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SOME RESULTS RELATED TO DENSELY HOMOGENEOUS SPACES

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ABSTRACT. We will give an example which is a normal Hausdorff countable dense homogeneous space but not a densely homogeneous space. Next, we will give a proof for the fact that every nondegenerate component of densely homogeneous spaces is open and densely homogeneous.

1. Introduction

In 1972, R. B. Bennett [1] first introduced and studied the concept of countable dense homogeneous spaces. Later, B. Fitzpatrick Jr. and N. F. Lauer [2] extended this concept to nonseparable spaces having σ-discrete dense subsets.

In [3], B. Fitzpatrick Jr. and Zhou H showed that an example Σ_B due to R. L. Moore of a regular separable connected 2-manifold which is not normal is a countable dense homogeneous space that is not densely homogeneous. In section 2, by showing that the real line in the lower limit topology is countable dense homogeneous but not densely homogeneous, we will provide another example which is normal.

It was shown in [3] that every component of a metric densely homogeneous space is open and densely homogeneous. And, these results were extended by R. Yongbin to perfectly normal, collectionwise Hausdorff densely homogeneous spaces. In section 3, we will show that any conditions are not needed for a component of a densely homogeneous space to be open and densely homogeneous.

For a space X, we will use $H(X)$ for the set of all autohomeomorphisms of X througout this paper.

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2. An example which is normal Hausdorff countable dense homogeneous but not densely homogeneous

A space X is said to be *countable dense homogeneous* (abbreviated *CDH*) if for any two countable dense subsets A and B of X, there is an $h \in H(X)$ such that $h(A) = B$. A space X is said to be *densely homogeneous* (abbreviated *DH*) if X has a σ-discrete dense subset such that for any two σ-discrete dense subsets A and B, there is an $h \in H(X)$ satisfying $h(A) = B$. Clearly, every separable DH space is CDH.

The following lemma is useful to prove the consequent results, Corollary 2.2 and Corollary 2.5.

LEMMA 2.1. *Let X be a CDH space. Then the following are equivalent:*

1. X is DH.
2. Every σ-discrete subset of X is countable.
3. Every uncountable subset of X contains at least one of its limit points.

PROOF. ($1 \Rightarrow 2$) If $A = \{a_n\}$ is a countable dense subset of X, then $A = \cup_{n \in \mathbf{Z}_+}\{a_n\}$ is a σ-discrete dense subset of X. Let B be a σ-discrete subset of X. Then $B \cup \left(\cup_{a \in A-B}\{a\}\right)$ is a σ-discrete dense subset of X, and hence there is an $h \in H(X)$ such that $h(A) = B \cup \left(\cup_{a \in A-B}\{a\}\right)$. This shows that B is countable.

($2 \Rightarrow 3$) Assume that A is an uncountable subset of X such that $A \cap A' = \emptyset$, where A' is the set of all limit points of A. Then each element a in A has a neighborhood U such that $U \cap A = \{a\}$. Thus A is a discrete subset of X, and therefore σ-discrete subset of X, contrary to the condition 2.

($3 \Rightarrow 1$) It suffices to show that every $\sigma-$discrete dense subset of X is countable. We will prove contrapositive: Suppose that $A = \cup_{n \in \mathbf{Z}_+} A_n$ is an uncountable σ-discrete dense subset of X. Then there is some $n \in \mathbf{Z}_+$ such that A_n is uncountable, so that A_n has a limit point of itself. This contradicts the fact that A_n is discrete.

From the well known fact that every uncountable subset of a second countable space contains uncountably many of its limit points, we obtain the following result.

COROLLARY 2.2. *Every metric CDH space is DH.*

Let $\mathbf{R}_l$ denote the set of real numbers in the lower limit topology. Clearly, $\mathbf{R}_l$ is a Hausdorff normal space.

From the countable dense homogeneity of the real line $\mathbf{R}$, we have the following lemma.

LEMMA 2.3. *For any two countable dense subsets A and B of $\mathbf{R}$, there is an $h \in H(\mathbf{R})$ such that $h(A) = B$ and h is strictly increasing.*

PROOF. Choose an $f \in H(\mathbf{R})$ such that $f(A) = B$. Suppose that f is strictly decreasing, and let $\mathbf{Q}$ be the set of rational numbers. Take $g_1, g_2 \in H(\mathbf{R})$ such that $g_1(\mathbf{Q}) = \mathbf{Q}$, $g_2(\mathbf{Q}) = B$, and g_1 is strictly decreasing. Then the map $h = g_2 \circ g_1 \circ g_2^{-1} \circ f$ satisfies the desired result.

COROLLARY 2.4. $\mathbf{R}_l$ *is CDH.*

PROOF. Let A and B be two countable dense subset of $\mathbf{R}_l$. Then they are countable dense subsets of $\mathbf{R}$. Since $\mathbf{R}$ is CDH, there is an $h \in H(\mathbf{R})$ such that $h(A) = B$. By Lemma 2.3, we may assume that h is strictly increasing. Thus for any basis element U of $\mathbf{R}_l$, $h(U)$ is also a basis element of $\mathbf{R}_l$. Consider a diagram

$$\begin{array}{ccc} \mathbf{R}_l & \xrightarrow{id} & \mathbf{R} \\ & & \downarrow h \\ \mathbf{R}_l & \xrightarrow{id} & \mathbf{R} \end{array}$$

Let $\overline{h} = id^{-1} \circ h \circ id$. Then $\overline{h} \in H(\mathbf{R}_l)$ such that $\overline{h}(A) = B$. Therefore $\mathbf{R}_l$ is CDH.

COROLLARY 2.5. $\mathbf{R}_l$ *is not DH.*

PROOF. Consider subsets of $\mathbf{R}$ defined as follows:

$$A_0 = (0,1].$$
$$A_1 = (0,\frac{1}{3}] \cup (\frac{2}{3},1].$$
$$A_2 = (0,\frac{1}{9}] \cup (\frac{2}{9},\frac{1}{3}] \cup (\frac{2}{3},\frac{7}{9}] \cup (\frac{8}{9},1].$$
$$\vdots$$
$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} (\frac{1+3k}{3^n}, \frac{2+3k}{3^n}].$$
$$\vdots$$

Then the intersection $A = \bigcap_{n\in\mathbf{Z}_+} A_n$ is an uncountable subset of $\mathbf{R}$ such that the set of all limit points of A in the lower limit topology disjoint from A. From Lemma 2.1, we conclude that $\mathbf{R}_l$ is not DH.

REMARK. 1. The Moore space Σ_B is SLH because it is a 2-manifold. But our example is not SLH.

2. Since our example is normal, we conclude that every Hausdorff normal CDH space need not be DH.

Note that the Moore space Σ_B is connected. On the other hand, the space $\mathbf{R}_l$ is totally disconnected. We do not know whether there is a connected normal Hausdorff CDH space that is not DH.

3. Every nondegenerate component of densely homogeneous spaces is open and densely homogeneous

Throughout this section, we will assume that all spaces under consideration are T_1 and have no degenerate (or trivial) components. For a subset A of a space X, we will write $Int(A)$ for the interior of A. We will use the notation $\mathbf{Z}_+$ for the set of all positive integers.

Define two points x and y of X to be equivalent if there is an $h \in H(X)$ such that $h(x) = y$. Trivially, this is an equivalence relation. We will use the notations $[x]$ as the equivalence class of x and $\tilde{X}$ the set of all equivalence classes. Clearly, for any $[x] \in \tilde{X}$ and any $h \in H(X)$, $h([x]) = [x]$.

B. Fitzparick Jr. and N. F. Lauer showed the following lemma. For the proof, see the Theorem of Section 3 in [3].

LEMMA 3.1. *If X is CDH, DH, or SLH, then every element of $\tilde{X}$ is both open and closed.*

By using this Lemma, B. Fitzpatrick Jr. and Zhou H showed the following lemma.(see Theorem 2.5 in [4])

LEMMA 3.2. *If X is CDH, DH, or SLH, then every element of $\tilde{X}$ is CDH, DH, or SLH, respectively.*

We need the following lemmas to prove main results of this section.

LEMMA 3.3. *If a space X has a σ-discrete dense subset, then every open subset of X has a σ-discrete dense subset.*

LEMMA 3.4. *Every element of $\tilde{X}$ is a union of components of X.*

PROOF. Clearly, every component of X is contained in some element of $\tilde{X}$. For a given component C of X, let $\mathcal{C}$ be the collection of all elements of $\tilde{X}$ that intersect C. Choose an element $[x]$ in $\mathcal{C}$. Let $U = [x]$ and $V = \cup\{[y] \in \mathcal{C} | [y] \neq [x]\}$. It follows from Lemma 3.1 that U and V are open subsets of X. Since $U \cap V = \emptyset$, we have that C is not connected unless $V = \emptyset$.

LEMMA 3.5. *If a component C of X contained in an element $[x]$ of $\tilde{X}$, then either C is open or $Int(C) = \emptyset$.*

PROOF. Suppose that $c_0 \in Int(C)$. Then there is a neighborhood U of c_0 contained in C. For each point $c \in C$, choose an element $h \in H(X)$ such that $h(c_0) = c$. Clearly, $h(U)$ is a neighborhood of c contained in C.

THEOREM 3.6. *Every component of a DH space X is an open subset of X.*

PROOF. Suppose that a component C of X is not open in X. By Lemma 3.4, there is an element $[x]$ of $\tilde{X}$ such that $C \subset [x]$. Let A be a σ-discrete dense subset of X. Using Lemma 3.5 we know that $\mathrm{Int}(C)$ is empty, and therefore the component C is nowhwere dense in X. Thus $A - C$ is a σ-discrete dense subset of X. Choose a point p in C and add it to $A - C$. Then $(A - C) \cup \{p\}$ is a σ-discrete dense subset of X, and there is no $h \in H(X)$ such that $h(A - C) = (A - C) \cup \{p\}$.

Combining Lemma 3.3 and Theorem 3.6, we conclude that every component of X has a σ-discrete dense subset.

THEOREM 3.7. *Every component of a DH space X is DH.*

PROOF. Let C be a component of X. Then by Lemma 3.4, there is an $[x] \in \tilde{X}$ such that $C \subset [x]$. Let $A = \bigcup_{n \in \mathbf{Z}_+} A_n$ and $B = \bigcup_{n \in \mathbf{Z}_+} B_n$ be σ-discrete dense subsets of C, and let $\{C_\alpha | \alpha \in J\}$ be the collection of components different from C such that $C \cup (\bigcup_{\alpha \in J} C_\alpha) = [x]$. For each α, choose an $h_\alpha \in H([x])$ such that $h_\alpha(C) = C_\alpha$. For each n, let $A'_n = A_n \cup (\bigcup_{\alpha \in J} h_\alpha(A_n))$ and $B'_n = B_n \cup (\bigcup_{\alpha \in J} h_\alpha(A_n))$. Then each A'_n and each B'_n are discrete, and hence $A' = \bigcup_{n \in \mathbf{Z}_+} A'_n$ and $B' = \bigcup_{n \in \mathbf{Z}_+} B'_n$ are σ-discrete dense subsets of $[x]$. By Lemma 3.2, $[x]$ is DH, and hence there is an $h \in H([x])$ such that $h(A') = B'$. Suppose that $h(A) = h_\alpha(B)$. Then $h_\alpha^{-1} \circ h \in H([x])$ such that $(h_\alpha^{-1} \circ h)|_C(C) = C$ and $(h_\alpha^{-1} \circ h)(A) = B$.

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