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ISOMETRY GROUP OF $SO(1,2)$

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ABSTRACT. We characterize the left invariant Riemannian metrics on $SO(1,2)$ which give rise to 3- or 4-dimensional isometry groups.

1. Introduction

Let G be the connected component of $SO(1,2)$, where

$$SO(1,2) = \{X \in SL(3,\mathbb{R}) : XSX^t = S\}$$

and

$$S = \begin{bmatrix} -1 & 0 \\ 0 & I \end{bmatrix} \in GL(3,\mathbb{R}), \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is well known that G is isomorphic to $PSL_2\mathbb{R}$. Therefore, if G is equipped with a certain Riemann metric, it can be viewed as the unit tangent bundle (or orthonomal frame bundle) of H, the Poincare upper half plane. It is also a model for a 3-dimensional geometry (see[4]). The geometric structures are determined not just by the group itself, but by a left invariant Riemann metric. It is very important to understand the symmetries of a space in general.

A left invariant metric on G is obtained by picking a basis for the linear space $\mathfrak{g}$, the Lie algebra of G, and declaring that it is orthonormal. Two bases give ries to the same metric if and only if they are related by an

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orthogonal matrix. We choose the following basis to parametrize all the other bases for $\mathfrak{g}$.

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Although this choice of basis may at first glance seem unusual, we justify it by noting that for any $Y \in G$, the matrix of $Ad(Y)$ with respect to $\{e_1, e_2, e_3\}$ is Y^t. Now, any basis of $\mathfrak{g}$ is related to $\{e_1, e_2, e_3\}$ by an element of $GL(3,\mathbb{R})$ and two such bases give rise to the same metric if and only if they are related by an element of $O(3)$. Therefore, the space of left invariant metrics can be identified with $O(3) \setminus GL(3,\mathbb{R}) = SO(3) \setminus GL^+(3,\mathbb{R})$, which is 6-dimensional. Here $GL^+(3,\mathbb{R})$ is the connected component of the identity. Let

$$K = \begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}$$

be the standard maximal compact subgroup of G. Let G be equipped with a left invariant metric. Then $I(G)$, the group of isometries of G, is either 3 or 4-dimensional. For example, G with the metric defined by $\{e_1, e_2, e_3\}$ as above, has a 4-dimensional isometry group. The purpose of this note is to understand for which left invariant metrics, $I(G)$ is 4-dimensional. See the main theorem for the result.

We are grateful to Professor K. B. Lee for invaluable suggestions to prepare this paper. In a forthcoming paper [5], we characterize the left invariant Riemannian metrics on other 3-dimensional Lie groups which give rise to 3-, 4-, or 6-dimensional isometry groups

THEOREM. *In the space $SO(3) \setminus GL^+(3,\mathbb{R})$ of left invariant metrics g on G, there exists a 4-dimensional subspace $(\mathbb{R}^+)^2 \times [K \setminus SO_0(1,2)] \approx (\mathbb{R}^+)^4$ with the following property: $I(G,g)$ is 4-dimensional if and only if $g \in (\mathbb{R}^+)^2 \times [K \setminus SO_0(1,2)]$. Furthermore, in this case, say $g \in (\mathbb{R}^+)^2 \times K'$ where $K' = K \cdot X$ is a K-coset, then $I_0(G,g) = l(G) \times r(XKX^{-1})$.*

As is customary, l(resp. r) denotes left (resp. right) translations on G. Let us denote by $I_0(G)$ (resp. $SO_0(1,2)$) the connected component of the identity in $I(G)$ (resp. $SO(1,2)$). The proof will follow from statements (A) and (B).

(A) Let $\{g_1, g_2, g_3\}$ be a basis for the vector space $\mathfrak{g}$ such that

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \Delta X \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$\Delta = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad X \in SO(1,2)$$

for $\alpha, \beta > 0$. Let G have the left invariant metric determined by the basis $\{g_1, g_2, g_3\}$. Then

$$I_0(G) = l(G) \times r(XKX^{-1})$$

PROOF. Let $Y \in SO(1,2)$. Consider the linear transformation $Ad(Y)$ on $\mathfrak{g}$. It is not hard to check that the matrix of $Ad(Y)$ with respect to the basis $\{e_1, e_2, e_3\}$ is just Y^t itself. Therefore, for any $k \in K$, we see that the matrix of $Ad(XkX^{-1})$ with respect to the basis $\{g_1, g_2, g_3\}$ is k^t. Since $k \in SO(3)$, this implies that $Ad(XKX^{-1})$ is an isometry of G. Thus, $I_0(G) = l(G) \times Ad(XKX^{-1}) = l(G) \times r(XKX^{-1})$.

(B) Suppose $I(G)$ is 4-dimensional. Then the left invariant metric is induced by a basis $\{g_1, g_2, g_3\}$ for $\mathfrak{g}$ of the form

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \Delta X \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

where

$$\Delta = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \quad \text{and} \quad X \in \mathrm{SO}(1,2)$$

for some $\alpha, \beta > 0$.

PROOF. Since the metric is left invariant and $I(G)$ is 4-dimensional, the connected component of its isotropy subgroup $I(G)_e$ at the identity of G is isomorphic to $SO(2)$. Choose an orthonormal basis $\{g_1, g_2, g_3\}$ of $\mathfrak{g}$. After Milnor [3], define a linear transformation $L : \mathfrak{g} \to \mathfrak{g}$ by $L(u \times v) = [u, v]$, where the cross product is defined by $g_1 \times g_2 = g_3, g_2 \times g_3 = g_1$ and $g_3 \times g_1 = g_2$. Since L is self adjoint, it can be diagonalized.

LEMMA. *At least two of the eigenvalues of L are the same.*

PROOF. Suppose L has three distinct eigenvalues. We may assume that g_1, g_2, g_3 are eigenvectors. Then this basis diagonalizes the Ricci quadratic transformation $\widehat{r}$ (see [3], Theorem 4.3). Since every isometry fixing the identity element of G must preserve the eigenspaces of $\widehat{r}$, elements of $I(G)_e$ map g_i to $\lambda_i g_i$. Since $I(G)_e$ contains a 1-dimensional Lie group, this implies that the representation of $I(G)_e$ into $\mathrm{Aut}(\mathfrak{g})$ via differentials is not faithful. This is a contradiction.

We resume the proof of (B). By the lemma and Milnor's classification in [3], precisely two of the eigenvalues are equal. Furthermore, since we are using $\mathrm{GL}^+(3, \mathbb{R})$, and $[e_2, e_3] = e_1, [e_3, e_1] = -e_2, [e_1, e_2] = -e_3$, the eigenvalues are necessarily of the form $\lambda, -\eta, -\eta (\lambda, \eta > 0)$. Let g_1, g_2, g_3 be corresponding independent eigenvectors. Suppose $\{g_1, g_2, g_3\}$ is related to $\{e_1, e_2, e_3\}$ by

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad A \in GL(3, \mathbb{R}).$$

From the identities

$$\begin{aligned} \lambda g_1 &= [g_2, g_3] \\ -\eta g_2 &= [g_3, g_1] \\ -\eta g_3 &= [g_1, g_2], \end{aligned}$$

it is not very hard to get

$$A^{-1} = S A^t S \Delta^{-2}$$

where

$$\Delta = \begin{bmatrix} \eta & 0 & 0 \\ 0 & \sqrt{\lambda\eta} & 0 \\ 0 & 0 & \sqrt{\lambda\eta} \end{bmatrix}, \quad S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $X = \Delta^{-1}A$. Then a simple calculation shows that $X \in \mathrm{SO}_o(1,2)$. Therefore $A = \Delta X$ as required. This finishes the proof of (B).

We complete the proof of the theorem as follows. If $g \in \mathrm{SO}(3) \setminus \mathrm{GL}^+(3,\mathbb{R})$ with $\dim I(G,g) = 4$, then as a coset g has a representative of the form ΔX where Δ is as above in (B) and $X \in \mathrm{SO}_o(1,2)$. This determines a representation of such g into $(\mathbb{R}^+)^2 \times \mathrm{SO}_o(1,2)$ so that $\Delta \in (\mathbb{R}^+)^2$ in an obvious way. Moreover, if $\Delta_1 X_1$ is another such representative of g, then clearly $\Delta_1 = \Delta$ and hence $X_1 = BX$ for some $B \in K$.

Consequently, the space of left invariant metrics whose isometry group is 4-dimensional is identified with $(\mathbb{R}^+)^2 \times \{K \setminus \mathrm{SO}_o(1,2)\} \approx (\mathbb{R}^+)^4$. This completes the proof.

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