paper on the evolution of random graphs is that "is almost every graph Hamiltonian?". A breakthrough was achieved by Pósa [15] and Korshunov [10]. They prove that for some constant c, almost every labelled graph with n vertices and at least $cn\log n$ edges is Hamiltonian. On the other hand, it would be useful to have a criterion by which to decide whether a specific graph behaves like a random graph, that is, has the property (of almost every graph) that interests us. Such a criterion gives the concepts of pseudo-random graphs ([8]) and quasi-random graphs ([18]) which is a special type of pseudo-random graphs. In [18], Thomason shows that a (p,α)-jumbled graph behaves like a random graph with edge probability p. A *Steinhaus graph* of order n is a labelled graph whose adjacency matrix has the Steinhaus property:

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$a_{i,j} + a_{i,j+1} \equiv a_{i+1,j+1} \pmod 2$. In fact, the adjacency matrix is completely determined by the first row in the matrix. Thus there are 2^{n-1} distinct Steinhaus graphs of order n. Now, we state the model in random graph theory. The theory of the random graphs which grew from the paper of Erdős and Rényi ([9]) is a striking example of the use of the probabilistic method in mathematics. We consider the sample space Ω_n consisting of all lablled graphs of order n. For each positive integer n and number $p = p(n)$ with $0 < p < 1$, the probability of a graph $G \in \Omega_n$ with m edges is given by $P(G) = p^m(1-p)^{\binom{n}{2}-m}$. Let Q be a property of graphs and consider the set $\mathcal{A}_n$ of graphs of order n that possess property Q. If $P(A_n) \to 1$ as $n \to \infty$ then we say that *almost every graph has property* Q. A graph G is said to be (p,α)-*jumbled* if p, α are real numbers satisfying $0 < p < 1 \leq \alpha$ and if every induced subgraph S of G satisfies

$$\left| e(S) - p\binom{|S|}{2} \right| \leq \alpha |S|,$$

where $e(S)$ is the number of edges in S. In particular, a $(\frac{1}{2}, o(n^2))$ graph G of order n is called a *quasi-random* graph. More precisely, for each subset S of G of order n, $e(S) = \frac{1}{4}|S|^2 + o(n^2)$. In this paper, we consider the sample Ω_n as the set of all Steinhaus graphs of order n, which is one of important models in random Steinhaus graph theory. The properties of random Steinhaus graphs and random generalized Steinhaus graphs have been investigated by Brand and many other authors in ([4], [5], [6]).

2. Hamilton cycles

Many theorems on Hamiltonian require a degree condition (see [2]). But not many graphs satisfy the degree conditions. For example, in [13] we see that almost all graphs G do not satisfy the condition that for every pair of nonadjacent vertices u and v, $d(u) + d(v) \geq n$. Also some theorems on Hamilton graphs require an edge condition ([14]). But not many graphs satisfy the edge conditions. For example, it is clear that almost all graphs G do not satisfy the condition that the number of edges of G is at least $\lfloor \frac{(n-1)(n-2)}{2} + 3 \rfloor$. Therefore, it is natural to consider that if we combine an edge condition (sometimes, called a global condition) with a degree condition then a graph satisfying both conditions may be Hamiltonian. From the definition of a quasi-random graph G with n vertices, we find a good global condition which we use through this paper. In [8], they showed that quasi-random graphs give the following property: all but $o(n)$ vertices of G have degree $\frac{1}{2}(1+o(1))n$. In this case we say that G is *almost-regular*. Note that the above property does not imply quasi-random property ([8]). Thus we conclude that a quasi-random graph has a good global condition but does not have a good degree condition. From now we assume that the probability of an edge is $\frac{1}{2}$. In [6] and [13], we can find that almost every Steinhaus graph satisfies the desired global and degree conditions

THEOREM 2.1. *([6]) Almost all Steinhaus graphs are quasi-random.*

THEOREM 2.2. ([6], [13]) *Let $\varepsilon > 0$. Then almost all Steinhaus graphs satisfy*

$$\frac{1}{2}(1-\varepsilon)n < d(v) < \frac{1}{2}(1+\varepsilon)n$$

for all of their vertices v.

In this paper, we present two proofs that almost all Steinhaus graphs are Hamiltonian. The first proof follows from a result in [18] and the second proof follows the standard method in the theory of random graphs ([2], [16]) with the above theorems. Let us give the first proof. Let G be a quasi-random and Steinhaus graph with n vertices. Let S be a subset of G. Then we have

$$e(S) = \frac{1}{4}|S|^2 + o(n^2)$$

and

$$e(S) - \frac{1}{2}\binom{|S|}{2} = \frac{1}{4}|S| + o(n^2) = o(n^2).$$

If $|S| \le o(n)$ then

$$\left|e(S) - \frac{1}{2}(\binom{|S|}{2})\right| = o(n^2) \le o(n)|S|.$$

By combining both cases, we show that the graph G is $(\frac{1}{2}, o(n))$-jumbled. Thus almost all Steinhaus graphs are $(\frac{1}{2}, o(n))$-jumbled. Since almost all the Steinhaus graphs G satisfy the degree condition in Theorem 2.2, $\delta(G) \geq (1+o(1))\frac{n}{2}$ where $\delta(G)$ is the minimum degree of G. This gives the first proof by the following theorem.

THEROEM 2.3. ([18]) *Let G be a (p,α)-jumbled graph, and P be a path in G of length $l \geq 0$ and $\delta(G)$ be the minimum degree of G. If $\delta(G) \geq 6\alpha p^{-1} + l$, then G has a Hamilton cycle containing P.*

Now we give the second proof. Let $G = (V,E)$ be a Steinhaus graph with n vertices. Also, assume that G is a quasi-random graph with the degree condition in Theorem 2.2. Let x_0 be a vertex in G. Let S be a longest x_0-path in G, that is a path beginning at x_0: $S = x_0x_1\ldots x_k$. Then the neighbor of x_k, $\Gamma(x_k)$, is contained in $\{x_0, x_1, \ldots, x_{k-1}\}$ since otherwise S could be continued to a longer path. If x_k is adjacent to x_j, $0 \leq j < k-1$, then $S' = x_0x_1\ldots x_jx_kx_{k-1}\ldots x_{j+1}$ is another longest x_0-path. We call S' *a simple transform of* S. Let L be the set of end vertices (different from x_0) of transforms of S and put $N = \{x_j \in S : x_{j-1} \in L \text{ or } x_{j+1} \in L\}$ and $R = V - N \cup L$. We are now ready to state Pósa's lemma.

THEROEM 2.4. ([2], [15]) *The graph G has no L-R edges.*

COROLLARY 2.5. *If $|L| \leq n/3$ then there are disjoint sets of size $|L|$ and $n - 3|L| + 1$, that are joined by no edges of G.*

PROOF. Consider L and R in Theorem 2.4. Then we have

$$|R| = n - |N \cup L| \geq n - 2|l| \geq n - 3|l| + 1.$$

Choose any subset W of R such that the size of W is $n - 3|L| + 1$. □

Let U and W be two subsets of G. Then from Theorems 2.1 and 2.2 by applying the quasi-random property to the subsets U, W, $U \cup W$, $U - W$ and $W - U$, we get the following corollaries.

COROLLARY 2.6. *Let k be the number of edges between U and $G - W$. Then k is given by*

$$k = \frac{1}{2}|U||W - U| + o(n^2).$$

COROLLARY 2.7. $|U \cup \Gamma(U)| \geq \frac{1}{2}(1 + o(n))n$.

Now we give a simple lemma for Theorem 2.10.

LEAAM 2.8. *Let $0 < \gamma < 1/3$ be a constant. Then for almost all Steinhaus graphs G, if U is a subset of G and $|U| \leq \gamma n$ then*

$$|U \cup \Gamma(U)| \geq 3|U|.$$

PROOF. Suppose that there is a quasi-random Steinhaus graph G with n vertices such that

$$|U \cup \Gamma(U)| < 3|U|$$

for some γ and some subset U of G and $|U| \leq \gamma n$. Let W be the complement of $U \cup \Gamma(U)$. Denote a, b and c by the size of subsets U, $\Gamma(U) - U$ and W respectively. By Corollary 2.6, $(1 + o(1))\frac{n}{6} < a$. Also, $\frac{ac}{2} = o(n^2)$ by Corollary 2.5. Since $a \leq \gamma n$ and $a + b < 3a$, we have $c \geq (1 - 3\gamma)n$. Thus we have

$$o(n^2) = \frac{ac}{2} \geq \frac{n^2}{12}(1 + o(1))(1 - 3\gamma).$$

This gives a contradiction for all n large enough. $\square$

Let $G = (V, E)$ be a Steinhaus graph with n vertices which is quasi-random. Denote D_t by the number of pairs (X, Y) of disjoint subsets of U such that $|X| = t$, $|Y| = n - 3t$ and G has no $X - Y$ edges. In fact, Corollary 2.5 provides an example of D_t. The following corollary comes from Lemma 2.8.

COROLLARY 2.9. *Let $D = \{G : D_t = 0$ for every $t, 1 \leq t \leq \gamma n\}$, where $0 < \gamma < \frac{1}{3}$ is a constant. If $\bar{D}$ is the complement of D in G then $P(\bar{D}) = o(1)$.*

Now we give the second proof.

THEROEM 2.10. *Almost all Steinhaus graphs contain a Hamiltonian path. More precisely, if x and y are arbitrary distinct vertices, then almost every Steinhaus graph contains a Hamilton path from x to y.*

PROOF. Since almost all Steinhaus graphs are quasi-random, we can assume that the Steinhaus graphs are quasi-random. Let us introduce the following notations for certain events whose general element is denoted by G.

- Let D be the collection of all Steinhaus graphs such that $D_t = 0$ for every t, where $1 \leq t \leq \gamma n$ and $0 < \gamma < \frac{1}{3}$.
- Let $E(W, x)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ of G has a path of maximal length whose end vertex is joined to x.
- Let $E(W, x|w)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ has a w-path of maximal length among the w-paths whose end vertex is joined to x.
- Let $F(x)$ be the collection of all Steinhaus graphs G such that every path of maximal length in G contains x.
- Let $H(W)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ of G has a Hamilton path.
- Let $H(x, y)$ be the collection of all Steinhaus graphs G such that G has a Hamilton x-y path.
- The complement of an event A is $\overline{A}$.

Note that by Corollary 2.9 we have

$$P(\overline{D}) = 1 - P(D) = o(1).$$

Let $|W| = n - 2$ or $n - 1$ and let us estimate the probability of the event $D \cap \overline{E}(W, x)$ and $P(D \cap \overline{E}(W, x))$ where x is not in W. Let $G \in D \cap \overline{E}(W, x)$ and consider a path $S = x_0 x_1 \ldots x_k$ of maximal length in $G[W]$. (By introducing an ordering in W, we can easily achieve that S is determined by $G[W]$.) Let $L = L(G[W])$ be the set of end vertices of the transforms of the x_0-path S and let R be as in Theorem 2.4 (applied to $G[W]$). Recall that $|R| \geq |W| + 1 - 3|L|$ and there is no L-R edge, so no L-$R \cup \{x\}$ edge either. Since $G \in D$ and $|R \cup \{x\}| \geq n - 3|L|$, we find that $|L| \geq \gamma n$. As L is independently of the edges incident with x, we have

$$P(D \cap \overline{E}(W, x)) \leq P(G \in D \text{and } x \text{ is not joined to } L(G[W])) \leq (\frac{1}{2})^{\lfloor \gamma n \rfloor}.$$

A similar argument shows that

$$P(D \cap \overline{E}(W, x|w) \leq (\frac{1}{2})^{\lfloor \gamma n \rfloor}$$

provided $|W| = n-2$ or $n-1$, $x \in W$ and $x \notin W$. Note that $\overline{F}(x) \subset \overline{E}(V-\{x\},\{x\})$, so

$$\begin{aligned}
P(\overline{H}(V)) =& P(\cup_{x\in V}\overline{F}(x)) \\
\leq& P(D \cap \cup_{x\in V}\overline{F}(x)) + P(\overline{D}) \\
\leq& \sum_{x\in V} P(D\cap \overline{F}(x)) + P(\overline{D}) \\
\leq& nP(D\cap \overline{E}(V-\{x\},x)) + P(\overline{D}) \\
\leq& n(\frac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).
\end{aligned}$$

This proves that almost every Steinhaus graph has a Hamilton path. Now let x and y be distinct vertices and put $W = V - \{x,y\}$. By the first part

$$P(\overline{H}(W)) \leq 2n(\frac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).$$

Since $H(x,y) \supset H(W) \cap E(W,y) \cap E(W,x|y)$ we have

$$\begin{aligned}
P(\overline{H}(x,y)) &\leq P(\overline{H}(W)) + P(D\cap \overline{E}(W,y)) + P(D\cap \overline{E}(W,x|y)) + P(\overline{D}) \\
&\leq 2n(\frac{1}{2})^{\lfloor \gamma n \rfloor} + 2(\frac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).
\end{aligned}$$

It shows that almost all Steinhaus graphs contain a Hamilton path from x to y. □

COROLLARY 2.11. *Almost all Steinhaus graphs are Hamiltonian.*

PROOF. Let $\varepsilon > 0$ be given. Choose k such that $(\frac{1}{2})^k < \frac{\varepsilon}{2}$. Let $H([n,i])$ be the collection of all Steinhaus graphs with n vertices which have a Hamilton path from the vertex 1 to the vertex i and $A([n,i])$ be the collection of all Steinhaus graphs such that the vertex 1 is adjacent to the vertex i for $2 \leq i \leq n$. Then by Theorem 2.10 there exists $n_0 > k$ such that

$$P(H([n,i])) > 1 - \frac{\varepsilon}{2^k}$$

for all $n \leq n_0$ and $2 \leq i \leq k$. Therefore, we have

$$\begin{aligned} P(\bigcap_{i=2}^{k} H([n,i]) \cap \bigcap_{i=2}^{k} A([n,i])) &> 1 - \sum_{i=2}^{k} \frac{\varepsilon}{2^i} - \frac{1}{2^k} \\ &> 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\ &= 1 - \varepsilon \end{aligned}$$

for all $n \geq n_0$. It shows that almost all Steinhaus graphs are Hamiltonian. □

We close by mentioning a Hamiltonian connected property on Steinhaus graphs. While almost all graphs are Hamiltonian connected ([2]), it is still not known that almost all Steinhaus graphs are Hamiltonian connected.

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