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ENUMERATING EMBEDDINGS OF A DARTBOARD GRAPH INTO SURFACES

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ABSTRACT. We enumerate the congruence classes of 2-cell embeddings of a dartboard graph into surfaces with respect to a group consisting of graph automorphisms of a dartboard graph.

1. Introduction and Preliminaries

Let G be a finite connected simple graph with a vertex set $V(G)$ and an edge set $E(G)$. Let $\mathrm{Aut}\,(G)$ denote the group of automorphisms of G and let $N(v)$ denote the neighborhood of a vertex $v \in V(G)$, *i.e.*,the set of all vertices adjacent to v. Any graph G can be regarded as a topological space in the following sense: Each vertex is represented by a distinct point and each edge by a distinct arc, homeomorphic to a closed interval, so that the boundary points of an arc represent the endpoints of the corresponding edge. Two graphs G and H are homeomorphic if and only if they have respective subdivisions G' and H' such that G' and H' are isomorphic graphs. Throughout this paper, all surfaces means closed connected surfaces that are either orientable or nonorientable.

An *embedding* of a graph G into a surface S is a continuous one-to-one function $i : G \to S$. It can be shown that for any embedding $i : G \to S$, $i(G)$ is contained in the 1-skeleton of a triangulation of the surface S and the graph embeddings in surfaces can be analyzed by combinatorial methods. If every component of $S - i(G)$, called a *region*, is homeomorphic to an open disk, then the embedding $i : G \to S$ is called a *2-cell embedding.* Every embedding treated in this paper is a 2-cell embedding. Two 2-cell embeddings i, j of a graph G into a surface

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S are said to be *congruent with respect to a subgroup* Γ of $\text{Aut}(G)$ if there are a homeomorphism $h : S \to S$ and an automorphism $\gamma \in \Gamma$ such that $h \circ i = j \circ \gamma$. If two embeddings are congruent with respect to $\text{Aut}(G)$, we say that they are *congruent*. If the surface S is oriented and the surface homeomorphism h preserves an orientation, we call it *oriented congruence*. Let $|\mathcal{C}_\Gamma(G)|$ denote the number of the congruence classes of 2-cell embeddings of a graph G into surfaces with respect to a subgroup Γ of $\text{Aut}(G)$ and $|\mathcal{C}_{\text{Aut}(G)}(G)|$ is written as $|\mathcal{C}(G)|$.

The oriented congruence of a graph G into oriented surfaces with respect to the full automorphism group of G was enumerated by Mull *et al* [5]. Recently, Kwak and Lee [3] gave some algebraic characterizations for congruence and formulas for enumerating the congruence classes of 2-cell embeddings of a graph G into surfaces, and the congruence classes of 2-cell embeddings of complete graphs were enumerated. In this paper, we enumerate the congruence classes of 2-cell embeddings of a dartboard graph into surfaces with respect to a subgroup of the automorphism group of a dartboard graph.

An *embedding scheme* (ρ, λ) for G consists of a *rotation scheme* ρ which assigns a cyclic permutation ρ_v on $N(v)$ to each $v \in V(G)$ and a *voltage map* λ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{1, -1\}$ to each $e \in E(G)$.

Stahl [7] showed that every embedding scheme for a graph G determines a 2-cell embedding of a graph into a surface S, and every 2-cell embedding of th graph into a surface is determined by such a scheme. Kwak and Lee [3] gave the following.

LEMMA 1.1. *Let (ρ, λ) and (τ, μ) be two embedding schemes for a graph G with the corresponding embeddings $i : G \to S$ and $j : G \to S$ respectively, and let Γ be a subgroup of $\text{Aut}(G)$. Then these two embeddings i, j are congruent with respect to Γ if and only if there are $\gamma \in \Gamma$ and a function $f : V(G) \to \mathbb{Z}_2$ such that $\tau_{\gamma(v)} = \gamma \circ (\rho_v)^{f(v)} \circ \gamma^{-1}$ and $\mu(\gamma(e)) = f(u)\lambda(e)f(v)$ for all $e = uv \in E(G)$.*

Suppose that any graph automorphism γ of a graph G is given. The subgroup $< \gamma >$ of $\text{Aut}(G)$ generated by γ acts on the vertex set $V(G)$ and the edge set $E(G)$ of G, and hence it gives a new graph G_γ with vertex set $V(G_\gamma) = \{[v] \mid v \in V(G)\}$ and edge set $E(G_\gamma) = \{[e] \mid e \in E(G)\}$, where $[v]$, $[e]$ are the orbits of v and e respectively under the

$<\gamma>$ action on $V(G)$ and $E(G)$.

Now, for any $\gamma \in \mathrm{Aut}(G)$ and $v \in V(G)$, let

$$P_{(v;\gamma)} = \{\sigma | \sigma \text{is a cycle permutation on} N(v) \text{and} \gamma^{|[v]|} \circ \sigma \circ \gamma^{-|[v]|} = \sigma\}$$

and

$$I_{(v;\gamma)} = \{\sigma | \sigma \text{is a cycle permutation on} N(v) \text{and} \gamma^{|[v]|} \circ \sigma \circ \gamma^{-|[v]|} = \sigma^{-1}\}.$$

Let $\mathrm{j}(\sigma) = (\mathrm{j}_1, \cdots, \mathrm{j}_n)$ denote the cycle type of a permutation σ in S_n, where j_k is the number of cycles of length k in a factorization of σ into disjoint cycles. As usual, $\phi(n)$ represents the value of n under the Euler phi-function.

According to the cycle type of $\gamma \in \mathrm{Aut}(G)$, $|P_{(v;\gamma)}|$ and $|I_{(v;\gamma)}|$ were computed as follows ([3], [5]), where $|X|$ represents the cardinality of a set X.

LEMMA 1.2. *If* $[v] \in V(G_\gamma)$ *and* $|N(v)| = n$, *then*

$$|P_{(v;\gamma)}| = \begin{cases} \phi(d)(\frac{n}{d} - 1)! d^{n/d-1} & \text{if } \mathrm{j}(\gamma^{|[v]|}|_{N(v)}) = (0, \cdots, 0, \mathrm{j}_d = \frac{n}{d}, 0, \cdots, 0) \\ 0 & \text{otherwise.} \end{cases}$$

and

$$|I_{(v;\gamma)}| = \begin{cases} (\frac{n-1}{2})! 2^{(n-1)/2} & \text{if } n \text{ is odd and } \mathrm{j}(\gamma^{|[v]|}|_{N(v)}) = (1, \frac{n-1}{2}, 0, \cdots, 0), \\ (\frac{n}{2})! 2^{n/2-1} & \text{if } n \text{ is even and } \mathrm{j}(\gamma^{|[v]|}|_{N(v)}) = (0, \frac{n}{2}, 0, \cdots, 0), \\ (\frac{n}{2} - 1)! 2^{n/2-1} & \text{if } n \text{ is even and } \mathrm{j}(\gamma^{|[v]|}|_{N(v)}) = (2, \frac{n}{2} - 1, 0, \cdots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

2. Congruence classes of 2-cell embeddings of a dartboard graph

For positive integers n (≥ 3) and k, a *dartboard graph* is a simple graph formed from k concentric cycles, a central vertex and n line segments joining the central vertex and the outmost circle as shown in Figure 1. The resulting graph is written as $W_{n,k}$. In particular, $W_{n,1}$ is a *wheel* . We label the vertices of the concentric cycles in $W_{n,k}$ as v_{11}, v_{21}, $\cdots$,

$v_{n1}, \cdots, v_{1k}, v_{2k}, \cdots, v_{nk}$ and the central vertex as v_0 so that the edges of $W_{n,k}$ are v_0v_{i1}, $v_{il}v_{jl}$ and $v_{il}v_{im}$, where $1 \le i, j \le n$, $1 \le l, m \le k$, $j \equiv i + 1 (\bmod n)$ and $m \equiv l + 1 (\bmod k)$ as Figure 1. The graph $W_{3,1}$ is the complete graph K_4 and $\mathrm{Aut}(W_4) = S_4$. The number of noncongruent 2-cell embeddings is 11 ([3]). From now on, we assume that $n \ge 3, k \ge 1$ and $n + k \ge 5$. Let $T_{n,k}$ be the spanning tree of $W_{n,k}$ with vertices and edges on the n line segments. The automorphism group of $W_{n,k}$ is naturally isomorphic to the symmetry group $\mathbb{D}_n$ of the regular n-gon with vertices $1, \cdots, n$ and so it is considered as $\mathbb{D}_n$. The identity automorphism of $\mathrm{Aut}(W_{n,k})$ is written as I. Note that every automorphism in $\mathrm{Aut}(W_{n,k})$ fixes $T_{n,k}$ and v_0. Let $\sigma = (12 \cdots n)$ be a fixed generator of the cyclic subgroup of order n of $\mathbb{D}_n$. The cyclic subgroup of $\mathbb{D}_n$ generated by σ is regarded as $\mathbb{Z}_n$.

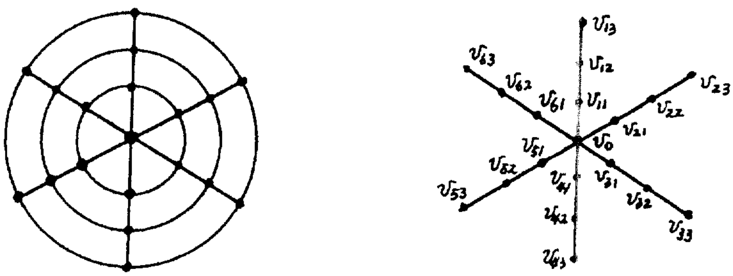

FIGURE 1. A dartboard graph $W_{6,3}$ and its spanning tree $T_{6,3}$.

We first devide $\mathbb{D}_n$ into the subsets A_d, B_i, where $d|n$ and $i = 0, 1, 2$, as follows:

$$A_d = \{\sigma^k \mid \gcd(k, n) = n/d,\ 1 \le k \le n\},$$

$$B_0 = \{\gamma \in \mathbb{D}_n \mid \mathrm{j}(\gamma) = (0, n/2, 0, \cdots, 0), \gamma \ne \sigma^{n/2}\},$$

$$B_1 = \{\gamma \in \mathbb{D}_n \mid \mathrm{j}(\gamma) = (1, (n-1)/2, 0, \cdots, 0)\},$$

$$B_2 = \{\gamma \in \mathbb{D}_n \mid \mathrm{j}(\gamma) = (2, n/2 - 1, 0, \cdots, 0)\}.$$

Then, $\mathrm{Aut}(W_{n,k})$ is $\cup_{d|n} A_d \cup B_1$ if n is odd and $\cup_{d|n} A_d \cup B_0 \cup B_2$ if n is even.

Let $\mathcal{E}_T(n, k)$ denote the set of all embedding schemes (ρ, λ) for $W_{n,k}$ with $\lambda(e) = 1$ for all $e \in E(T_{n,k})$. Now, we introduce an useful group action on $\mathcal{E}_T(n, k)$ which will give an equivalence relation characterizing the congruence classes in the 2-cell embeddings of $W_{n,k}$. The group action of $\Gamma \times \mathbb{Z}_2$ on $\mathcal{E}_T(n,k)$ is defined as follows:

$$(\gamma,\alpha)(\rho,\lambda) = ((\gamma,\alpha)\rho, \lambda_\gamma)$$

for any $(\gamma,\alpha) \in \Gamma \times \mathbb{Z}_2$ and $(\rho,\lambda) \in \mathcal{E}_T(n,k)$, where

$$[(\gamma,\alpha)\rho]_v = \gamma \circ (\rho_{\gamma^{-1}(v)})^\alpha \circ \gamma^{-1} \quad \text{and} \quad \lambda_\gamma(e) = \lambda(\gamma^{-1}(e))$$

for any $v \in V(W_{n,k})$ and $e \in E(W_{n,k})$.

Let $\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,\alpha)}$ denote the set $\{(\rho,\lambda) \in \mathcal{E}_T(n,k) \mid (\gamma,\alpha)(\rho,\lambda) = (\rho,\lambda)\}$.

For $\gamma \in A_d$, let

$$V_{d,j} = \{[v] \in V(W_{n,k_\gamma}) - \{v_0\} \mid |[v]| = d, |N(v)| = j\},$$

where $j = 3, 4$. For $\gamma \in B_i$, let

$$V^i_{l,j} = \{[v] \in V(W_{n,k_\gamma}) - \{v_0\} \mid |[v]| = l, |N(v)| = j\},$$

where l =1, 2 and j = 3, 4.

We observe that if $(\rho,\lambda) \in \mathrm{Fix}\,^{T_{n,k}}_{(\gamma,\alpha)}$ and i is a natural number, then $\lambda(\gamma^i(e)) = \lambda(e)$ for any $e \in E(W_{n,k})$, and $\rho_{\gamma^i(v)} = \gamma^i \circ (\rho_v)^{\alpha^i} \circ \gamma^{-i}$ for any $v \in V(W_{n,k})$. In particular, $\lambda(\gamma^i(e)) = 1$ for any $e \in E(T_{n,k})$. Let $E^T(W_{n,k_\gamma})$ be the set of edges of $E(W_{n,k_\gamma})$ which can not be represented by edges of $E(T_{n,k})$. Then the values of λ on $[e] \in E^T(W_{n,k_\gamma})$ are completely determined by the value $\lambda(e)$ of e in $\mathbb{Z}_2$, and the assigned cycle permutations of ρ on $[v] \in V(W_{n,k_\gamma})$ is determined by the cycle permutation ρ_v on $N(v)$. Moreover, $\rho_v \in P_{(v;\gamma)}$ if $\alpha^{|[v]|} = 1$ and $\rho_v \in I_{(v;\gamma)}$ if $\alpha^{|[v]|} = -1$. Thus we have the following identities. If $\gamma \in A_d$, then

$$|\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,1)}| = 2^{|E^T(W_{n,k_\gamma})|}|P_{(v_0;\gamma)}| \prod_{[v]\in V_{d,3}} |P_{(v;\gamma)}| \prod_{[v]\in V_{d,4}} |P_{(v;\gamma)}|$$

and

$$|\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,-1)}| = 2^{|E^T(W_{n,k_\gamma})|}|I_{(v_0;\gamma)}| \prod_{[v]\in V_{d,3}} |P_{(v;\gamma)}| \prod_{[v]\in V_{d,4}} |P_{(v;\gamma)}|.$$

If $\gamma \in B_i$, then

$$|\mathrm{Fix}\,{}^{T_{n,k}}_{(\gamma,1)}| = 2^{|E^T(W_{n,k_\gamma})|}|P_{(v_0;\gamma)}| \prod_{\substack{[v]\in V^i_{l,j}\\ l\in\{1,2\},j\in\{3,4\}}} |P_{(v;\gamma)}|$$

and

$$|\mathrm{Fix}\,{}^{T_{n,k}}_{(\gamma,-1)}| = 2^{|E^T(W_{n,k_\gamma})|}|I_{(v_0;\gamma)}| \prod_{\substack{[v]\in V^i_{1,j}\\ j\in\{3,4\}}} |I_{(v;\gamma)}| \prod_{\substack{[v]\in V^i_{2,j}\\ j\in\{3,4\}}} |P_{(v;\gamma)}|,$$

where the product over empty index set is defined to be 1. On the other hand, for any $\gamma \in \mathrm{Aut}\,(W_{n,k})$, the number $|E^T(W_{n,k_\gamma})|$ is given as follows:

$$|E^T(W_{n,k_\gamma})| = \begin{cases} \frac{kn}{d} & \text{if } \gamma \in A_d, \\ \frac{k(n+1)}{2} & \text{if } n \text{ is odd and } \gamma \in B_1, \\ \frac{k(n+2)}{2} & \text{if } n \text{ is even and } \gamma \in B_0, \\ \frac{kn}{2} & \text{if } n \text{ is even and } \gamma \in B_2. \end{cases}$$

Using Lemma 1.2, we obtain the numbers $|P_{(v;\gamma)}|$ and $|I_{(v;\gamma)}|$ as follows: (i) For $\gamma \in A_d$ where $d|n$, $|P_{(v;\gamma)}|$ is 2 if $[v] \in V_{d,3}$ and 6 if $[v] \in V_{d,4}$; $|P_{(v_0;\gamma)}|$ is $\phi(d)(n/d-1)!d^{n/d-1}$; $|I_{(v_0;\gamma)}|$ is $\phi(d)(n/2)!2^{n/2-1}$ if n is even and d is 2, and 0 otherwise. (ii) For $\gamma \in B_i$ where $i = 1$ or 2, $|P_{(v;\gamma)}|$ is 2 if $[v] \in V^i_{2,3}$ and 6 if $[v] \in V^i_{2,4}$, and 0 if $[v] \in V^i_{1,3}(\neq \emptyset)$; $|I_{(v_0;\gamma)}|$ is $((n-i)/2)!2^{(n-i)/2}$; $|I_{(v;\gamma)}|$ is 2 if $v \neq v_0$. (iii) For $\gamma \in B_0$, $|P_{(v_0;\gamma)}|$ is $(n/2-1)!2^{n/2-1}$; $|I_{(v_0;\gamma)}|$ is $(n/2)!2^{n/2-1}$; $|P_{(v;\gamma)}|$ is 2 if $[v] \in V^i_{2,3}$ and 6 if $[v] \in V^i_{2,4}$.

Therefore, we have the following lemma.

LEMMA 2.1.

(1) *If $\gamma \in A_d$ where $d|n$,*

$$|\mathrm{Fix}\,{}^{T_{n,k}}_{(\gamma,1)}| = \phi(d)(\frac{n}{d}-1)!d^{n/d-1}2^{2kn/d}3^{(k-1)n/d},$$

and

$$|\text{Fix}\,{}^{T_{n,k}}_{(\gamma,-1)}| = \begin{cases} (\frac{kn}{2})!2^{(2k+1)/2-1}3^{(k-1)n/2} & \text{if } n \text{ is even and } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If n is odd and $\gamma \in B_1$,*

$$|\text{Fix}\,{}^{T_{n,k}}_{(\gamma,1)}| = 0, \quad |\text{Fix}\,{}^{T_{n,k}}_{(\gamma,-1)}| = (\frac{n-1}{2})!2^{((2k+1)n+2k-1)/2}3^{(k-1)(n-1)/2}.$$

(3) *If n is even and $\gamma \in B_0$,*

$$|\text{Fix}\,{}^{T_{n,k}}_{(\gamma,1)}| = (\frac{n}{2} - 1)!2^{((2k+1)n+2(k-1))/2}3^{(k-1)n/2},$$

$$|\text{Fix}\,{}^{T_{n,k}}_{(\gamma,-1)}| = (\frac{n}{2})!2^{((2k+1)n+2(k-1))/2}3^{(k-1)n/2}.$$

(4) *If n is even and $\gamma \in B_2$*

$$|\text{Fix}\,{}^{T_{n,k}}_{(\gamma,1)}| = 0, \quad |\text{Fix}\,{}^{T_{n,k}}_{(\gamma,-1)}| = (\frac{n}{2} - 1)!2^{(2k+1)n/2+k-1}3^{(k-1)(n-2)/2}.$$

To an embedding scheme (ρ, λ) for $W_{n,k}$, we associate a new embedding scheme $(\dot{\rho}, \dot{\lambda})$ in $\mathcal{E}_T(n,k)$ as follows: for any $v_{il}v_{jl} \in E(W_{n,k})$ where $1 \leq i, j \leq n$ and $j \equiv i + 1 \pmod{n}$, define

$$\dot{\rho}_{v_{jl}} = (\rho_{v_{jl}})^{\lambda(v_0 v_{j1}) \cdots \lambda(v_{jl-1} v_{jl})}$$

and

$$\dot{\lambda}(v_{il}v_{jl}) = \lambda(v_0 v_{i1} \cdots \lambda(v_{il-1}v_{il})\lambda(v_{il}v_{jl})\lambda(v_0 v_{j1}) \cdots \lambda(v_{jl-1}v_{jl}).$$

Then we can drive the following from Lemma 1.1.

LEMMA 2.2. *If (ρ, λ) and (τ, μ) are embedding schemes for $W_{n,k}$ with the corresponding embeddings $i : W_{n,k} \to S$ and $j : W_{n,k} \to S$ respectively, then these embeddings i, j are congruent with respect to a subgroup Γ of $\mathbb{D}_n$ if and only if $(\gamma, \alpha)(\dot{\rho}, \dot{\lambda}) = (\dot{\tau}, \dot{\mu})$ for some $(\gamma, \alpha) \in \Gamma \times \mathbb{Z}_2$.*

By Lemmas 2.1 , 2.2 and Burnside's lemma,

$$\begin{aligned}
|\mathcal{C}_\Gamma(W_{n,k})| &= \frac{1}{2|\Gamma|}\sum_{\gamma\in\Gamma}(|\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,1)}| + |\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,-1)}|) \\
&= \frac{1}{2|\Gamma|}\sum_{d|n}\sum_{\gamma\in\Gamma\cap A_d}(|\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,1)}| + |\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,-1)}|) \\
&\quad + \frac{1}{2|\Gamma|}\sum_{i\in J}\sum_{\gamma\in\Gamma\cap B_i}(|\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,1)}| + |\mathrm{Fix}\,^{T_{n,k}}_{(\gamma,-1)}|),
\end{aligned}$$

where J is $\{1\}$ if n is odd and $\{0,2\}$ if n is even. Therefore we have the following theorems.

THEOREM 2.3. *Let n and k be positive integers with $n \geq 3$ and $n + k \geq 5$. Then*

$$|\mathcal{C}(W_{n,k})| = \begin{cases}
\frac{1}{n}\sum_{d|n}(\phi(d))^2(\frac{n}{d}-1)!d^{n/d-1}2^{2kn/d-2}3^{(k-1)n/d} & \\
+ (\frac{n-1}{2})!2^{((2k+1)n+2k-5)/2} & \text{if } n \text{ is odd,} \\
\frac{1}{n}\sum_{d|n}(\phi(d))^2(\frac{n}{d}-1)!d^{n/d-1}2^{2kn/d-2}3^{(k-1)n/d} & \\
+ (\frac{n}{2}-1)!2^{(2k+1)n/2-4}(1+2^k+2^k3^{1-k}+2^{k-1}n) & \text{if } n \text{ is even.}
\end{cases}$$

THEOREM 2.4. *Let n and k be positive integers with $n \geq 3$ and $n + k \geq 5$. Then*

$$|\mathcal{C}_{\mathbb{Z}_n}(W_{n,k})| = \begin{cases}
\frac{1}{n}\sum_{d|n}(\phi(d))^2(\frac{n}{d}-1)!d^{n/d-1}2^{2kn/d-1}3^{(k-1)n/d} & \text{if } n \text{ is odd,} \\
\frac{1}{n}\sum_{d|n}(\phi(d))^2(\frac{n}{d}-1)!d^{n/d-1}2^{2kn/d-1}3^{(k-1)n/d} & \\
+\frac{1}{n}(\frac{n}{2})!2^{(2k+1)n/2-2}3^{(k-1)n/2} & \text{if } n \text{ is even.}
\end{cases}$$

THEOREM 2.5. *Let n and k be positive integers with $n \geq 3$ and $n+k \geq 5$. Then*

$$|\mathcal{C}_{\{I\}}(W_{n,k})| = (n-1)!2^{2kn-1}3^{(k-1)n}.$$

Table 1 shows the number of the congruence classes of 2-cell embeddings of a wheel $W_{n,1}$ for small numbers n, which are calculated from the above theorems.

n	4	5	6	7	8	$\cdots$
$\lvert\mathcal{C}_{\mathbb{Z}_n}(W_{n,1})\rvert$	206	2464	41148	842616	20647700	$\cdots$
$\lvert\mathcal{C}(W_{n,1})\rvert$	135	1360	21214	424380	10342282	$\cdots$

Table 1.

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