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APPROXIMATE FIBRATIONS AND NON-APPROXIMATE FIBRATIONS IN PL CATEGORY

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ABSTRACT. This paper provides examples which can not be approximate fibrations and shows that if N^n is a closed aspherical manifold, $\pi_1(N)$ is hyperhophian, normally cohophian, and $\pi_1(N)$ has no non-trivial Abelian normal subgroup, then the product of N^n and a sphere S^m satisfies the property that all PL maps from an orientable manifold M to a polyhedron B for which each point preimage is homotopy equivalent to $N^n \times S^m$ necessarily are approximate fibrations

1. Introduction

In the study of proper maps between manifolds, approximate fibrations, introduced and studied by Coram and Duvall[1], form an important class of mappings, nearly as effective as Hurewicz fibrations. This paper examines the PL version of the subject.

A proper map $p : M \to B$ between locally compact ANRs is called an approximate fibration if it has the following homotopy property: Given an open cover ϵ of B, an arbitrary space X and two maps $g : X \to M$ and $F : X \times I \to B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \to M$ such that $G_0 = g$ and $p \circ G$ is ϵ-close to F.

When $p : M \to B$ is an approximate fibration, there is a homotopy exact sequence developed by Coram and Duvall[1];

$$\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}b) \to \pi_i(M) \to \pi_i(B) \to \cdots$$

just like the one for Hurewicz fibrations, relating homotopy data of the total space, base space, and typical fiber.

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Throughout this paper, we fix for all the setting and notation to be used: M is an orientable PL $(n+k)$-manifold, B is a polyhedron, and $p: M \to B$ is a PL map such that each point preimage $p^{-1}b$ has the homotopy type of a closed, connected, orientable manifold.

Restriction to the PL category offers some advantages. The target spaces are standard geometric objects, obviously finite dimensional and locally contractible, features which a priori dispel potentially troublesome issues lurking in the background of the general (non-PL)setting[4]. The chief benefit is not the simplicial structure of the image, however, but rather the potential for inductive arguments, as in the classical PL topology, which apply to the restriction of p over certain links in the target and bring about the lowering of fiber codimension without changing fiber character.

When N is fixed and orientable, such a PL map $p: M \to B$ is called N-like if each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N.

We call N a codimension k PL fibrator if, for all orientable $(n+k)$-manifolds M and N-like PL maps $p: M \to B$, p is an approximate fibration. If N has this property for all $k > 0$, call N simply a PL fibrator.

Most closed manifolds are known to be codimension 1 fibrators[3]. For codimension 2 fibrators, we have fairly rich data[4,5,6,8,10,11]. In particular, every closed surface except torus is a codimension 2 fibrator[4]. Also manifolds that satisfy a certain hophian property are codimension 2 fibrators if they have either non-zero Euler characteristic or hyperhophian fundamental groups[6].

Daverman and Walsh [8] showed that a sphere S^m is a codimension m fibrator, but not a codimension $(m+1)$ fibrator. Until now, all known nonfibrators either failed in codimension 2 or had a 2-sphere as cartesian factor. Daverman [7] proved that if N^n is a closed aspherical manifold with some fundamental group condition, then N is a PL fibrator.

In this paper, we show that real projective (n odd) is a codimension n PL fibrator but not one in codimension $n+1$. Also, we show that a product of a closed apherical manifold with some fundamental condition and a sphere S^m is a codimension m PL fibrator.

A group G is hophian if every epimorphism $\Theta: G \to G$ is necessarily an isomorphism, while a finitely presented group G is hyperhophian if every homomorphism $\Psi : G \to G$ with $\Psi(G)$ normal and $G/\Psi(G)$ cyclic is an automorphism. A group G is normally cohophian if every monomorphism $\Phi : G \to G$ with normal image is an automorphism.

A closed manifold N is hophian if it is orientable and every degree one map $N \to N$ which induces a π_1-automorphism is a homotopy equivalence. This term plays an important role in determining approximate fibrations. Swarup [14] has shown this hophian feature for closed orientable n-manifolds N with $\pi_1(N) = 0$ for $1 < i < n-1$, and Hausmann has done the same for all closed orientable 4-manifolds[9].

The symbol χ is used to denote Euler characteristic.

The (absolute) degree of a map $f : N \to N$, where N is a closed, connected, orientable n-manifold, is the non-negative integer d such that the induced endomorphism of $H_n(N : Z) \cong Z$ amounts to multiplication by d, up to sign.

A PL map $p : M \to B$ has Property $R_\# \cong$ ($R_* \cong$) if, for each $b \in B$, a retraction $R : U \to p^{-1}b$ defined on some open set $U \supset p^{-1}b$ induces $\pi_1(H_1)$-isomorphisms $(R|)_\# : \pi_1(p^{-1}b') \to \pi_1(p^{-1}b)$ ($(R|)_* : H_1(p^{-1}b') \to H_1(p^{-1}b)$) for all $b' \in B$ sufficiently close to b.

In section 2, we show important examples which are not codimension k fibrators.

In section 3, we present results about codimension k fibrators.

2. Examples of nonfibrators

Remarkably, until now only two types of manifolds are known not to be PL fibrators: those that alredy fail in codimension 2, and those that have a sphere as Cartesian factor.

In this section, we show that the real projective n-space (n odd) is a codimension n PL fibrator but not one in codimension $(n+1)$.

Let $J^k(N)$ be the group of all self-homeomorphisms of $N \times S^k$ and $J^k_{he}(N)$ the subgroup consisting of those $h \in J^k(N)$ for which the composite

$$N \xrightarrow{incl} N \times * \subset N \times S^k \xrightarrow{h} N \times S^k \xrightarrow{proj} N$$

is a homotopy equivalence. Given $h \in J^k(N)$, we use $\psi_h : N \to N$ to denote the composite above and define $C^k(N)$ as $J^k(N)/J^k_{he}(N)$.

THEOREM 2.1. *If $|C^k(N)| \neq 1$, then N fails to be a codimension $(k+1)$ PL fibrator.*

PROOF. Let $M = N \times R^{k+1}$. Parameterize $M - (N \times 0)$ as $N \times S^k \times (0, \infty)$. Pick $h \in J^k(N) - J^k_{he}(N)$, and partition M into copies of N via

$$\{N \times 0\} \cup \{h(N \times z) \times r | z \in S^k, r \in (0, \infty)\}.$$

For the retraction $R : M \to N \times 0 = N$ which essentially equals projection,

$$R| : h(N \times z) \times r \to N \times 0 = N$$

behaves just like ψ_h and thus, fails to be a homotopy equivalence. According to [2], the associated decomposition map $p : M \to B$ is not an approximate fibration.

THEOREM 2.2. *RP^{2n+1} is a codimension $(2n+1)$ PL fibrator which fails to be a codimension $(2n+2)$ PL fibrator.*

PROOF. Let $\eta : S^{2n+1} \to RP^{2n+1}$ denote the standard covering map. It suffices to find a homeomorphism θ of $S^{2n+1} \times S^{2n+1}$ to itself which is equivariant with respective to $\eta \times Id : S^{2n+1} \times S^{2n+1}$ and for which the composite

$$S^{2n+1} \xrightarrow{incl} S^{2n+1} \times * \subset S^{2n+1} \times S^{2n+1} \xrightarrow{\theta} S^{2n+1} \times S^{2n+1} \xrightarrow{proj} S^{2n+1}$$

fails to be a homotopy equivalence, as θ then induces a comparable automorphism $h_\theta = (\eta \times Id) \circ \theta \circ (\eta \times Id)^{-1}$ on $RP^{2n+1} \times S^{2n+1}$. Since θ is equivariant with respect to $(\eta \times Id)$, $(\eta \times Id)(\theta(x, y))$ and $(\eta \times Id)(\theta(-x, y))$ are the same. Therefore, $(\eta \times Id) \circ \theta \circ (\eta \times Id)^{-1}$ is well defined.

The existence of such a θ follows directly from a construction of Wall[15]. Let $(p, q) \in S^{2n+1} \times S^{2n+1}$. On the great circle in S^{2n+1} through p, q, let $q'(p)$ be the other point at the same distance from p as q is. Note that if $-p$ is the point of S^{2n+1} antipodal to p, then $q'(-p) = q'(p)$; furthermore, if r is the point antipodal to q, then $r'(p)$ and $q'(p)$ are antipodal. This means that the map $\theta_2 : S^{2n+1} \times S^{2n+1}$ to itself sending (p, q) to $(p, q'(p))$ is equivariant with respect to $\eta \times Id$, as is the map $\theta_1 : S^{2n+1} \times S^{2n+1}$ to itself sending (q,p) to $(q'(p),p)$. Wall points out that the automorphism of $H_{2n+1}(S^{2n+1} \times S^{2n+1})$ induced by θ_2 is given by matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

so the matrix associated with the composite $\theta = \theta_1\theta_2$ is

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$$

Hence ψ_θ fails to be a homotopy equivalence, as it has degree 3, and $|C^{2n+1}(RP^{2n+1})| \neq 1$. Consequently, RP^{2n+1} is not a codimension $(2n+2)$ PL fibrator by Theorm 2.1.

3. Codimension k fibrators

It is well-known that if N^n is a closed, aspherical manifold and if $\pi_1(N)$ is a hyperhophian and normally cohophian group with no nontrivial normal subgroup, then N is a PL fibrator[7]. Also any sphere S^m is a codimension m fibrator, but not a codimension $m+1$ fibrator.

The main result of this section is that a product of such a manifold N and a sphere S^m can be a codimension m fibrator.

Throughout the rest of the paper, for a PL map $p : M \to B$, v will denote a vertex of B, $L = \text{link}(v, B)$, $S = \text{star}(v, B) = v * L$, $L' = p^{-1}L$, and $S' = p^{-1}S$. These are understood to arise in the first barycentric subdivision of triangulations on which p is simplicial.

LEMMA 3.1. *If X is a CW-complex such that $\pi_i(X) = 0$ for $1 < i \leq k$ and if the map $f : X \to X$ induces an isomorphism $\pi_1(X) \to \pi_1(X)$, then f also induces isomorphisms*

$$f_* : H_i(X) \to H_i(X) \text{ and } f^* : H^i(X) \to H^i(X) \ (i \leq k)$$

PROOF. Build an Eilenberg-MacLane space $K = K(\pi_1(X), 1) \supset X$ by attaching cells of dimension at least $k+2$ to X. There is no obstruction to extending $f : X \to X$ to a map $F : K \to K$. Dimension restrictions pertaining to the attached cells cause the inclusion $X \to K$ to induce homotopy and homology isomorphisms for $i \leq k$. Since then F, like f, induces a fundamental group isomorphism, it is a homotopy equivalence; moreover, the consequence that $F_* : H_i(K) \to H_i(K)$ is an isomorphism shows $f_* : H_i(X) \to H_i(X)$ is one as well, provided $i \leq k$. The identical argument works for cohomology.

LEMMA 3.2. *[7] Let N be a closed n-manifold which is a codimension $k-1$ PL fibrator, $k > 2$, and whose fundamental group is normally cohophian and contains no Abelian normal subgroup, and let $p : M \to B$ be an N-like PL map defined on a PL $(n+k)$-manifold. Then p has Property $R \cong$.*

LEMMA 3.3. *[12] Suppose N^n is a closed aspherical manifold with hophian fundamental group. Then $N^n \times S^m$ is a hophian manifold.*

THEOREM 3.4. *Suppose N^n is a closed, aspherical manifold and $\pi_1(N)$ is a hyperhophian and normally cohophian group with no non-trivial Abelian normal subgroup. Then $\tilde{N} = N^n \times S^m$ is a codimension m PL fibrator.*

PROOF. For $m = 2$, in view of [12], $\tilde{N}$ is a codimension 2 fibrator.

Inductively assume that $\tilde{N}$ is a codimension i fibrator for $i \leq m-1$. Let $p : M^{n+2m} \to B^m$ be any PL $\tilde{N}$-like map. Because of $\pi_1(\tilde{N}) \cong \pi_1(N)$, Lemma 3.2 confirms that p has Property $R \cong$.

By the inductive hypothesis, $p|L' : L' \to L$ is an approximate fibration. From the m-movability criterion [2], it suffices to show that $R : p^{-1}c \to p^{-1}v$ is a homotopy equivalence for any $c \in L$. Since $\tilde{N}$ is a hophian manifold by Lemma 3.3, it is enough to show that R is a degree one map.

The fact that N is an aspherical manifold implies $\pi_i(\tilde{N}) = 0$ for $1 < i \leq m-1$. Applying Lemma 3.1, R induces isomorphisms

$$R_* : H_i(\tilde{N}) \to H_i(\tilde{N}) \text{ and } R^* : H^i(\tilde{N}) \to H^i(\tilde{N}) \text{ for } 1 < i \leq m-1.$$

Then B^m is an m-dimensional manifold by [7,Corallary 3.6]. This implies that L^{m-1} is a homotopy $(m-1)$-sphere.

First, the homology sequence of (S', L') shows

$$H_m(S', L') \to H_{m-1}(L') \to H_{m-1}(S') \to H_{m-1}(S', L'),$$

where the first term is $H_m(S', L') \cong H^{n+m}(\tilde{N}) \cong Z$ and the last term is $H_{m-1}(S', L') \cong H^{n+m+1}(\tilde{N}) \cong 0$ by the Alexander duality.

Consider the following sequence

$$(*) \qquad \begin{array}{ccccccc} H_m(S', L') & \longrightarrow & H_{m-1}(L') & \longrightarrow & H_{m-1}(S') & \longrightarrow & 0 \\ \downarrow p_* & & \downarrow p'_* & & & & \\ H_m(S, L) & \xrightarrow{\cong} & H_{m-1}(L) & \longrightarrow & H_{m-1}(S) \cong 0 & & \end{array}$$

Now, we show that $p'_* : H_{m-1}(L') \to H_{m-1}(L)(\cong Z)$ is an ephimorphism. Since $p|L' : L' \to L$ is an approximate fibration, we have the homotopy exact sequence;

$$\cdots \to \pi_{m-1}(\tilde{N}) \to \pi_{m-1}(L') \to \pi_{m-1}(L) \to \pi_{m-2}(\tilde{N}) \to \cdots$$

If $m \geq 4$, then $\pi_I(\tilde{N}) = 0$ for $i = m-2$ and $m-1$. Thus, $p_\# : \pi_{m-1}(L') \to \pi_{m-1}(L)$ is an isomorphism.

If $m = 3$, by using the fact that $\pi_1(\tilde{N})$ has no non-trivial Abelian normal subgroup, we obtain that $p_\#$ is an isomorphism.

On the other hand, we have the natural following diagram

$$\begin{array}{ccc} \pi_{m-1}(L') & \xrightarrow{\cong} & \pi_{m-1}(L) \\ \downarrow & & \downarrow \cong \\ H_{m-1}(L') & \xrightarrow{p'_*} & H_{m-1}(L) \end{array}$$

where the vertical map is an isomorphism because L is a homotopy $(m-1)$-sphere. This shows that $p'_* : H_{m-1}(L') \to H_{m-1}(L)$ is an epimorphism.

Next, we apply the Wang sequences for approximate fibration $p|L' : L' \to L$[7];

$$\cdots \to H_{m-1}(\tilde{N}) \to H_{m-1}(L') \to H_0(\tilde{N}) \to H_{m-2}(\tilde{N}) \to H_{m-2}(L').$$

Here the last term is $H_{m-2}(L') \cong H_{m-2}(S')$ due to the homology exact sequence of (S', L') and the last homomorphism is an isomorphism since S' collapses to $p^{-1}v$ and $R : p^{-1}c \to p^{-1}v$ is an isomorphism. Therefore, $H_{m-1}(L') \cong \mathrm{Im}H_{m-1}(\tilde{n}) \oplus Z$ and we easily check that in (*), $p_* : H_m(S', L') \to H_m(S, L)$ is an isomorphism by the diagram chasing, and then we see that $p_* : H_m(S', S'-p^{-1}v) \to H_m(S, S-v)$ is an isomorphism. Similarly, we obtain an isomorphism $p_* : H_m(S', S' - p^{-1}c) \to H_m(S, S - c)$ for any $c \in S$ sufficiently close to v.

Then the following commutative diagram holds, where U is a connected open neighborhood of v in$S$ having compact closure and $c \in U$.

$$\begin{array}{ccc}
H^{n+m}(p^{-1}v) \cong H_m(S', S' - p^{-1}v) & \xrightarrow{\cong} & H_m(S, S - v) \cong H^0(v) \\
\uparrow & & \uparrow \cong \\
H_m(S', S' - cl(p^{-1}U)) & \xrightarrow{\cong} & H_m(S, S - cl(U)) \cong H^0(clU) \\
\downarrow & & \downarrow \cong \\
H^{n+m}(p^{-1}c) \cong H_m(S', S' - p^{-1}c) & \xrightarrow{\cong} & H_m(S, S - c) \cong H^0(c)
\end{array}$$

This implies that R^* on the cohomology is constantly 1 near v, and implies the same on the homology by the universal coefficient theorem. As a consequence, we conclude that $\tilde{N}$ is a codimension m PL fibrator.

COROLLARY 3.5. *Suppose F^2 is any closed orientable surface with negative Euler characteristic. Then $F^2 \times S^m$ is a codimension m PL fibrator.*

PROOF. A fundamental group of a closed orientable surface with negative Euler characteristic is a hyperhophian[6] and a normally cohophian group with no nontrivial Abelian normal subgroup[7]. Therefore, the conclusion follows from Theorem 3.4.

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