Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 1123–1135

MAXIMAL INDUCED CYCLES IN STEINHAUS GRAPHS

DAEKEUN LIM

ABSTRACT. In this paper, we introduce some properties of Steinhaus graphs of order n, and prove that the size of some special type of induced cycles in Steinhaus graphs of order n is bounded by $\lfloor \frac{n+3}{2} \rfloor$.

1. Introduction

A *Steinhaus graph* is a labelled graph of order n whose adjacency matrix $(a_{i,j})$ satisfies the condition that $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2$ for each $1 \leq i < j \leq n$. The triangle $(a_{i,j})_{2 \leq i < j \leq n}$ in the adjacency matrix is called the *Steinhaus triangle* of G. It is clear that a Steinhaus graph is completely determined by the first row of the Steinhaus triangle and the first row $(a_{1,j})_{j=2}^{n}$ is called the *generating string* for the graph G. It is obvious that there are exactly 2^{n-1} Steinhaus graphs of order n. Let G be a Steinhaus graph of order n with the generating string $(a_{1,j})_{j=2}^{n}$. The *partner* of $G, P(G)$, is the Steinhaus graph generated by the string $(a_{n-i+1,n})_{i=2}^{n}$. It is obvious that G and its partner $P(G)$ have the same graphical structure, which means that they are isomorphic. A Steinhaus graph G is said to be *doubly symmetric* if $G = P(G)$. Steinhaus graphs have several interesting properties which are not shared by all graphs. For example, the diameter of all Steinhaus graphs with n vertices except the path P^n and the empty graph E^n is at most $\lfloor \frac{1}{2}(n+2) \rfloor$ ([5]), the order of a large clique in any Steinhaus graph with n vertices is at most $\lceil \frac{1}{2}(n+3) \rceil$ ([5]) and a Steinhaus graph is bipartite if and only if it has no

Received April 11, 1996. Revised July 5, 1996.
1991 AMS Subject Classification: 05C05.
Key words and phrases: Key words, Steinhaus graph, simple induced cycle, Pascal triangle, doubly symmetric.
The research has been conducted by the Bisa Research Grant of Keimyung University in 1995.

triangle ([7]). More on the properties of Steinhaus graphs are found in [9], [10]. Also, it seems reasonable that the size of a large induced cycle in any Steinhaus graph with n vertices might have an desired upper bound. In this paper, we investigate the maximum size of an induced cycle in a Steinhaus graph which is simple in the following sence:

DEFINITION 1. Let G be a Steinhaus graph of order n. An induced cycle $C = x_1x_2\ldots x_l$ of G is said to be simple if $x_1 = 1$, $x_n = n$ and $x_i < x_j$ whenever $i < j$.

Let G be a Steinhaus graph and C be a simple induced cycle in G. For our convenience, we decompose the Steinhaus graph G with the simple induced cycle C in G as follows: Let $\Phi_C = \{A_i : A_i \subset C\}$ be the partition of C such that each A_i is a maximal subset of C consisting of consecutive vertices and the largest vertex in A_i is joined to the smallest vertex in A_{i+1}. Thus each induced subgraph A_i in G is a path. Let $\overline{\Phi}_C = \{B_j : B_j \subset G - C\}$ be the partition of $G - C$ such that each B_j is the largest subset of $G - C$ consisting of consecutive vertices. Then $\Phi = \Phi_C \cup \overline{\Phi}_C$ is said to be the *cover of the Steinhaus graph* G with the simple induced cycle C in G. Let us give an example of the cover of a Steinhaus graph with a simple induced cycle.

EXAMPLE 2. Let G be the Steinhaus graph which is generated by the string (01000001111). Then $C=\{1,2,3,4,5,11\}$ is a simple induced cycle of G. The cover of G with C is given by $A_1 = \{1,2,3,4,5\}$, $A_2 = \{11\}$ and $B_1 = \{6,7,8,9,10\}$.

Let us mention the following simple lemma and some facts about Pascal's triangle.

LEMMA 3. *Let G be a doubly symmetric Steinhaus graph with generating string $(a_{1,i})_{i=2}^n$. Then $a_{1,n} = 0$.*

PROOF. If $a_{1,n} = 1$, then $a_{1,n-1} + a_{2,n} \equiv 1 \pmod 2$. So G is not doubly symmetric. □

We now present some facts concerning Pascal's triangle modulo two that will be needed later. The rows of the triangle are labelled $R_1, R_2, \ldots$, and so the k^{th} element of R_n is $\binom{n-1}{k-1} \pmod 2$ if $1 \le k \le n$. A Pascal triangle is said to be *of dimension* n if the triangle consists of the n rows

$R_1, R_2, \ldots, R_n$, and is denoted by $(a_{i,j})_{1\le j\le i\le n}$. More on the properties of Pascal triangles are found in [8].

EXAMPLE 4. Here, we give the Pascal triangle of dimension 6.

$$\begin{array}{llllllll} R_1 & \to & 1 \\ R_2 & \to & 1 & 1 \\ R_3 & \to & 1 & 0 & 1 \\ R_4 & \to & 1 & 1 & 1 & 1 \\ R_5 & \to & 1 & 0 & 0 & 0 & 1 \\ R_6 & \to & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

LEMMA 5. *Let $(a_{i,j})$ be the Pascal triangle of dimension n. If $a_{n,j} = 1$ for all $j \le \lfloor \frac{1}{3}(n+4) \rfloor$, then n is a power of 2.*

PROOF. We will use an induction on n. Since $a_{n,j} \equiv \binom{n-1}{j-1} \equiv \binom{n-1}{n-j} \equiv a_{n,n-j+1} \pmod 2$, $a_{n,n-j} = 1$ for all $j \ge \lfloor \frac{1}{3}(n+4) \rfloor$. Let $n = 2^m + k$ for some $0 \le k < 2^m$. We want to show that k is equal to 0. Suppose that k is greater than or equal to 1. Then the Pascal triangle of dimension k satisfies the condition in lemma. So k is a power of 2 by the induction hypothsis. Since $a_{n,j} = 1$ for $1 \le j \le n - \lfloor \frac{1}{3}(n+4) \rfloor + 1$, k is equal to 2^{m-1}. Then $a_{n,k+1} = 0$. This gives a contradiction since $k \le \lfloor \frac{1}{3}(n+4) \rfloor$. It proves lemma. □

LEMMA 6. *Let $a_{n,j} = 1$ for some $1 < j < n$ in the Pascal triangle of dimension n. Then n is odd if and only if $a_{n,j-1} = 0$ and $a_{n,j+1} = 0$.*

PROOF. Suppose that n is odd. Since $\binom{n-1}{j-1}$ is odd and $n-1$ is even, $j-1$ is even by Luscus' Theorem (see [11]). By applying Luscus' Theorem to the 0-th binary digits of $j-2, j$ and $n-1$, we have that $\binom{n-1}{j-2}$ and $\binom{n-1}{j}$ are even. So $a_{n,j-1} = a_{n,j+1} = 0$.

Conversely, suppose that n is even. So $n-1$ is odd. Therefore $n-1$ has 1 as its 0-th binary digit. Now, $j-1$ has either 0 or 1 as its 0-th binary digit. If $j-1$ has 0 as its 0-th binary digit, then j has 1 as its 0-th binary digit. So $\binom{n-1}{j}$ is odd by Luscus' Theorem. If $j-1$ has 1 as its 0-th binary digit, then $j-2$ has 0 as its binary digit. So $\binom{n-1}{j-2}$ is odd also by Luscus' Theorem. Thus either $a_{n,j-1}=1$ or $a_{n,j+1}=1$. □

2. Simple maximal induced cycles in Steinhaus graphs

Let G be a Steinhaus graph with n vertices and $(a_{i,j})$ be the adjacency matrix of G. Let C be a simple maximal induced cycle in G and $\Phi = \Phi_C \cup \overline{\Phi}_C$ be the cover of G with C. In particular, in C, the largest vertex in A_i is joined to the smallest vertex in A_{i+1}.

Now we give a series of lemmas in order to estimate the size of B_i in $\overline{\Phi}_C = \{B_i : i = 1, 2, \dots, t\}$. Let α_i be the smallest vertex and β_i be the largest vertex in A_i respectively. Note that α_1 is the vertex 1 and β_i is joined to α_{i+1} for each $1 \le i \le t$. Also, $B_i = \{\beta_i+1, \beta_i+2, \dots, \alpha_{i+1}-1\}$. Let a_i be the size of A_i. Then $b_i = \alpha_{i+1} - \alpha_i - 1$ is the size of B_i respectively.

Let us observe the following simple facts about strings in the Steinhaus triangle by using the above notations.

fact 1 Since A_i is the path $\alpha_i \alpha_i + 1 \dots \beta_i$, the string $(a_{\alpha_i,j})_{\alpha_i \le j \le \beta_i}$ is $(010\dots0)$ for each i. Thus for all $\alpha_i \le s \le s' \le \beta_i$,

$$a_{s,s'} = \begin{cases} 1 & \text{if } s' = s+1; \\ 0 & \text{otherwise.} \end{cases}$$

fact 2 For each $1 \le i \le t$, the transpose of $(a_{k,\alpha_{i+1}})_{\alpha_i \le k \le \beta_i}$, is the transpose of $(00\dots01)$. Therefore, $(a_{\alpha_i,j})_{\alpha_{i+1}-a_i+1 \le j \le \alpha_{i+1}}$ is $(10\dots0)$.

fact 3 Since β_i is joined to α_{i+1}, the string $(a_{\beta_i,j})_{\alpha_{i+1} \le j \le \beta_{i+1}}$ is $(10\dots0)$.

fact 4 Either a_1 or a_t is equal to 1. (Otherwise, the entries $a_{1,n-1}$ and $a_{1,n}$ are equal to 1 since $a_{1,n}=1$ and $a_{2,n}=0$. Then C is not a cycle.)

In the following lemmas, we will use the above facts.

LEMMA 7. *For each i, $b_i \ge max\{a_i - 1, a_{i+1} - 1\}$.*

PROOF. Without loss of generality, we assume that a_i is greater than or equal to a_{i+1} by considering its partner, $P(G)$, of G. Suppose that b_i is less than $\max\{a_i - 1, a_{i+1} - 1\}$. Consider the string in fact 2. So the entry $a_{\beta_i - b_i - 1, \beta_i}$ is equal to 1 by the Steinhaus property. Since $b_i < a_i - 1$, the entry $a_{\beta_i - b_i - 1, \beta_i}$ is in the subtriangle generated by the string $(a_{\alpha_i, j})_{\alpha_i \le j \le \beta_i}$. Therefore, $\beta_i - b_i - 1 + 1 = \beta_i$ by fact 1. We have $b_i = 0$, which gives a contradiction. *square*

LEMMA 8. *If a_i is equal to a_{i+1}, then*

$$b_i \ge \begin{cases} a_i & \text{if } a_i \text{ is a power of } 2; \\ a_i + & \text{otherwise.} \end{cases}$$

PROOF. First, b_i is at least $a_i - 1$, by Lemma 7. Suppose that b_i is equal to $a_i - 1$. Then the string $(a_{\alpha_i, j})_{\alpha_i \le j \le \beta_{i+1}}$ in the α_i^{th} row in the Steinhaus triangle is clearly $(010\ldots0)$. Since b_i is equal to $a_i - 1$, the entry $a_{\alpha_{i+1} - 1, \beta_{i+1}}$ is equal to 1. Since $(a_{\alpha_{i+1}, j})_{\alpha_{i+1} \le j \le \beta_{i+1}}$ in the α_{i+1}^{th} row is equal to $(010\ldots0)$, the string $(a_{\alpha_{i+1} - 1, j})_{\alpha_{i+1} - 1 \le j \le \beta_{i+1}}$ is $(001\ldots1)$ by the Steinhaus property. But the triangle $(a_{k,j})_{\alpha_i \le k \le \alpha_{i+1} - 1, \beta_i + 1 \le j \le k + a_i}$ is the Pascal triangle of dimension $2a_i - 1$. By Lemma 5, $2a_i - 1$ is a power of 2, which gives a contradiction. Suppose that a_i is not a power of 2. Assume that b_i is equal to a_i.

Case 1 $a_{\alpha_i, \beta_i + 1}$ is equal to 0.

If $a_{\alpha_{i+1} - 1, \beta_{i+1}}$ is 0, $(a_{\alpha_{i+1} - 2, j})_{\alpha_{i+1} - 2 \le j \le \beta_{i+1}}$ in the α_{i+1}^{th} row is $(001\ldots1)$ by the Steinhaus property with the α_{i+1}^{th} row. By the same argument as the above, $2a_i - 1$ is a power of 2. This gives a contradiction. If $a_{\alpha_{i+1} - 1, \beta_{i+1}}$ is 1, $(a_{\alpha_{i+1} - 1, j})_{\alpha_{i+1} - 1 \le j \le \beta_{i+1}}$ in the α_{i+1}^{th} row is $(001\ldots1)$ by the Steinhaus property. By the same arguement as the above, $2a_i$ is a power of 2, which gives a contradiction.

Case 2 $a_{\alpha_i, \beta_i + 1}$ is equal to 1.

If $a_{\alpha_{i+1} - 1, \beta_{i+1}}$ is equal to 0, then $2a_i$ is a power of 2 by the same arguement as in Case 1. This gives a contradiction. Similarly, if $a_{\alpha_{i+1} - 1, \beta_{i+1}}$ is equal to 1, then $2a_i + 1$ is a power of 2. This gives a contradiction also.

By combining both cases, we prove lemma. $\square$

LEMMA 9. *If $|a_{i+1} - a_i|$ is equal to 1, then*

$$b_i \geq \begin{cases} max\{a_i, a_{i+1}\} & if\ min\{a_i, a_{i+1}\}\ is\ a\ power\ of\ 2; \\ max\{a_i, a_{i+1}\} + 1 & otherwise. \end{cases}$$

PROOF. Without loss of generality, we can assume that a_i is greater than a_{i+1}, by considering its partner, $P(G)$. First, b_i is greater than or equal to a_{i+1} by Lemma 7. Suppose that b_i is equal to a_{i+1}. Assume that a_{i+1} is not a power of 2. The string $(a_{\alpha_i,j})_{\alpha_i \leq j \leq \beta_{i+1}}$ in the α_i^{th} row is $(010\ldots010\ldots0)$ where $a_{\alpha_i,\beta_i+1} = 1$.

Case 1 $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 0.

By the Steinhaus property with $a_{\alpha_{i+1}-2,\beta_{i+1}} = 1$, $a_{\alpha_{i+1}-1,\beta_{i+1}} = 0$ and the α_{i+1}^{th} row in the Steinhaus triangle, the string $(a_{\alpha_{i+1}-2,j})_{\alpha_{i+1}-2 \leq j \leq \beta_{i+1}}$ is $(001\ldots1)$. Then $(a_{k,j})_{\alpha_i \leq k \leq \alpha_{i+1}-2, \beta_i+1 \leq j \leq k+a_i}$ is the Pascal triangle of dimension $a_{i+1} + b_i$ satisfying the condition in Lemma 5. So $a_{i+1} + b_i = 2a_{i+1}$ is a power of 2, which gives a contradiction.

Case 2 $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 1.

Again, by the same arguement in the Case 1, $(a_{\alpha_{i+1}-1,j})_{\alpha_{i+1}-1 \leq j \leq \beta_{i+1}}$ in the $(\alpha_{i+1}-1)^{th}$ row is $(001\ldots1)$. So $a_i + b_i = 2a_{i+1}+1$ is a power of 2 by Lemma 5, which gives a contradiction.

By combining both cases, we prove lemma. □

From Lemmas 8 and 9, we observe the followings: Let $\{A_i : i = 1, 2, \ldots, t+1\} \cup \{B_i : i = 1, 2, \ldots, t\}$ be the cover of G with a simple induced cycle C with $|A_{t+1}| = 1$. First, if b_i is greater than or equal to a_i for all $1 \leq i \leq t$, then it is clear that the order of C is at most $\lfloor \frac{1}{2}(n+3) \rfloor$. Second, if b_i is less than a_i for some i, we can not guarantee that the order of C is at most $\lfloor \frac{1}{2}(n+3) \rfloor$. It is the case from Lemmas 7 and 9 that there exists i such that b_i is equal to either $a_i - 1 = a_{i+1}$, where a_{i+1} is a power of 2 or $a_i - 1$, where $a_{i+1} \leq a_i - 2$. Thus we give better estimations regarding the second observation in the following two lemmas.

LEMMA 10. *Suppose that a_i is equal to $a_{i+1} + 1$ and that b_i is equal to a_{i+1} for some i. Let a_{i+1} be a power of 2 which is greater than 1.*

Let k be the smallest number such that $k \geq i+1$, $a_k \geq 2$ and for all $i+1 \leq l \leq k-1$

$$a_l = a_{i+1}$$

and

$$a_k \neq a_{k-1}.$$

Then either

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

or

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

PROOF. First, since a_{i+1} is equal to a_{i+2} and a_{i+1} is a power of 2, by Lemma 8, we have $b_{i+1} \geq a_{i+1}$. Observe that if $b_{i+1} = a_{i+1} + 1$ then $b_{i+2} \geq a_{i+2} + 1$. By continuing this process, we have inequality

$$\sum_{l=i}^{k-2} a_l \leq \sum_{l=i}^{k-2} b_l + 1.$$

Suppose that we have the inequality

$$\sum_{l=i}^{k-1} a_l > \sum_{l=i}^{k-1} b_l.$$

Then from the above inequalities, we have $b_l = a_l$ for all $i+1 \leq l \leq k-2$. Since the a_l's are all the same and a power of 2, the string $(a_{\alpha_l, j})_{\alpha_l \leq j \leq \alpha_{l+1}}$ in the α_l^{th} row is $(010\ldots0110\ldots0)$ for $i+1 \leq l \leq k-1$. Then we have $b_{k-1} \geq a_{k-1}$. Otherwise, we have $b_{k-1} = a_{k-1} - 1$. Therefore the vector $(a_{\alpha_{k-1}, j})_{\alpha_{k-1} \leq j \leq \alpha_k}$ in the α_{k-1}^{th} row is $(010\ldots010\ldots0)$, which is impossible by Lemma 6. Hence $b_{k-1} = a_{k-1}$. Moreover, we have $b_k \geq a_k$ by the same argument as above. Next, we want to show that b_k is greater than or equal to $a_k + 1$, which gives the inequality

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Assume that b_k is equal to a_k.

Case 1 $a_k > a_{k-1}$.

First, if $a_k \geq a_{k-1}+2$ then $b_{k-1} \geq a_{k-1}+1$ by Lemma 7. This is impossible because $b_k = a_k$. Therefore, a_k must be equal to $a_{k-1}+1$. So the entry a_{α_k-1,β_k} is 1. By applying the Steinhaus property to the α_k^{th} row in the Steinhaus triangle, the string $(a_{\alpha_{k-1}-1,j})_{\alpha_{k-1}-1\leq j\leq \beta_{k-1}}$ in the $(\alpha_{k-1}-1)^{th}$ row is $(01\ldots 1)$ because a_{k-1} is a power of 2. Since the string $(a_{l,\alpha_k})_{\alpha_{k-2}\leq l\leq \alpha_k-1}$ in the α_k^{th} column is $(00\ldots 01)$, the entry $a_{\alpha_{k-1}-2,\beta_{k-1}}$ is equal to 0, which gives a contradiction by fact 3.

Case 2 $a_k < a_{k-1}$.

Since the string $(a_{\beta_{k-1},j})_{\beta_{k-1}\leq j\leq \alpha_{k+1}}$ in the β_{k-1}^{th} row is $(00\ldots 010\ldots 0)$ and a_k is less than a_{k-1}, we get the Pascal triangle $(a_{l,j})$ of dimension $a_{k-1}+2$ such that the entry a_{α_k,β_k+1} is 0 where $\beta_{k-1}\leq l\leq \alpha_k$ and $j\leq k+a_{k-1}+1$. Since this entry a_{α_k,β_k+1} is in an even row in the above Pascal triangle, Lemma 6 implies that the entries a_{α_k,β_k+j} are either all 0's or all 1's where $j=2,3$. In both cases, $b_k \geq a_k+1$. This gives a contradiction.

By combining both cases, we prove lemma. □

LEMMA 11. *Suppose that a_i is greater than or equal to $a_{i+1}+2$ and that b_i is equal to a_i-1. Let k be the smallest number such that $k\geq i+2$, for all $i+1\leq l\leq k-1$*

$$a_{l-1} \geq a_l$$

and

$$2 \leq a_{k-1} \leq a_k - 1.$$

Then either

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

or

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

PROOF. First, by Lemma 7, we have $b_{i+1} \geq a_{i+1} - 1$. Moerover, $b_{i+1} \geq a_{i+1}$ by the following argument. If $b_{i+1} = a_{i+1} - 1$ then we have $a_{\alpha_{i+1},\beta_{i+1}} = 0, a_{\alpha_{i+1},\beta_{i+1}+1} = 1$ and $a_{\alpha_{i+1},\beta_{i+1}+2} = 0$. But the entries are in the Pascal triangle $(a_{l,j})_{\alpha_i \leq l \leq \alpha_{i+1}, j \leq l+a_{i+1}}$ of dimension a_i+b_i+1. But by Lemma 6, $a_i + b_i + 1 = 2a_i$ must be odd, which gives a contradiction. Next, if b_j is equal to a_j for some $i+1 \leq j \leq k-2$, then $b_{j+1} \geq a_{j+1}$ by the same argument as above. Therefore, by continuing this process we have inequality

$$\sum_{j=i}^{k-2} a_l \leq \sum_{l=i}^{k-2} b_l + 1.$$

Since $a_k \geq a_{k-1} + 1$, we have $b_{k-1} \geq a_{k-1}$ by Lemma 7. Therefore, we have inequality

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l + 1.$$

If $a_k \geq a_{k-1} + 2$ then we have $b_{k-1} \geq a_{k-1} + 1$ by Lemma 7. Therefore, we have inequality

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

and this inequality gives the proof of theorem. So we assume that a_k is equal to $a_{k-1} + 1$. Suppose that we have the inequality

$$\sum_{l=i}^{k-1} a_l > \sum_{l=i}^{k-1} b_l.$$

We want to show that b_k is greater than or equal to $a_k + 1$, which gives the inequality

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Assume that b_k is equal to a_k. Then by the ineqality in the above, *i.e.*

$$\sum_{j=i}^{k-1} a_j \leq \sum_{j=i}^{k-1} b_j + 1,$$

we have $b_l = a_l$ for all $i+1 \leq l \leq k-1$. Therefore, the string $(a_{\alpha_l,j})_{\alpha_l \leq j \leq \alpha_{l+1}}$ in the α_l^{th} row is $(010\ldots0110\ldots0)$ for each l. Since the entry $a_{\alpha_k-1,\beta_k}=0$, the string $(a_{\alpha_k-1,j})_{\alpha_k-1\leq j\leq\beta_k}$ in the $(\alpha_k-1)^{th}$ row is $(001\ldots1)$ by the Steinhaus property along with the α_k^{th} row. Thus we have the Pascal triangle $(a_{m,j})_{\alpha_{k-1}\leq m\leq\alpha_k-1, m\leq j}$ of dimension a_k whose a_k^{th} row is $(11\ \ldots\ 1)$. Hence a_k is a power of 2. Since $|a_k - a_{k-1}|$ is equal to 1 and b_{k-1} is equal to a_{k-1}, we conclude that a_{k-1} is a power of 2. This gives a contradiction. This proves that b_k is at least a_k+1. □

Now we prove the main theorem.

THEOREM 12. *Let G be a Steinhaus graph with n vertices and let C be a simple maximal induced cycle in G. Then the order of C is less than or equal to $\lfloor \frac{n+3}{2} \rfloor$.*

PROOF. Let $\{A_i\ :\ i=1,2,\ldots,t+1\} \cup \{B_i\ :\ i=1,2,\ldots,t\}$ be the cover of the Steinhaus G with the simple induced cycle C. Without loss of generality, we assume that a_1 is greater than or equal to a_{t+1} by considering its partner $P(G)$. Then a_{t+1} is equal to 1 by fact 4. It is enough to show that

$$\sum_{i=1}^{t} a_i \leq \sum_{i=1}^{t} b_i + 2,$$

because this inequality gives

$$\begin{aligned} 2\Big(\sum_{i=1}^{t+1} a_i\Big) &\leq \sum_{i=1}^{t+1} a_i + \sum_{i=1}^{t} b_i + 3 \\ &= n+3. \end{aligned}$$

SUBLEMMA. Let a_i be equal to 2 and b_i be equal to 1. Let i_0 be the smallest number such that $i_0 \geq i+1$, $a_{i_0} \geq 2$ and $a_j = 1$ for all $i+1 \leq j \leq i_0 - 1$. Then

$$\sum_{j=i}^{i_0-1} a_j \leq \sum_{j=i}^{i_0-1} b_j.$$

PROOF OF SUBLEMMA. Consider the subtriangle generated by string $(a_{\alpha_i,k})$ in the Steinhaus triangle of G where $\alpha_i \le k \le \beta_{i_0}$. Note that if $b_j \le 2$ for all $i \le j \le i_0 - 1$, then the generating string in the above subtriangle is $(0110\ldots0)$ by the Steinhaus property. Thus for $\alpha_i \le s \le \beta_{i_0}$, the pair $(a_{s,s+1}, a_{s,s+2})$ is given by

$$(a_{s,s+1}, a_{s,s+2}) = \begin{cases} (0,1) & \text{if} s - \alpha_i \text{is odd}, \\ (1,1) & \text{if} s - \alpha_i \text{is even}. \end{cases}$$

Assume that we have the inequality

$$\sum_{j=i}^{i_0-1} a_j > \sum_{j=i}^{i_0-1} b_j.$$

Then $b_j = 1$ for all $i+1 \le j \le i_0 - 1$. Since $a_{i_0} = 2$ and $b_{i_0-1} = 1$, we have $(a_{\alpha_{i_0}-1,\alpha_{i_0}}, a_{\alpha_{i_0}-1,\alpha_{i_0}+1}) = (1,1)$. Thus $\alpha_{i_0} - \alpha_i - 1$ is even. This gives a contradiction because $\alpha_{i_0} - \alpha_i$ is even.

Now, we claim the following inequality which we asked.

LEMMA.

$$\sum_{j=1}^{t} a_j \le \sum_{j=1}^{t} b_j + 2.$$

PROOF OF LEMMA. If t is equal to 1, then $a_1 \le b_1 + 2$. Also if $t = 2$, it is not difficult to show that

$$a_1 + a_2 \le b_1 + b_2 + 1$$

by considering all cases. From now on, we assume that $t \ge 3$. If $a_j \le 2$ for all $1 \le j \le t$, then we are done by Sublemma. Therefore, we assume that there exists j such that $a_j \ge 3$. Suppose that i is the largest number such that

$$\sum_{j=1}^{i} a_j \le \sum_{j=1}^{i} b_j + 1.$$

We want to show that i is equal to t. Suppose that i is less than t. If there is no $j \geq i$ such that $a_j = 1$, then we have

$$\sum_{j=i}^{t} a_j \leq \sum_{j=i}^{t} b_j$$

by applying either Lemma 10 or Lemma 11 sucessfully, which gives a contradiction to the choice of i. Therefore, there exists a smallest number k such that $k \geq i+1$ and $a_k = 1$. First, if $a_i, \ldots, a_{k-1}$ satisfy the conditions in Lemma 10, then $a_{k-1} \leq b_{k-1}$ and the string $(a_{\alpha_k,j})_{\alpha_k \leq j}$ in the α_k^{th} row is $(000\ldots1\ldots)$ because a_{k-1} is a power of 2. Thus $b_k \geq 2$, which gives a contradiction by the choice of i. Next, suppose that $a_i, \ldots, a_{k-1}$ satisfy the conditions in Lemma 11. Note that $b_{k-1} \geq a_{k-1}$ by Lemma 11. If there is some $k_0 > k$ such that $a_{k_0} \geq 2$ then

$$\sum_{l=i}^{s} a_l \leq \sum_{l=i}^{s} b_l$$

for some $s \geq j_0$ by Sublemma, which gives a contradiction by the choice of i. If $a_j = 1$ for all $k \leq j \leq t$, then i is equal to t, which gives a contradiction. Finally, if $a_l \leq 2$ for all $j \leq l \leq t$, then by applying Sublemma, we have a contradiction by the choice of i. By considering all cases, we prove the lemma.

By the above Lemma, we prove theorem. □

The proof of Theorem 12 shows that if $t \geq 2$ then the order of any induced cycle C can not achieve the upper bound. Therefore, we get the following:

COROLLARY 13. *Let G be a Steinhaus graph with n vertices and C be a simple induced cycle in G. If the order of C is $\lfloor \frac{1}{2}(n+3) \rfloor$ then C is either $\{1, 2, \ldots, \lfloor \frac{1}{2}(n+1) \rfloor, n\}$ or $\{1, n - \lfloor \frac{1}{2}(n+1) \rfloor, \ldots, n\}$.*

Now, we give an example of simple induced cycle which achieves the bound in the theorem. Let G be the Steinhaus graph with generating string $(a_{1,j})_{2 \leq j \leq n}$ given by

$$a_{1,j} = \begin{cases} 0 & \text{if} j = 3, 4, \cdots, \lceil \frac{n}{2} \rceil, \\ 1 & \text{otherwise.} \end{cases}$$

Then G has the induced cycle $\{1, 2, \ldots, \lceil \frac{n}{2} \rceil, n\}$ of order $\lfloor \frac{1}{2}(n+3) \rfloor$. We close by mentioning the size of maximal induced cycles in Steinhaus graphs. The question is that "Does the order of any induced cycles in Steinhaus graphs have a reasonable bound like in Theorem 12 ?". But for all $n \leq 30$, it is not difficult to show that the maximum size of an induced cycle in Steinhaus graphs with n vertices is $\lfloor \frac{n+3}{2} \rfloor$. Thus we give a conjecture.

CONJECTURE. The size of any induced cycles in a Steinhaus graph with n vertices is at most $\lfloor \frac{n+3}{2} \rfloor$.

References

1. C. Berge, *Graphs*, North-Holland, Amsterdam, 1985.
2. B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.
3. B. Bollobás, *Graph Theory*, Springer-Verlag, New York,, 1979.
4. R. Brigham and R. Dutton, *Distances and diameters in Steinhaus graphs*, Congressus Numerantium **76** (1990), 7-14.
5. R. Brigham, N. Deo and R. Dutton, *Some properties of Steinhaus graphs*, (preprint).
6. V. Chvatál and P. Erdős, *A note on Hamiltonian circuits*, Discrete Mathematics **2** (1972), 111-113.
7. W. Dymacek, *Cycles in Steinhaus graphs*, Congressus Numerantium **70** (1990), 41-45.
8. W. Dymacek, *Generating strings for bipartite Steinhaus graphs*, (preprint).
9. H. Harborth, *Solution of Steinhaus'problem with plus and minus signs*, J. of Combinatorial Theory, Ser. **A 122** (1972), 253-259.
10. J. Molluzo, *Steinhaus graphs*, Theory and Applications of Graphs (Y. Alavi and D. Lick ed.), Springer-Verlag, New York,, 1978.
11. R. Stanley, *Enumerative Combinatorics Volume I*, Wadsworth and Brooks/Cole Advanced Books and Software, Belmont, 1986.
12. H. Steinhaus, *One hundred Problems in Elementary Mathematics*, Elinsford, New York, 1963.

Department of Mathematics
Keimyung University
Taegu, 704-701 Korea