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SOME LIPSCHITZ REGULARITY OF THE CAUCHY TRANSFORM ON A CONVEX DOMAIN IN $\mathbb{C}^2$ WITH REAL ANALYTIC BOUNDARY

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ABSTRACT. Let $D \subset \mathbb{C}^2$ be a bounded convex domain with real analytic boundary. We get some Lipschitz regularity of the Cauchy transform on the convex domain D.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. The Cauchy transform $\mathbf{C}(f)$ of a function $f \in C^\infty(bD)$ is defined by

$$(\mathbf{C}f)(z) = \frac{1}{2\pi i}\int_{bD} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \in D$. It is well-known that the Cauchy transform maps $C^\infty(bD)$ into $A^\infty(\overline{D}) = \mathcal{O}(D) \cap C^\infty(\overline{D})$, and so it can be viewed as an operator on $C^\infty(bD)$. In this paper, we consider the Cauchy kernel and the Cauchy transform in higher dimensional cases.

For a domain D in $\mathbb{C}^n$ we denote $\mathcal{O}(D)$ by the space of holomorphic functions on D equipped with the natural topology in which convergent sequences are precisely those which converge compactly.

Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ and let $C(\zeta, z)$ be the Cauchy kernel of D (see [7]). Then for $f \in A(D) = \mathcal{O}(D) \cap C(\overline{D})$, we have

$$f(z) = \int_{bD} f(\zeta) C(\zeta, z)$$

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for $z \in D$. We can generalize some classical results for the Cauchy kernel in $\mathbb{C}$ to the Cauchy kernel in the strongly pseudoconvex domain D.

THEOREM 1.1 ([2, 4, 7]). *Define*

$$(\mathbf{C}f)(z) = \int_{bD} f(\zeta)C(\zeta, z).$$

Then

(i) $\mathbf{C}(f) \in \mathcal{O}(D)$ *for all* $f \in L^1(bD)$.
(ii) $\mathbf{C} : L^p(bD) \to \mathcal{O}(D)$ *is continuous for* $1 \leq p \leq \infty$.
(iii) $\mathbf{C} : \Lambda_\alpha(bD) \to \mathcal{O}(D) \cap \Lambda_\alpha(D)$ *is bounded for* $0 < \alpha < 1$.

REMARK. It is not true, even in the case $n = 1$, that $\mathbf{C}(f)$ extends continuously to $\overline{D}$ if f is only continuous on bD; (iii) gives a useful sufficient condition for the continuity of $\mathbf{C}(f)$ on $\overline{D}$. Some regularity properties of the Cauchy kernel on the Hardy space $H^p(D)$ were studied by Stout [9].

Let $D \Subset \mathbb{C}^n$ be a convex domain with C^2 defining function r. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{bD} f(\zeta) \frac{\partial r(\zeta) \wedge (\overline{\partial}\partial r(\zeta))^{n-1}}{< \partial r(\zeta), \zeta - z >^n} \tag{1.1}$$

for $f \in A(D) = \mathcal{O}(D) \cap C(\overline{D})$ and $z \in D$ ([7], IV-3).

DEFINITION 1.2. Define

$$C(\zeta, z) = \frac{\partial r(\zeta) \wedge (\overline{\partial}\partial r(\zeta))^{n-1}}{< \partial r(\zeta), \zeta - z >^n}$$

and

$$(\mathbf{C}f)(z) = \frac{1}{(2\pi i)^n} \int_{bD} f(\zeta)C(\zeta, z)$$

for $f \in L^1(bD)$. Then $C(\zeta, z)$ is called the Cauchy kernel and $\mathbf{C}(f)$ the Cauchy transform of f on the convex domain D.

Let $D \Subset \mathbb{C}^2$ be a convex domain with real analytic boundary. We define $\phi(\zeta, z) =< \partial r(\zeta), \zeta - z >$. By Range [6], there are a positive number $\epsilon > 0$ and a positive even integer $m \geq 2$ such that

$$|\phi(\zeta, z)| \gtrsim |\text{Im } \phi(\zeta, z)| + \text{ dist } (z, bD) + |\zeta - z|^m$$

for $(\zeta, z) \in bD \times \overline{D}$ with $|\zeta - z| < \epsilon$.

For any bounded domain E, we denote by $\Lambda_\alpha(E)$ the classical Lipschitz space of order α (See [8]). Now we state our main results.

THEOREM 1.3.

(i) $\mathbf{C}(f) \in \mathcal{O}(D)$ *for all* $f \in L^1(bD)$.

(ii) $\mathbf{C} : L^p(bD) \to \mathcal{O}(D)$ *is continuous for* $1 \leq p \leq \infty$.

(iii) *If* $1 - \frac{2}{m} < \alpha < 1$, *then there exists* $0 < \beta < \alpha$ *such that* $\mathbf{C} : \Lambda_\alpha(bD) \to \mathcal{O}(D) \cap \Lambda_\beta(D)$ *is bounded.*

2. Elementary estimates

We first prove elementary estimates which will be useful in proving the main results.

LEMMA 2.1. *For* $\beta > 1$, $\delta > 0$, *and a positive even integer* $m \geq 2$,

$$\int_{\substack{x \in \mathbb{R}^2 \\ |x|<1}} \frac{dx}{(\delta + |x|^m)^\beta} \lesssim \delta^{-\beta + \frac{2}{m}}.$$

PROOF. By using the polar coordinates, we get

$$\int_{\substack{x \in \mathbb{R}^2 \\ |x|<1}} \frac{dx}{(\delta + |x|^m)^\beta} \lesssim \int_0^1 \frac{\rho d\rho}{(\delta + \rho^m)^\beta} \lesssim \delta^{-\beta + \frac{2}{m}} \int_0^\infty \frac{s ds}{(1 + s^m)^\beta}.$$

Since $\beta > 1$ and $m \geq 2$, the last integral converges, and we are done.

We now show that Im $\phi(\zeta, z)$ can be used as a local coordinate on bD. This can be proved by Range ([7], V-3). But for the reader's convenience, we give the proof.

LEMMA 2.2. *There are positive constants* M, a, *and* $\eta \leq \epsilon$, *and, for each* z *with dist* $(z, bD) \leq a$, *there is a* C^∞ *local coordinate system* $(t_1, t_2, t_3, t_4) = t = t(\zeta, z)$ *on* $B(z, \eta)$ *such that the following hold:*

$$t_1(\zeta, z) = r(\zeta) \quad \text{and} \quad t(z, z) = (r(z), 0, 0, 0),$$

$$t_2(\zeta, z) = \text{Im } \phi(\zeta, z),$$

$$|t(\zeta, z)| < 1 \quad \text{for} \quad \zeta \in B(z, \eta),$$

$$|J_{\mathbb{R}}(t(\cdot, z))| \leq M \quad \text{and} \quad |\det J_{\mathbb{R}}(t(\cdot, z))| \geq \frac{1}{M}.$$

PROOF. Fix $z \in bD$. Since

$$\phi(\zeta, z) = \sum_{j=1}^{2} \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j),$$

it follows that

$$\begin{aligned} d_\zeta \phi(z, z) &= \frac{\partial \phi}{\partial \zeta_1}(z, z) d\zeta_1 + \frac{\partial \phi}{\partial \zeta_2}(z, z) d\zeta_2 \\ &= \frac{\partial r}{\partial \zeta_1}(z) d\zeta_1 + \frac{\partial r}{\partial \zeta_2}(z) d\zeta_2 \\ &= \partial_\zeta r(z). \end{aligned} \tag{2.1}$$

Thus, at the point $\zeta = z$, one obtains

$$\begin{aligned} d_\zeta \operatorname{Im} \phi \wedge d_\zeta r &= \frac{1}{2i}(\partial_\zeta r - \overline{\partial}_\zeta r) \wedge (\partial_\zeta r + \overline{\partial}_\zeta r) \\ &= \frac{1}{i} \partial r \wedge \overline{\partial} r \neq 0. \end{aligned}$$

Hence, we have the result.

LEMMA 2.3.

(i) $\displaystyle \int_{bD} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)|} \lesssim 1.$

(ii) $\displaystyle \int_{bD} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)|^{3-\alpha}} \lesssim |r(z)|^{-2+\alpha+\frac{2}{m}}$ *if* $\displaystyle 1 - \frac{2}{m} < \alpha < 1.$

PROOF. We shall use the special real coordinate system (t_1, t_2, t_3, t_4) on the ball $B(z, \eta)$ introduced in Lemma 2.2. With the notation chosen there, it is clearly enough to prove the estimates for $|r(z)| \leq a$, and the region of integration replaced by $bD \cap B(z, \eta)$. By using the estimate $|\phi| \gtrsim |t_2| + |r(z)| + |(t_3, t_4)|^m$ for $\zeta \in bD \cap B(z, \eta)$, it follows that

$$\int_{bD} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)|} \lesssim \int_{\substack{|t'| \leq 1 \\ t' = (t_2, t_3, t_4)}} \frac{dt_2 dt_3 dt_4}{|t_2| + |r(z)| + |(t_3, t_4)|^m} \lesssim 1.$$

For the case (ii), by Lemma 2.1, we get

$$\begin{aligned}\int_{bD}\frac{d\sigma(\zeta)}{|\phi(\zeta,z)|^{3-\alpha}} &\lesssim \int_{\substack{|t'|<1\\ t'=(t_2,t_3,t_4)}}\frac{dt_2dt_3dt_4}{(|t_2|+|r(z)|+|(t_3,t_4)|^m)^{3-\alpha}}\\ &\lesssim \int_0^1\frac{dt_2}{(t_2+|r(z)|)^{3-\alpha-\frac{2}{m}}}\\ &\lesssim |r(z)|^{-2+\alpha+\frac{2}{m}}.\end{aligned}$$

3. Proof of Theorem 1.3

Since (i) is trivial, we prove only (ii) and (iii). For a compact subset K of D, it follows that

$$\begin{aligned}|\mathbf{C}(f)-\mathbf{C}(g)|_K &= \sup_{z\in K}|(\mathbf{C}f)(z)-(\mathbf{C}g)(z)|\\ &= \sup_{z\in K}\left|\int_{bD}(f(\zeta)-g(\zeta))C(\zeta,z)\right|\\ &\lesssim \|f-g\|_{L^p(bD)},\quad 1\le p\le\infty.\end{aligned}$$

This proves (ii).

For the proof of (iii), let $z=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$. Then we have

$$\frac{\partial(\mathbf{C}f)}{\partial x_j}(z)=\frac{1}{(2\pi i)^2}\int_{bD}f(\zeta)\frac{\partial}{\partial x_j}C(\zeta,z)=\int_{bD}f(\zeta)\frac{K(\zeta,z)}{\phi^3(\zeta,z)}d\sigma(\zeta)$$

where $K(\zeta,z)\in C^\infty(\overline{D}\times\overline{D})$ and $d\sigma(\zeta)$ is the surface measure on bD. Let $\gamma>0$ be the smaller of the constants a and η in Lemma 2.2. It suffices to show that $|I(z)|\le C_\alpha|r(z)|^{\beta-1}|f|_{\Lambda_\alpha(bD)}$ for $z\in D$, where $\beta=\alpha+\frac{2}{m}-1$ and

$$I(z)=\int_{bD\cap B(z,\gamma)}f(\zeta)\frac{\chi(\zeta)K(\zeta,z)}{\phi^3(\zeta,z)}d\sigma(\zeta),$$

where χ is a compactly supported cut off function in $B(z,\gamma)$.

We decompose

$$f(t) = [f(t_1, t_2, t_3, t_4) - f(0, 0, t_3, t_4)] + f(0, 0, t_3, t_4). \tag{3.1}$$

Corresponding to (3.1), we have $I(z) = I_1(z) + I_2(z)$, where, with $t' = (t_2, t_3, t_4)$,

$$|I_1(z)| \lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'|\leq 1} \frac{|t_2|^\alpha}{|\phi|^3} dt_2 dt_3 dt_4$$

and

$$I_2(z) = \int_{|t'|\leq 1} \frac{f(0, 0, t_3, t_4)\chi(t')K(t', z)}{\phi^3} dt_2 d_3 d_4. \tag{3.2}$$

From (ii) in Lemma 2.3 it follows that

$$\begin{aligned} |I_1(z)| &\lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'|\leq 1} \frac{|t_2|^\alpha}{|\phi|^3} dt_2 dt_3 dt_4 \\ &\lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'|\leq 1} \frac{1}{|\phi|^{3-\alpha}} dt_2 dt_3 dt_4 \\ &\lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{-2+\alpha+\frac{2}{m}}. \end{aligned}$$

In order to estimate (3.2), we first integrate by parts in t_2, using the fact that $-\frac{1}{2}\frac{\partial^2}{\partial t_2^2}\frac{1}{\phi} = \frac{1}{\phi^3}$. We give some explanation for the identity. From (2.1) one also obtain that $2\, d_\zeta$ Re $\phi = d_\zeta r$ at $\zeta = z$, therefore

$$d_\zeta \text{ Re } \phi \wedge d_\zeta \text{ Im } \phi \neq 0.$$

If we choose coordinates $t_1 = r(\zeta)$ and $t_2 = \text{Im } \phi(\zeta, z)$ as in Lemma 2.2, then

$$\frac{\partial \text{ Re } \phi}{\partial t_2} = 0 \quad \text{and} \quad \frac{\partial \text{ Im } \phi}{\partial t_2} = 1.$$

Thus, it follows that $\frac{\partial \phi}{\partial t_2} = i$, therefore

$$\begin{aligned} -\frac{1}{2}\frac{\partial^2}{\partial t_2^2}\frac{1}{\phi} &= -\frac{1}{2}\frac{\partial}{\partial t_2}\left(-\frac{1}{\phi^2}\frac{\partial \phi}{\partial t_2}\right) \\ &= \frac{i}{2}\left(-\frac{2}{\phi^3}\frac{\partial \phi}{\partial t_2}\right) \\ &= \frac{1}{\phi^3}. \end{aligned}$$

Since χ has compact support in $B(z,\gamma)$, by integration by parts, one obtains

$$I_2(z) = -\frac{1}{2}\int_{|t'|\leq 1}\frac{f(0,0,t_3,t_4)}{\phi}\frac{\partial^2}{\partial t_2^2}(\chi(t')K(t',z))dt_2dt_3dt_4$$

which leads to, by (i) in Lemma 2.3,

$$|I_2(z)| \lesssim \|f\|_{L^\infty(bD)}.$$

Thus we get (iii).

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