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A PARTIAL ORDERING OF WEAK POSITIVE QUADRANT DEPENDENCE

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ABSTRACT. A partial ordering is developed among weakly positive quadrant dependent(WPQD) bivariate random vectors. This permits us to measure the degree of WPQD-ness and to compare pairs of WPQD random vectors. Some properties and closures under certain statistical operations are derived. An application is made to measures of dependence such as Kendall's τ and Spearman's ρ.

1. Introduction

A bivariate random vector (X,Y) or its distribution H is positive quadrant dependent(PQD) if $P[X > x, Y > y] - P(X > x)P(Y > y) \geq 0$ for all x, y. Lehmann(1966) introduced this concept of positive quadrant dependence(PQD) together with some other dependence concepts. For review of some dependence concepts, one may consult Barlow and Proschan(1981) or Tong(1980). For recent literature one may also consult Jogdeo(1982).

Dependence concepts introduced in the literature are mostly stronger than positive quadrant dependence. Recently, Alzaid(1990) introduced a notion of weak positive quadrant dependence(WPQD) between two random variables. The importance of this concept of dependence lies in the fact that it is weaker than the positive quadrant dependence and it enjoys various properties. For many purposes, in addition to knowledge of the nature of dependence, it is also important to know the degree of WPQD-newss and to compare pairs of WPQD random vectors as to their WPQD-ness. Ahmed et al.(1979) have studied very extensively

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the partial ordering of positive quadrant dependence which permits us to compare pairs of positive quadrant dependent bivariate random vectors with specified marginals as to their PQD-ness. Kimeldorf, G. and Sampson, A.R(1987) presented a systematic basis for studying orderings of bivariate distributions according to their degree of positive dependence and introduced a general concept of a positive dependence ordering.

In this paper a partial ordering of weak positive quadrant dependence is developed to compare pairs of weakly positive quadrant dependent bivariate random vectors. The definitions and some basic properties of WPQD ordering are introduced in Section 2. Some preservation results of WPQD ordering are derived in Section 3. It is shown that the WPQD ordering is preserved under convolution, mixture of a certain type, transformation, and limit in distribution. We also make an application of WPQD ordering to some well known quantitative measures of dependence such as Kendall's τ and Spearman's ρ.

2. Preliminaries

In this section we present definitions, and basic facts used in the sequel.

DEFINITION 2.1. (Alzaid, 1990) A bivariate random vector (X, Y) or its distribution function H is said to be weakly positive quadrant dependent of the first type(WPQD1) (second type (WPQD2)) if for all x, y,

$$(2.1) \quad \int_x^\infty \int_y^\infty [P(X > u, Y > t) - P(X > u)P(Y > t)]dtdu \geq 0.$$

$$(2.1)' \quad \left(\int_{-\infty}^x \int_{-\infty}^y [P(X > u, Y > t) - P(X > u)P(Y > t)]dtdu \geq 0\right)$$

A bivariate random vector (X, Y) or its distribution H is said to be weakly positive quadrant dependent(WPQD) if it is WPQD1 and WPQD 2. Let $\beta = \beta(F, G)$ denote the class of bivariate distribution functions H on $\mathbb{R}^2$ having marginals F and $G, \bar{H}(x, y) = P(X > x, Y > y)$, and $\bar{\beta}$ be the subclass of β where H is WPQD.

DEFINITION 2.2. Let H_1 and H_2 both blong to β. The bivariate random vector (X_1, X_2) or its distribution H_1 is said to be more weakly positive quadrant dependent of the first type(second type) than the bivariate random vector (Y_1, Y_2) or its distribution H_2 if for all x, y,

$$\int_x^\infty \int_y^\infty \bar{H}_1(u,t)dtdu \geq \int_x^\infty \int_y^\infty \bar{H}_2(u,t)dtdu. \tag{2.2}$$

$$\left(\int_{-\infty}^x \int_{-\infty}^y \bar{H}_1(u,t)dtdu \geq \int_{-\infty}^x \int_{-\infty}^y \bar{H}_2(u,t)dtdu.\right) \tag{2.2$'$}$$

where the integrals exist.
We write $H_1 \geq {}^{WPQD1} H_2$ or $(X_1, X_2) \geq {}^{WPQD1} (Y_1, Y_2) (H_1 \geq {}^{WPQD2} H_2$ or $(X_1, X_2) \geq {}^{WPQD2} (Y_1, Y_2))$. We say that the bivariate random vector (X_1, X_2) or its distribution H_1 is more weakly positive quadrant dependent than (Y_1, Y_2) or its distribution H_2 if H_1 is more WPQD1 than H_2 and more WPQD2 than H_2. We write $H_1 \geq {}_{WPQD} H_2$ or $(X_1, X_2) \geq {}_{WPQD} (Y_1, Y_2)$.

REMARK. An equivalent form of (2.2) ((2.2)$'$) is

$$\int_x^\infty \int_y^\infty H_1(u,t)dtdu \geq \int_x^\infty \int_y^\infty H_2(u,t)dtdu. \tag{2.3}$$

$$\left(\int_{-\infty}^x \int_{-\infty}^y H_1(u,t)dtdu \geq \int_{-\infty}^x \int_{-\infty}^y H_2(u,t)dtdu\right) \tag{2.3$'$}$$

where the integrals exist.

PROOF. Since $\bar{H}_i(s,t) = 1 - F(s) - G(t) + H_i(s,t)$ for $i = 1, 2$, $\bar{F} = 1 - F$ and $\bar{G} = 1 - G \bar{H}_i(u,t) - \bar{F}(u)\bar{G}(t) = H_i(u,t) - F(u)G(t)$ for $= 1, 2$. Hence

$$\begin{aligned}
&\int_x^\infty \int_y^\infty \bar{H}_1(u,t)dtdu \geq \int_x^\infty \int_y^\infty \bar{H}_2(u,t)dtdu \\
&\Rightarrow \int_x^\infty \int_y^\infty [\bar{H}_1(u,t) - \bar{F}(u)\bar{G}(t)]dtdu \\
&\qquad\qquad \geq \int_x^\infty \int_y^\infty [\bar{H}_2(u,t) - \bar{F}(u)\bar{G}(t)]dtdu
\end{aligned}$$

$$\Rightarrow \int_x^\infty \int_y^\infty [H_1(u,t) - F(u)G(t)]dtdu$$
$$\geq \int_x^\infty \int_y^\infty [H_2(u,t) - F(u)G(t)]dtdu$$
$$\Rightarrow \int_x^\infty \int_y^\infty H_1(u,t)dtdu \geq \int_x^\infty \int_y^\infty H_2(u,t)dtdu$$

PROPERTY P1. Let H_1 and H_2 belong to $\bar{\beta}$. Assume that H_1 is more positively quadrant dependent than H_2. Then H_1 is more weakly positive quadrant dependent than H_2.

PROOF. From the definition of positive quadrant dependent ordering(See Ahmed et al.(1979)) and Definition 2.2 the result follows.

From Defintion 2.2 we have the following property:

PROPERTY P2. Let H_1, H_2, an H_3 belong to $\bar{\beta}$. Then $H_1 \geq^{WPQD} H_2$ and $H_2 \geq^{WPQD} H_3$ imply $H_1 \geq^{WPQD} H_3$.

PROPERTY P3. Let (X,Y) and (U,V) have distributions H_1 and H_2, respectively where H_1 and H_2 belong to $\bar{\beta}$. Assume (X,Y) is more WPQD than (U,V). Then (Y,X) is more WPQD than (V,U).

PROOF. First, note that both (Y,X) and (V,U) have marginals G and F.

$$\int_y^\infty \int_x^\infty P(Y \leq u, X \leq t)dtdu = \int_x^\infty \int_y^\infty P(X \leq t, Y \leq u)dudt$$
$$\geq \int_x^\infty \int_y^\infty P(U \leq t, V \leq u)dudt = \int_y^\infty \int_x^\infty P(V \leq u, U \leq t)dtdu.$$

Hence $(Y,X) \geq^{WPQD1} (V,U)$. Similarly, $(Y,X) \geq^{WPQD2} (V,U)$ and the proof is complete.

PROPERTY P4. Let H_1, H_2 belong to $\bar{\beta}$. Assume that H_1 is more WPQD than H_2. Then, for $0 \leq a \leq 1$,

$$H_1 \geq^{WPQD} aH_1 + (1-a)H_2 \geq^{WPQD} H_2. \tag{2.4}$$

PROOF. For $a = 0, 1$ it is clear that (2.4) holds. For $0 < a < 1$,

$$\lim_{x \to \infty} \{aH_1(x,y) + (1-a)H_2(x,y)\} = G(y),$$
$$\lim_{y \to \infty} \{aH_1(x,y) + (1-a)H_2(x,y)\} = F(x),$$

Since $aH_1 + (1-a)H_2$ is clearly WPQD, it belongs to $\bar{\beta}$. Next note that

$$\begin{aligned} \int_x^\infty \int_y^\infty H_1(s,t)dtds &= \int_x^\infty \int_y^\infty [aH_1(s,t) + (1-a)H_1(s,t)]dtds \\ &\geq \int_x^\infty \int_y^\infty [aH_2(s,t) + (1-a)H_2(s,t)]dtds \\ &= \int_x^\infty \int_t^\infty H_2(s,t)dtds. \end{aligned}$$

Thus $H_1 \geq {}^{WPQD1} aH_1 + (1-a)H_2 \geq {}^{WPQD1} H_2$, for $0 \leq a \leq 1$. Similary, $H_1 \geq {}^{WPQD2} aH_1 + (1-a)H_2 \geq {}^{WPQD2} H_2$, for $0 \leq a \leq 1$. Thus the proof is complete.

DEFINITION 2.3. A family of WPQD distributions $\{H_\lambda(x,y) | \lambda \in \Lambda \subset R\}$ is said to be increasingly WPQD in λ if $\lambda' > \lambda$ implies $H_{\lambda'} \geq {}^{WPQD} H_\lambda$.

EXAMPLE 2.4. A bivariate family of $H_\lambda(x,y), 0 < \lambda < 1$ is increasingly WPQD in λ, where $H_\lambda(x,y) = \lambda H(x,y) + (1-\lambda)FG$ and $H \in \bar{\beta}$. It is celar that $H_\lambda \subset \bar{\beta}$ by Property P4. For $0 < \lambda_1 < \lambda_2 < 1$

$$\begin{aligned} &\int_{-\infty}^x \int_{-\infty}^y [\lambda_2 H(u,t) + (1-\lambda_2)F(u)G(t) - F(u)G(t)]dtdu \\ &= \int_{-\infty}^x \int_{-\infty}^y \lambda_2 [H(u,t) - F(u)G(t)]dtdu \\ &\geq \int_{-\infty}^x \int_{-\infty}^y \lambda_1 [H(u,t) - F(u)G(t)]dtdu \\ &= \int_{-\infty}^x \int_{-\infty}^y [\lambda_1 H(u,t) + (1-\lambda_1)F(u)G(t) - F(u)G(t)]dtdu \end{aligned}$$

which yields

$$\int_{-\infty}^{x}\int_{-\infty}^{y}[\lambda_2 H(u,t)+(1-\lambda_2)F(u)G(t)]dtdu$$
$$\geq \int_{-\infty}^{x}\int_{-\infty}^{y}[\lambda_1 H(u,t)+(1-\lambda_1)F(u)G(t)]dtdu.$$

Thus $H_\lambda(x,y)$ is increasingly WPQD1 in λ. Similarly, $H_\lambda(x,y)$ is increasingly WPQD2 in λ and the proof is complete.

3. Some Preservation Results with Application

In this section we establish preservation of the WPQD ordering under combination, mixture, transformations of random variables by increasing function, limit in distribution and other operation in statistics.

LEMMA 3.1. *Let $\underline{X}=(X_1,X_2)$ and $\underline{Y}=(Y_1,Y_2)$ have distributions H_1 and H_2, respectively, where H_1 and H_2 belong to β such that $H_1 \geq^{WPQD} H_2$ and let $\underline{Z}=(Z_1,Z_2)$ with an arbitrary WPQD distribution function H be independent of both $\underline{X}$ and $\underline{Y}$. Then $\underline{X}+\underline{Z} \geq^{WPQD} \underline{Y}+\underline{Z}$.*

PROOF. First we will show that $\underline{X}+\underline{Z}$ and $\underline{Y}+\underline{Z}$ and WPQD. For nonnegative increasing convex functions f, g

$$\begin{aligned}&\mathrm{Cov}(f(X_1+Z_1),g(X_2+Z_2))\\&=\mathrm{Cov}[E\{f(X_1+Z_1)|\underline{Z}\},E\{g(X_2+Z_2)|\underline{Z}\}]\\&+E_{\underline{Z}}\{\mathrm{Cov}(f(X_1+Z_1),g(X_2+Z_2)|\underline{Z})\}\end{aligned}$$

observe that according to Theorem 3 of Alzaid(1990) the second term of the right hand side of the above equation is nonnegative while the conditional expectations in the first term have the same properties in Z_1 and Z_2 as do the functions f and g, and the first term of the right hand side is also nonnegative. Thus $\underline{X}+\underline{Z}$ is WPQD1.
A similar result holds for the WPQD2 and thus $\underline{X}+\underline{Z}$ is WPQD. Similarly, it is proved that $\underline{Y}+\underline{Z}$ is also WPQD. Next, we need show for

each $(v,w) \in \mathbb{R}^2$,

$$\int_v^\infty \int_w^\infty P[X_1 + Z_1 > s, X_2 + Z_2 > t]dtds \geq \int_v^\infty \int_w^\infty P[Y_1 + Z_1 > s, Y_2 + Z_2 > t]dtds \tag{3.1}$$

$$\int_{-\infty}^v \int_{-\infty}^w P[X_1 + Z_1 > s, X_2 + Z_2 > t]dtds \geq \int_{-\infty}^v \int_{-\infty}^w P[Y_1 + Z_1 > s, Y_2 + Z_2 > t]dtds \tag{3.2}$$

Note that

$$\begin{aligned} &\text{left side of } (3.1) \\ &= \int_v^\infty \int_w^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty P[X_1 > s - z_1, X_2 > t - z_2]dH(z_1, z_2)dtds \\ &\geq \int_v^\infty \int_w^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty P[Y_1 > s - z_1, Y_2 > t - z_2]dH(z_1, z_2)dtds \\ &= \int_v^\infty \int_w^\infty P[Y_1 + Z_1 > s, Y_2 + Z_2 > t]dtds = \text{right side of } (3.1). \end{aligned}$$

The above inequality follows from the assumption that $\underline{X} \geq^{WPQD} \underline{Y}$. Similarly, (3.2) is proved. Thus we complete the proof.

THEOREM 3.2. *Let (X_i, Y_i) and (U_i, V_i) be WPQD for $i = 1, 2$. Let (X_i, Y_i) be more WPQD than (U_i, V_i) for $i = 1, 2$. Further, let (X_1, Y_1) and (X_2, Y_2) be independent, (U_1, V_1) and (U_2, V_2) independent, and (X_2, Y_2) and (U_1, V_1) independent. Then*

$$(X_1 + X_2, Y_1 + Y_2) \geq^{WPQD} (U_1 + U_2, V_1 + V_2)$$

PROOF. By assumption, $(X_1, Y_1) \geq^{WPQD} (U_1, V_1)$. Specifying $\underline{Z}$ to be (X_2, Y_2), we apply Lemma 3.1 to obtain

$$(X_1 + X_2, Y_1 + Y_2) \geq^{WPQD} (U_1 + X_2, V_1 + Y_2). \tag{3.3}$$

Next, we use the assumption $(X_2, Y_2) \geq^{WPQD} (U_2, V_2)$, specify $\underline{Z}$ to be (U_1, V_1), and again use Lemma 3.1, yielding

$$(3.4) \qquad (U_1 + X_2, V_1 + Y_2) \geq^{WPQD} (U_1 + U_2, V_1 + V_2).$$

By combining (3.3) and (3.4), we complete the proof according to P2. From Theorem 3 of Alzaid(1990) it follows that (X_1, X_2) is more WPQD1 (WPQD2) than (Y_1, Y_2) if and only if

$$(3.5) \qquad \mathrm{Cov}_{H_1}(f(X_1), g(X_2)) \geq \mathrm{Cov}_{H_2}(f(Y_1), g(Y_2))$$

for all increasing nonnegative convex(nonpositive concave) functions f and g, where H_1 and H_2 are the distributions of (X_1, X_2) and (Y_1, Y_2) respectively and H_1 and H_2 belong to $\bar{\beta}$.

THEOREM 3.3. *Let* $\underline{X} = (X_1, X_2), \underline{Y} = (Y_1, Y_2)$ *have distributions* H_1 *and* H_2*,where* H_1 *and* H_2 *belong to* $\bar{\beta}$ *so that* $(X_1, X_2) \geq^{WPQD1} (Y_1, Y_2)$. *Then* $(f(X_1), X_2) \geq^{WPQD1} (f(Y_1), Y_2))$ *for all increasing nonnegative convex function* f.

PROOF. Let h and g be increasing, nonnegative convex functions. Since hf is an increasing nonnegative convex function for all increasing nonnegative convex function f
$\mathrm{Cov}_{H_1}(h(f(X_1)), g(X_2)) \geq \mathrm{Cov}_{H_2}(h(f(Y_1)), g(Y_2))$ according to (3.5). Hence $(f(X_1), X_2) \geq^{WPQD1} (f(Y_1), Y_2)$ and the proof is complete.

COROLLARY 3.4. *Let* $\underline{X} = (X_1, X_2)$ *and* $\underline{Y} = (Y_1, Y_2)$ *have distributions* H_1 *and* H_2 *respectively, where* H_1 *and* H_2 *belong to* $\bar{\beta}$, *such that* $(X_1, X_2) \geq^{WPQD1} (Y_1, Y_2)$. *Then*

$$(f(X_1), g(X_2)) \geq^{WPQD1} (f(X_1), g(Y_2))$$

for all increasing nonegative convex functions f *and* g.

DEFINITION 3.5.(EBRAHIMI, GHOSH, 1981). A random vector $\underline{Y}$ is stochastically increasing in the random vector $\underline{X}$ if $E(f(\underline{Y})|\underline{X} = \underline{x})$ is increasing in $\underline{x}$ for all real valued increasing function f. We shall use the abbreviation SI for stochastically increasing.

The following theorem gives a sufficient condition for WPQD-ness.

THEROEM 3.6. *Let (a) (X_1, X_2) given λ, a scalar random variable, be conditionally WPQD, and (b) X_1 and X_2 be SI in λ. Then (X_1, X_2) is WPQD.*

PROOF. Observe that

$$\operatorname{Cov}[f(X_1), g(X_2)] = E[\operatorname{Cov}\{f(X_1), g(X_2)|\lambda\}] + \operatorname{Cov}[E\{f(X_1)|\lambda\}, E\{g(X_2)|\lambda\}] \tag{3.6}$$

for increasing nonnegative convex(nonpositive concave) functions f and g. Since conditioned on λ (X_1, X_2) is WPQD1(WPQD2) by Theorem 3 of Alzaid(1990) the first term on the right hand side of (3.6) is nonnegative. From Definition 3.5 the conditional expectations in the second term on the right hand side of (3.6) are increasing functions.
Since λ is associated, according P_4 of Esary, et al.(1967), the covariance of the conditional expectations in the second term is nonnegative. It follows that $\operatorname{Cov}[f(X_1), g(X_2)] \geq 0$. Thus WPQD1(WPQD2).

We may now define the class $\bar{\beta}_\lambda$ by

$$\bar{\beta}_\lambda = \{H_\lambda : H(x, \infty|\lambda) = F(x|\lambda), H(\infty, y|\lambda) = G(y|\lambda), H_\lambda|\lambda \text{is WPQD, and both } F \text{ and } G \text{are SI in} \lambda\}$$

The following theorem shows that if two elements of $\bar{\beta}_\lambda$ are ordered accordering to $\geq^{WPQD}$, than after mixing on λ, the resulting elements in $\bar{\beta}$ preserve the same order.

THEOREM 3.7. *Let $(X_1, X_2)|\lambda$ and $(Y_1, Y_2)|\lambda$ belong to $\bar{\beta}_\lambda$. Assume $(X_1, X_2)|\lambda \geq^{WPQD} (Y_1, Y_2)|\lambda$. Then, unconditionally*

$$(X_1, X_2) \geq^{WPQD} (Y_1, Y_2)$$

PROOF. From Theroem 3.6 (X_1, X_2) and (Y_1, Y_2) are WPQD. For showing $(X_1, X_2) \geq^{WPQD} (Y_1, Y_2)$ we have to show (3.5), i.e.
$E(f(X_1), g(X_2)) \geq E(f(Y_1)g(Y_2))$ for all increasing, non-negative convex (non-positive concave) functions f and g. Now

$$\begin{aligned} E(f(X_1), g(X_2)) &= E_\lambda\{E(f(X_1)g(X_2)|\lambda)\} \\ &\geq E_\lambda\{E(f(Y_1)g(Y_2)|\lambda)\} = E(f(Y_1)g(Y_2)). \end{aligned}$$

Thus $(X_1, X_2) \geq^{WPQD} (Y_1, Y_2)$ and the proof is complete.

LEMMA 3.8. *Suppose* $(U, V) \leq^{WPQD} (X, Y)$. *Let* $(Z_1, Z_2), (X, Y)$ *and* (U, V) *be independent random vectors and* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *increasing. Then* $(f(U, Z_1), g(V, Z_2)) \leq^{WPQD} (f(X, Z_1), g(Y, Z_2))$.

PROOF. By the monotonicity of f and g, the set $\{(u, a) : f(u, z_1) \leq a, g(v, z_1) \leq b\}$ is a lower rectangle and hence

$$\begin{aligned}
&\int_\alpha^\infty \int_\beta^\infty \mathrm{P}\,[f(U, Z_1) \leq a, g(V, Z_2) \leq b] dbda \\
&= \int_\alpha^\infty \int_\beta^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \mathrm{P}\,[f(U, z_1) \leq a, g(V, z_2) \leq b] dH(z_1, z_2) dbda \\
&\leq \int_\alpha^\infty \int_\beta^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \mathrm{P}\,[f(X, z_1) \leq a, g(Y, z_2) \leq b] dH(z_1, z_2) dbda \\
&= \int_\alpha^\infty \int_\beta^\infty \mathrm{P}\,[f(X, Z_1) \leq a, g(Y, Z_2) \leq b] dbda,
\end{aligned}$$

where H is the distribuion of (Z_1, Z_2). Thus $(f(U, Z_1), g(V, Z_2) \leq^{WPQD1} (f(X, Z_1), g(Y, Z_2))$. Similarly, $(f(U, Z_1), g(V, Z_2)) \leq^{WPQD2} (f(X, Z_1), g(Y, Z_2))$ and the proof is complete.

THEOREM 3.9. *Suppose* $(U, V_1)) \leq^{WPQD} (X_1, Y_1)$ *and* $(U_2, V_2) \leq^{WPQD} (X_2, Y_2)$. *Then for independent random vectors* $(X_1, Y_1), (X_2, Y_2), (U_1, V_1)(U_2, V_2)$ *and for increasing functions* f *and* g,

$$(f(U_1, U_2), g(V_1, V_2)) \leq^{WPQD} (f(X_1, X_2), g(Y_1, Y_2)). \tag{3.7}$$

PROOF. For increasing functions f and g define $f'(s, t) = f(t, s)$ and $g'(s, t) = g(t, s)$. Then f' and g' are also increasing. Apply Lemma 3.8 to deduce that $(f(X_1, X_2), g(Y_1, Y_2))$ is more WPQD than $(f(U_1, X_2), g(V_1, Y_2)) = (f'(X_2, U_1), g'(Y_2, V_1))$ which is more WPQD than $(f'(U_2, U_1), g'(V_2, V_1) = (f(U_1, U_2), g(V_1, V_2))$ by Lemma 3.8.

THEOREM 3.10. *Let* $\{(X_n, Y_n) : n \geq 1\}$ *and* $\{(U_n, V_n) : n \geq 1\}$ *be sequences of nonnegative WPQD1 bivariate random vectors with same marginals. Assume* (i) $(X_n, Y_n) \geq^{WPQD1} (U_n, V_n)$, (ii) $Cov(X_n, Y_n) \to Cov(X, Y)$, (iii) $Cov(U_n, V_n) \to Cov(U, V)$, (iv) (X_n, Y_n) *is weakly convergent to* (X, Y), (v) (U_n, V_n) *is weakly convergent to* (U, V). *Then* $(X, Y) \geq^{WPQD1} (U, V)$

PROOF. Observe that

$$\mathrm{Cov}(X_n, Y_n) - \int_0^x \int_0^y [P(X_n > s, Y_n > t) - P(X_n > s)P(Y_n > t)]dtds$$

$$= \int_x^\infty \int_y^\infty [P(X_n > s, Y_n > t) - P(X_n > s)P(Y_n > t)dtds$$

$$\geq \int_x^\infty \int_y^\infty [P(U_n > s, V_n > t) - P(U_n > s)P(V_n > t)]dtds =$$

$$\mathrm{Cov}(U_n, Y_n) - \int_0^x \int_0^y [P(U_n > s, V_n > t) - P(U_n > s)P(V_n > t)]dtds.$$

Taking the limit and using the dominated convergence theorem and the assumptions of theorem concerning the convergences of $\mathrm{Cov}(X_n, Y_n)$ and $\mathrm{Cov}(U_n, V_n)$ we get the required result.

THEOREM 3.11. *Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be independent random vectors having WPQD1 distribution H_1, and let $(U_1, V_1), \cdots, (U_n, V_n)$ be independent random vectors having WPQD1 distribution H_2. Let f and g be nonnegative increasing convex for the ith coordinate $(i = 1, \cdots, n)$. Assume that $H_1 \geq^{WPQD1} H_2$. Then*

$$\begin{aligned} &Cov_{H_1}[f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n)] \\ &\geq Cov_{H_2}[f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n)]. \end{aligned} \tag{3.8}$$

PROOF. According to Theorem 5 in Alzaid(1990) $(f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n))$ is WPQD1 and $(f(U_1, \cdots, U_n), g(V_1, \cdots, V_n))$ is WPQD1. Thus (3.8) holds according to (3.5).

Among the most familiar measure of dependence there are (i) Spearman's $\rho, \rho(X, Y) = 3Cov(sgn(X_2 - X_1), (sgn(Y_3 - Y_1))$, and (ii) Kendall's $\tau, \tau(X, Y) = Cov(sgn(X_2 - X_1), sgn(Y_2 - Y_1))$ where $(X_1, Y_1), (X_2, Y_2)$ and (X_3, Y_3) are independent random vectors having the same distribution as (X, Y) . An application of Theorem 5 of Alzaid(1990) implies that all of the above measures are nonnegative under the weaker assumption that (X, Y) is WPQD1. An immediate consequence of Theorem 3.11 is the following example:

EXAMPLE 3.12. Let H_1 and H_2 be such that $H_1 \geq^{WPQD1} H_2$. Then Kendalls τ, and Spearman's ρ satisfy $\tau_{H_1} \geq \tau_{H_2}$ and $\rho_{H_1} \geq \rho_{H_2}$.

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